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ON SOLVABILITY AND NILPOTENCY OF ALGEBRAS WITH BRACKET

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Dedicated to Manuel Ladra on occasion of his 60^{th} *birthday*

Abstract. We analyze properties of solvable and nilpotent algebras with bracket. The class of solvability and nilpotency of the tensor square of an algebra with bracket is obtained. Homological characterizations of nilpotent algebras with bracket are presented.

1. Introduction

Algebras with bracket were introduced in [9] as a generalization of Poisson algebras where the dot operation is associative, but not necessarily commutative, and the bracket operation does not satisfy any condition, except the distribution Poisson law (1). Since the seminal paper [9], several algebraic properties of this structure, in particular the homological ones, were analyzed in different articles [6, 7, 8].

The goal of the present article is to continue with the study of the algebras with bracket structure, in particular solvability and nilpotency properties. To do so, we organize the paper as follows: in Section 2 we recall from [6, 9] some basic notions needed in the forthcoming sections and we classify 2-dimensional AWB in Example 2.1 ix). In Section 3 we study solvable algebras with bracket by means of derived sequences and we introduce the notion of right representation M over an algebra with bracket A. Then we construct $\mathrm{AWB}(f)$, the algebra with bracket structure over M associated to a homomorphism of right A-representations $f : \mathrm{M} \to \mathsf{A}$. When this homomorphism satisfies the property $f(\mathrm{M}) \subseteq \mathsf{A}^{(j)}$ and A is solvable of class k, then $\mathrm{AWB}(f)$ is solvable of class $k - j + 1$. This result allow us to establish the solvability class of $\mathsf{A}^{\otimes 2}$ and $\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}$ for particular algebras with bracket A. In Section 4 we analyze nilpotent algebras with bracket in terms of lower and upper central series and

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by means of the normalizer condition. The class of nilpotency corresponding to $\mathsf{A}^{\otimes 2}$ is obtained. Finally, Section 5 is devoted to obtain homological characterizations of nilpotent algebras with bracket, in particular by means of the Schur c-multiplier of an algebra with bracket.

2. Preliminaries on algebras with bracket

We fix a field $\mathbb{K}$. All vector spaces are taken over $\mathbb{K}$. In what follows Hom and $\otimes$ means $\mathsf{Hom}_{\mathbb{K}}$ and $\otimes_{\mathbb{K}}$ respectively.

Definition ([9])**.** An *algebra with bracket* or simply an *AWB* is an associative (not necessarily commutative) algebra A equipped with a bilinear map $[-,-] : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A}$, $(a \otimes b) \mapsto [a,b]$ satisfying the following relation:

$$[a \cdot b, c] = [a,c] \cdot b + a \cdot [b,c] \tag{1}$$

for all $a, b, c \in \mathsf{A}$.

We denote by AWB the category whose objects are algebras with bracket and whose morphisms are $\mathbb{K}$-linear maps preserving the $\cdot$ and $[-,-]$ operations. It is a routine task to check that AWB is a semi-abelian category [5], so classical results as Five Lemma, 3×3 Lemma and the Second Noether Isomorphism Theorem hold for AWB.

Example 2.1.

i) It is clear that any Poisson algebra is an AWB. In fact, the category Poiss of commutative Poisson algebras is a subcategory of AWB. The inclusion functor $\mathsf{Poiss} \hookrightarrow \mathsf{AWB}$ has as left adjoint the functor given by $\mathsf{A} \mapsto \mathsf{A}_{\mathsf{Poiss}}$, where $\mathsf{A}_{\mathsf{Poiss}}$ is the maximal quotient of A, such that the following relations hold: $a \cdot b - b \cdot a \sim 0$, $[a,a] \sim 0$ and $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] \sim 0$.

ii) Let A be an associative $\mathbb{K}$-algebra equipped with a linear application $D : \mathsf{A} \to \mathsf{A}$. Then A is an AWB with respect to the bracket $[a,b] := aD(b) - D(b)a$.

For $D = 0$ one obtains an AWB, which has the trivial bracket. The more interesting is the case when $D = Id$, then A is an AWB with respect to the usual bracket for associative algebras. This particular AWB is said to be the *tautological* AWB associated to an associative algebra A.

iii) Another example comes from dendriform algebras (see [18]). If A is a dendriform algebra, then $(\mathsf{A}, \star, [-,-])$ is an AWB, where $a \star b = a \prec b + a \succ b$ and $[a,b] = a \star b - b \star a = a \prec b + a \succ b - b \prec a - b \succ a$.

iv) For examples coming from Physics we refer to [16].

v) Novikov algebras were firstly introduced in the study of Hamiltonian operators concerning integrability of certain nonlinear partial differential equations (see [12]). They are algebras $(\mathsf{A}, \cdot)$ over a field of characteristic zero satisfying the following identities:

a) $x \cdot (y \cdot z) - (x \cdot y) \cdot z = y \cdot (x \cdot z) - (y \cdot x) \cdot z$
b) $(x \cdot y) \cdot z = (x \cdot z) \cdot y$

Novikov algebras satisfying the additional condition

c) $(x \cdot y) \cdot z = z \cdot (y \cdot x)$

can be endowed with an AWB structure by means of the operations $x \circ y = x \cdot y + y \cdot x$ and $[x, y] = x \cdot y - y \cdot x$.

Let us observe that the $\circ$ operation endows A with an associative algebra structure. Following example ii), a Novikov algebra satisfying the additional condition c) can be endowed with another AWB structure, but the bracket operation is trivial since the $\circ$ operation is commutative.

vi) Let A be an algebra with bracket, then the $\mathbb{K}$-vector space $\mathsf{A} \otimes \mathsf{A}$ endowed with the operations

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 b_1) \otimes (a_2 b_2),$$

$$[a_1 \otimes a_2, b_1 \otimes b_2] = [a_1, [b_1, b_2]] \otimes a_2 + a_1 \otimes [a_2, [b_1, b_2]]$$

for all $a_i, b_i \in \mathsf{A}, i = 1, 2$, is an algebra with bracket.

vii) Let A be an algebra with bracket, then the $\mathbb{K}$-vector space $\mathsf{A} \otimes \mathsf{A}$ endowed with the operations

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 \otimes (a_2 (b_1 b_2)),$$

$$[a_1 \otimes a_2, b_1 \otimes b_2] = a_1 \otimes [a_2, b_1 \cdot b_2] + [a_1, b_1 \cdot b_2] \otimes a_2$$

for all $a_i, b_i \in \mathsf{A}, i = 1, 2$, is an algebra with bracket.

viii) 1-dimensional algebras with bracket with basis $\{e\}$ over a field $\mathbb{K}$ of characteristic 0, up to isomorphism, are abelian (see below) AWB or an element of the following non pairwise isomorphism classes:

a) $e \cdot e = 0, [e, e] = e$,
b) $e \cdot e = e, [e, e] = 0$.

ix) Having in mind the classification of 2-dimensional associative algebras given in [20], 2-dimensional algebras with bracket with basis $\{e, f\}$ over the field of complex numbers $\mathbb{C}$, up to isomorphism, are abelian AWB (see below) or an element of the following non pairwise isomorphism classes:

a) AWB_2^1: Dot operation is trivial together with any bracket operation.
b) AWB_2^2: Any dot associative operation together with a trivial bracket operation.
c) AWB_2^3:

$$\begin{aligned} e \cdot e = f; \quad & [e, e] = \alpha_1 e + \beta_1 f; \\ & [e, f] = \alpha_2 e + \beta_2 f; \\ & [f, e] = 2\alpha_1 f; \\ & [f, f] = 2\alpha_2 f. \end{aligned}$$

d) AWB_2^4:

$$\begin{array}{ll} e \cdot e = e; & [e,f] = \beta_1 f; \\ e \cdot f = f; & [f,e] = \beta_2 f; \\ & [f,f] = \beta_3 f. \end{array}$$

e) AWB_2^5:

$$\begin{array}{ll} e \cdot e = e; & [e,e] = \beta_1 f; \\ f \cdot e = f; & [e,f] = \beta_2 f; \\ & [f,e] = \beta_3 f; \\ & [f,f] = \beta_4 f. \end{array}$$

f) AWB_2^6:

$$\begin{array}{ll} e \cdot e = e; & [f,e] = \beta_1 f; \\ e \cdot f = f; & [f,f] = \beta_2 f; \\ f \cdot e = f. & \end{array}$$

Note that all non written operations are trivial.

The following notions for an algebra with bracket A are given in [6] and they agree with the corresponding notions in semi-abelian categories. A subalgebra I of A is a $\mathbb{K}$-subspace which is closed under $\cdot$ and $[-,-]$ operations, that is, $\mathsf{I} \cdot \mathsf{I} \subseteq \mathsf{I}$ and $[\mathsf{I},\mathsf{I}] \subseteq \mathsf{I}$. I is said to be a *right* (*respectively, left*) *ideal* if $\mathsf{A} \cdot \mathsf{I} \subseteq \mathsf{I}$, $[\mathsf{A},\mathsf{I}] \subseteq \mathsf{I}$ (respectively, $\mathsf{I} \cdot \mathsf{A} \subseteq \mathsf{I}$, $[\mathsf{I},\mathsf{A}] \subseteq \mathsf{I}$). If I is both left and right ideal, then it is said to be a *two-sided ideal*. In this case, the quotient A/I is endowed with an AWB structure naturally induced from the operations on A.

Let I,J be two-sided ideals of A. The *commutator ideal* of I and J is the two-sided ideal of I and J

$$[[\mathsf{I},\mathsf{J}]] = \langle \{ i \cdot j, j \cdot i, [i,j], [j,i] \mid i \in \mathsf{I}, j \in \mathsf{J} \} \rangle.$$

Obviously $[[\mathsf{I},\mathsf{J}]] \subseteq \mathsf{I} \bigcap \mathsf{J}$. Observe that $[[\mathsf{I},\mathsf{J}]]$ is not a two-sided ideal of A, except when $\mathsf{I} = \mathsf{A}$ or $\mathsf{J} = \mathsf{A}$. In the particular case $\mathsf{I} = \mathsf{J} = \mathsf{A}$, one obtains the definition of derived algebra of A, i.e.,

$$[[\mathsf{A},\mathsf{A}]] = \langle \{ a \cdot b, [a,b] \mid a,b \in \mathsf{A} \} \rangle.$$

We define the *center* of an algebra with bracket A as the two-sided ideal

$$\mathsf{Z}(\mathsf{A}) = \{ a \in \mathsf{A} \mid a \cdot b = 0 = b \cdot a, [a,b] = 0 = [b,a] \text{ for all } b \in \mathsf{A} \}.$$

An *abelian* algebra with bracket A is an AWB with trivial $\cdot$ and $[-,-]$ operations, i.e., $\mathsf{A} \cdot \mathsf{A} = 0 = [\mathsf{A},\mathsf{A}]$. Hence an algebra with bracket A is abelian if and only if $\mathsf{A} = \mathsf{Z}(\mathsf{A})$.

3. Solvable algebras with bracket

Definition. Let I be a two-sided ideal of an algebra with bracket A. Assume that there is a finite sequence $\mathsf{I}_i, 1 \leq i \leq k$, of two-sided ideals of A such that

$$\mathsf{I} = \mathsf{I}_0 \trianglelefteq \mathsf{I}_1 \trianglelefteq \cdots \trianglelefteq \mathsf{I}_{k-1} \trianglelefteq \mathsf{I}_k = \mathsf{A}.$$

This chain of two-sided ideals is said to be a series of length k from I to A. The two-sided ideals I_i are said to be the terms of the series and the quotient $\mathsf{I}_i/\mathsf{I}_{i-1}$ the factors of the series. A series of A will mean a series from 0 to A. A series is said to be central if all its factors are central, that is $\mathsf{I}_i \cdot \mathsf{A} \subseteq \mathsf{I}_{i-1}, \mathsf{A} \cdot \mathsf{I}_i \subseteq \mathsf{I}_{i-1}, [\mathsf{I}_i, \mathsf{A}] \subseteq \mathsf{I}_{i-1}, [\mathsf{A}, \mathsf{I}_i] \subseteq \mathsf{I}_{i-1}$, equivalently $\mathsf{I}_i/\mathsf{I}_{i-1} \subseteq Z(\mathsf{A}/\mathsf{I}_{i-1})$. A series is said to be abelian if $\mathsf{I}_i \cdot \mathsf{I}_i \subseteq \mathsf{I}_{i-1}, [\mathsf{I}_i, \mathsf{I}_i] \subseteq \mathsf{I}_{i-1}$, that is, $[[\mathsf{I}_i/\mathsf{I}_{i-1}, \mathsf{I}_i/\mathsf{I}_{i-1}]] = 0$.

When the terms of a series are subalgebras instead of two-sided ideals of A, then the series are said to be a sequence.

Definition. An algebra with bracket A is said to be solvable if it has an abelian sequence. Let k be the minimal length of such series, then k is said to be the class of solvability of A.

We shall now show that among all the abelian sequences of a solvable AWB there is one which descends most rapidly.

Definition. The sequence of subalgebras of A defined recursively by

$$\mathsf{A}^{(0)} = \mathsf{A}; \qquad \mathsf{A}^{(i)} = [[\mathsf{A}^{(i-1)}, \mathsf{A}^{(i-1)}]],\ i \geq 1$$

is said to be the derived sequence of the algebra with bracket A.

Remark 3.1. Let us observe that $\mathsf{A}^{(i)}$ is a two-sided ideal of $\mathsf{A}^{(i-1)}$, but is not a two-sided ideal of A in general.

Theorem 3.2.

i) *Let* A *be an algebra with bracket and let* $\mathsf{I} = \mathsf{I}_0 \trianglelefteq \mathsf{I}_1 \trianglelefteq \cdots \trianglelefteq \mathsf{I}_{j-1} \trianglelefteq \mathsf{I}_j = \mathsf{A}$ *be an abelian sequence of* A*, then* $\mathsf{A}^{(i)} \subseteq \mathsf{I}_{j-i}, 0 \leq i \leq j$.

ii) *An algebra with bracket* A *is solvable with class of solvability* k *if and only if* $\mathsf{A}^{(k)} = 0$ *and* $\mathsf{A}^{(k-1)} \neq 0$.

Proof. i) This is a routine checking by using induction on i.

ii) If $\mathsf{A}^{(k)} = 0$ and $\mathsf{A}^{(k-1)} \neq 0$, then $0 = \mathsf{A}^{(k)} \trianglelefteq \mathsf{A}^{(k-1)} \trianglelefteq \cdots \trianglelefteq \mathsf{A}^{(1)} \trianglelefteq \mathsf{A}^{(0)} = \mathsf{A}$ is an abelian sequence and by i) its length is minimal. Therefore A is a solvable algebra with bracket of class k. The converse statement is a direct consequence of i). $\square$

Abelian algebras with bracket are solvable algebras with bracket. The 2-dimensional tautological algebra with bracket with basis $\{e, f\}$ and operations $e \cdot e = f$ and trivial bracket (see Example 2.1 ii)), is a solvable algebra with bracket of class 2. Algebras with bracket of the class AWB_2^4 in Example 2.1 ix) are non-solvable. Subalgebras and images by homomorphisms of solvable algebras with bracket are solvable algebras with bracket as well. If A/H is a solvable algebra with bracket, where H is a solvable two-sided ideal of A, then A is a solvable algebra with bracket as well. If H and K are solvable two-sided ideals of an algebra with bracket A, then $\mathsf{H} \cap \mathsf{K}$ and $\mathsf{H} + \mathsf{K}$ are solvable two-sided ideals of A as well. The proof of the last three arguments use classical arguments (see [13, 15]) and we omit it.

Proposition 3.3. *If* H *and* K *are solvable two-sided ideals of an algebra with bracket* A*, then* $[[\mathsf{H}, \mathsf{K}]]$ *is a solvable two-sided ideal of* H *or* K*.*

Proof. $[[\mathsf{H}, \mathsf{K}]] \subseteq \mathsf{H} \cap \mathsf{K}$ and $\mathsf{H} \cap \mathsf{K}$ is a solvable two-sided ideal of H or K. $\square$

Definition. Let A be an algebra with bracket. A $\mathbb{K}$-vector space M is said to be a right A-representation if M is a right module over A equipped with a bilinear map

$$\begin{aligned} [-,-]: \quad & \mathrm{M} \times \mathsf{A} \to \mathrm{M} \\ & (m, a) \mapsto [m, a], \end{aligned}$$

satisfying the following identity

$$[m \cdot a, b] = m \cdot [a, b] + [m, b] \cdot a$$

for all $m \in \mathrm{M}$ and $a, b \in \mathsf{A}$.

Example 3.4.

i) Let A be an algebra with bracket, then $\mathrm{M}=\mathsf{A}$ is a right A-representation, where the operations $\cdot$ and $[-,-]$ are the structural ones for A.
ii) Representations of algebras with bracket [9] are right A-representations.
iii) Let A be a tautological algebra with bracket associated to an associative algebra A (see Example 2.1 ii)) and let M be a right A-module. If we put $[m, a] = m \cdot a - a \cdot m, m \in \mathrm{M}, a \in \mathsf{A}$, then M is a right A-representation.
iv) Let A be an algebra with bracket, then the $\mathbb{K}$-vector space $\mathsf{A}^{\otimes 2}$ is endowed with a structure of right A-representation with respect the actions

$$(a_1 \otimes a_2) \cdot a = a_1 \otimes (a_2 a),$$
$$[a_1 \otimes a_2, a] = a_1 \otimes [a_2, a] + [a_1, a] \otimes a_2$$

for all $a_1, a_2, a \in \mathsf{A}$.

Definition. Let M and N be right A-representations. A linear application $f : \mathrm{M} \to \mathrm{N}$ is said to be a homomorphism of right A-representations if the following identities hold for any $m \in \mathrm{M}$ and $a \in \mathsf{A}$:

$$f(m \cdot a) = f(m) \cdot a,$$
$$f[m, a] = [f(m), a].$$

Example 3.5. For an algebra with bracket A, consider the right A-representation $\mathsf{A}^{\otimes 2}$ given in Example 3.4 iv). Then the $\mathbb{K}$-linear map $f : \mathsf{A}^{\otimes 2} \to \mathsf{A}$ given by $f(a_1 \otimes a_2) = a_1 \cdot a_2$ is a homomorphism of right A-representations.

Let $f : \mathrm{M} \to \mathsf{A}$ be a homomorphism of right A-representations from M to the structural representation of A, then the operations

$$m_1 \cdot m_2 := m_1 \cdot f(m_2),$$
$$[m_1, m_2] := [m_1, f(m_2)],$$

define an algebra with bracket structure on M. We denote by $\mathrm{AWB}(f)$ this particular algebra with bracket.

One observes that AWB(f) is a non-commutative Leibniz-Poisson algebra [7] if and only if the following relations hold for any $m_1, m_2, m_3 \in \mathrm{M}$:

$$[m_1, [f(m_2), f(m_3)]] - [[m_1, f(m_2)], f(m_3)] + [[m_1, f(m_3)], f(m_2)] = 0.$$

On the other hand, AWB(f) is a Poisson algebra if in addition the following relations hold for any $m_1, m_2 \in \mathrm{M}$:

$$m_1 \cdot f(m_2) = m_2 \cdot f(m_1),$$
$$[m_1, f(m_2)] + [m_2, f(m_1)] = 0.$$

Remark 3.6. Observe that for an algebra with bracket A, the structure of algebra with bracket AWB(f) provided by the homomorphism $f : \mathsf{A}^{\otimes 2} \to \mathsf{A}$ given in Example 3.5 is nothing else that the structure of algebra with bracket of $\mathsf{A}^{\otimes 2}$ given in Example 2.1 vii).

Proposition 3.7. *Let $f : \mathrm{M} \to \mathsf{A}$ be a homomorphism of right A-representations with the property $f(\mathrm{M}) \subset \mathsf{A}^{(j)}$ and $\mathsf{A}^{(k)} = 0$, then* AWB$(f)^{(k-j+1)} = 0$. *In other words, if $f(\mathrm{M}) \subset \mathsf{A}^{(j)}$ and A is a solvable* AWB *of class k, then* AWB(f) *is a solvable* AWB *of class $k - j + 1$.*

Proof. f : AWB$(f) \to \mathsf{A}$ is a homomorphism of algebras with bracket and $f(\mathrm{AWB}(f)) \subset \mathsf{A}^{(j)}$, then $f(\mathrm{AWB}(f)^{(m)}) \subset \mathsf{A}^{(m+j)}$. Hence $f(\mathrm{AWB}(f)^{(k-j)}) \subset \mathsf{A}^{(k)} = 0$.

By definition we have,

$$\begin{aligned}\mathrm{AWB}(f)^{(k-j+1)} &= [[(\mathrm{AWB}(f))^{(k-j)}, (\mathrm{AWB}(f))^{(k-j)}]] \\ &= [[(\mathrm{AWB}(f))^{(k-j)}, f(\mathrm{AWB}(f))^{(k-j)}]] \\ &= 0.\end{aligned}$$ □

Corollary 3.8. *Let A be a solvable algebra with bracket of class k, then the algebra with bracket $\mathsf{A}^{\otimes 2}$ given in Example 2.1* vii) *is solvable of class k.*

Proof. In Proposition 3.7 take the homomorphism of right A-representations $f : \mathsf{A}^{\otimes 2} \to \mathsf{A}$ given in Example 3.5; obviously $f(\mathsf{A}^{\otimes 2}) \subseteq \mathsf{A}^{(1)}$. □

Example 3.9.

i) For an algebra with bracket A, consider two copies of the $\mathbb{K}$-vector space $\mathsf{A}^{\otimes 2}$ and take the $\mathbb{K}$-vector space $\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}$; in order to distinguish the elements in every copy, we will denote $a_1 \otimes a_2$ the elements in the left hand side copy and $a_1 \circ a_2$ the elements in the right hand side copy.

The $\mathbb{K}$-vector space $\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}$ is endowed with a right A-representation structure with respect to the actions

$$\begin{aligned}(a_1 \circ a_2) \cdot a &= (a_1 a) \circ a_2 - a_1 \otimes [a, a_2], \\ [a_1 \circ a_2, a] &= [a_1, a] \circ a_2 + a_1 \circ [a_2, a], \\ (a_1 \otimes a_2) \cdot a &= a_1 \otimes (a_2 a), \\ [a_1 \otimes a_2, a] &= a_1 \otimes [a_2, a] + [a_1, a] \otimes a_2,\end{aligned} \tag{2}$$

Then the linear map

$$\varphi : \mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2} \to \mathsf{A} \tag{3}$$

given by

$$\varphi(a_1 \otimes a_2) = a_1 \cdot a_2, \quad \varphi(a_1 \circ a_2) = [a_1, a_2]$$

is a homomorphism of right A-representations. If A is a solvable algebra with bracket of class k, since $\varphi(\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}) \subseteq \mathsf{A}^{(1)}$, then AWB$(\varphi)$ is a solvable algebra with bracket of class k by Proposition 3.7.

ii) The tautological algebra with bracket corresponding to the class AWB_2^3 in Example 2.1 ix) (that is, $\alpha_i = \beta_i = 0, i = 1, 2$) is a solvable AWB of class 2. Consider the homomorphism of right A-representations given in (3) by $\varphi(e \otimes e) = f$ and zero elsewhere, where the right A-representation structure provided by (2) is $(e \circ e) \cdot e = f \circ e, (e \circ f) \cdot e = f \circ f, (e \otimes e) \cdot e = e \otimes f, (f \otimes e) \cdot e = f \otimes f$. Since $\varphi(\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}) \subseteq \mathsf{A}^{(1)}$ and A is solvable of class 2, then AWB(φ) is solvable of class 2 by Proposition 3.7.

4. Nilpotent algebras with bracket

Definition. An algebra with bracket A is said to be nilpotent if it has a central series. Let k be the length of that series, then $k + 1$ is called the class of nilpotency of A.

We shall now show that among all the central series of a nilpotent AWB there is one which descends most rapidly.

Definition. The sequence of two-sided ideals defined recursively by

$$\mathsf{A}^{[1]} = \mathsf{A}; \qquad \mathsf{A}^{[k]} = [[\mathsf{A}^{[k-1]}, \mathsf{A}]], \; k \geq 2$$

is said to be the lower central series of an algebra with bracket A.

Theorem 4.1.

i) *An* AWB A *is nilpotent with class of nilpotency k if and only if* $\mathsf{A}^{[k+1]} = 0$.

ii) *Suppose that* A *is nilpotent. Then for any central series of* A*, say* $0 = \mathsf{M}_0 \trianglelefteq \mathsf{M}_1 \trianglelefteq \cdots \trianglelefteq \mathsf{M}_k = \mathsf{A}$*,* $\mathsf{A}^{[i]} \subseteq \mathsf{M}_{k-i+1}, 1 \leq i \leq k+1$.

Proof. If $\mathsf{A}^{[k+1]} = 0$, then $0 = \mathsf{A}^{[k+1]} \trianglelefteq \mathsf{A}^{[k]} \trianglelefteq \cdots \trianglelefteq \mathsf{A}^{[2]} \trianglelefteq \mathsf{A}^{[1]} = \mathsf{A}$ is a central series of A of length k.

Conversely, if A is nilpotent, say $0 = \mathsf{M}_0 \trianglelefteq \mathsf{M}_1 \trianglelefteq \cdots \trianglelefteq \mathsf{M}_k = \mathsf{A}$ is a central series of A, then a straightforward induction shows that $\mathsf{A}^{[i]} \subseteq \mathsf{M}_{k-i+1}$. Particularly, $\mathsf{A}^{[k+1]} \subseteq \mathsf{M}_0 = 0$. Same argument proves the second statement. $\square$

Definition. Let M be a two-sided ideal of an algebra with bracket A. The two-sided ideal of A

$$C_{\mathsf{A}}(\mathsf{A}, \mathsf{M}) = \{a \in \mathsf{A} \mid a \cdot a', a' \cdot a, [a, a'], [a', a] \in \mathsf{M} \;\text{ for all } a' \in \mathsf{A}\}$$

is said to be the centralizer of A and M on A.

If $\mathsf{M} = 0$, then $C_{\mathsf{A}}(\mathsf{A}, 0) = Z(\mathsf{A})$.

Definition. We call upper central series of an algebra with bracket A to the sequence of two-sided ideals defined recursively by

$$\mathcal{Z}_0(\mathsf{A}) = 0; \qquad \mathcal{Z}_i(\mathsf{A}) = C_{\mathsf{A}}(\mathsf{A}, \mathcal{Z}_{i-1}(\mathsf{A})),\ i \geq 1.$$

Let us observe that $\mathcal{Z}_1(\mathsf{A}) = Z(\mathsf{A})$ and that $\mathcal{Z}_i(\mathsf{A})$ is a two-sided ideal of A.

Lemma 4.2. *Let* M *and* N *be two-sided ideals of an algebra with bracket* A*. If* $\mathsf{M} \cdot \mathsf{A} \subseteq \mathsf{N}, \mathsf{A} \cdot \mathsf{M} \subseteq \mathsf{N}, [\mathsf{M}, \mathsf{A}] \subseteq \mathsf{N}, [\mathsf{A}, \mathsf{M}] \subseteq \mathsf{N}$*, then* $\mathsf{M} \subseteq C_{\mathsf{A}}(\mathsf{A}, \mathsf{N})$.

We shall now show that among all the central series of a nilpotent algebra with bracket there is one which ascends most rapidly.

Proposition 4.3. *For a central series of a nilpotent algebra with bracket* A*, say* $0 = \mathsf{M}_0 \trianglelefteq \mathsf{M}_1 \trianglelefteq \cdots \trianglelefteq \mathsf{M}_k = \mathsf{A}$*, one has* $\mathsf{M}_i \subseteq \mathcal{Z}_i(\mathsf{A}),\ 0 \leq i \leq k$.

Proof. The assertion is true for $i = 0$. Proceeding by induction on i, we assume that it is true for $i - 1$, then using the centrality of the series and Lemma 4.2 we obtain that $\mathsf{M}_i \subseteq C_{\mathsf{A}}(\mathsf{A}, \mathsf{M}_{i-1}) \subseteq C_{\mathsf{A}}(\mathsf{A}, Z_{i-1}(\mathsf{A})) = Z_i(\mathsf{A})$. □

Theorem 4.4. *An algebra with bracket* A *is nilpotent with class of nilpotency* k *if and only if* $\mathcal{Z}_k(\mathsf{A}) = \mathsf{A}$ *and* $\mathcal{Z}_{k-1}(\mathsf{A}) \neq \mathsf{A}$.

Proof. If A is a nilpotent algebra with bracket with class of nilpotency k, then $\mathsf{A} = \mathsf{M}_k \subseteq \mathcal{Z}_k(\mathsf{A}) \subseteq \mathsf{A}$ by Proposition 4.3. Moreover, in this case $0 = \mathcal{Z}_0(\mathsf{A}) \trianglelefteq \mathcal{Z}_1(\mathsf{A}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}(\mathsf{A}) \trianglelefteq \mathcal{Z}_k(\mathsf{A}) = \mathsf{A}$ is a central series of length k of A. Hence $\mathcal{Z}_{k-1}(\mathsf{A}) \neq \mathsf{A}$.

Conversely, if $\mathcal{Z}_k(\mathsf{A}) = \mathsf{A}$ and $\mathcal{Z}_{k-1}(\mathsf{A}) \neq \mathsf{A}$, then $0 = \mathcal{Z}_0(\mathsf{A}) \trianglelefteq \mathcal{Z}_1(\mathsf{A}) \trianglelefteq \cdots \trianglelefteq \mathcal{Z}_{k-1}(\mathsf{A}) \trianglelefteq \mathcal{Z}_k(\mathsf{A}) = \mathsf{A}$ is a central series of A and by Proposition 4.3 its length is minimal. □

Abelian algebras with bracket are examples of nilpotent algebras with bracket. Clearly $\mathsf{A}^{(1)} = \mathsf{A}^{[2]} = [[\mathsf{A}, \mathsf{A}]]$ and $\mathsf{A}^{(n)} \subseteq \mathsf{A}^{[n]}, n \geq 1$, so nilpotent algebras with bracket are solvable algebras with bracket. The tautological algebra with bracket corresponding to the class AWB_2^3 in Example 2.1 ix), that is $e \cdot e = f$ and trivial bracket (see Example 2.1 ii)), is a nilpotent algebra with bracket of class 2. The two-dimensional algebra with bracket with trivial product and bracket operation given by $[e, f] = f$, corresponding to the class AWB_2^1 in Example 2.1 ix), is a solvable of class 2, but non nilpotent algebra with bracket. Subalgebras and images by homomorphisms of nilpotent algebras with bracket are nilpotent algebras with bracket. If $\mathsf{A}/Z(\mathsf{A})$ is a nilpotent algebra with bracket, then A is a nilpotent algebra with bracket. If A is a nilpotent and non trivial algebra with bracket, then $Z(\mathsf{A}) \neq 0$.

Example 4.5.

i) Let A be an abelian – so nilpotent – algebra with bracket, then $\mathsf{A}^{\otimes 2}$ with respect to both structures given in Example 2.1 vi) (respectively, Example 2.1 vii)) is abelian – so nilpotent – algebra with bracket.

ii) Let A be the algebra with bracket in the class AWB_2^1 given in Example 2.1 ix), then $\mathsf{A}^{\otimes 2}$ with the structure given in Example 2.1 vii) is an abelian algebra with bracket, hence is nilpotent.

Proposition 4.6. *Let* A *be a nilpotent algebra with bracket of class* $\leq k$*, then the algebra with bracket* $\mathsf{A}^{\otimes 2}$ *with the structure given in Example 2.1* vi) (*respectively, Example 2.1* vii)) *is nilpotent of class* $\leq 2k$.

Proof. With a straightforward induction and having in mind the structure given in Example 2.1 vi) (respectively, Example 2.1 vii)), it is easy to check the following inclusion

$$(\mathsf{A}^{\otimes 2})^{[k]} \subseteq \sum_{i=1}^{k} \mathsf{A}^{[k+1-i]} \otimes \mathsf{A}^{[i]}.$$

The statement is a direct consequence of this inclusion. □

Proposition 4.7.

i) *Let* H *be a two-sided ideal of an algebra with bracket* A *such that* $\mathsf{H} \subseteq Z(\mathsf{A})$*. Then* A *is nilpotent if and only if* A/H *is nilpotent.*

ii) *Let* $f : \mathsf{A} \twoheadrightarrow \mathsf{B}$ *be a central extension* (*that is,* $Ker(f) \subseteq Z(\mathsf{A})$) *of an algebra with bracket* B*. Then* A *is nilpotent if and only if* B *is nilpotent.*

Proof. i) The quotient of nilpotent algebras with bracket is nilpotent as well. Conversely, there exist k such that $(\mathsf{A}/\mathsf{H})^{[k]} = 0$, hence $\mathsf{A}^{[k]} \subseteq H \subseteq Z(A)$, then $\mathsf{A}^{[k+1]} = 0$.

ii) Direct consequence of i). □

Definition. The left normalizer of a subset H of an algebra with bracket A is the set $N^l_{\mathsf{A}}(\mathsf{H}) = \{a \in \mathsf{A} \mid a \cdot h \in \mathsf{H}; [a,h] \in \mathsf{H};\ \text{for all } h \in \mathsf{H}\}$.

The right normalizer of a subset H of an algebra with bracket A is the set $N^r_{\mathsf{A}}(\mathsf{H}) = \{a \in \mathsf{A} \mid h \cdot a \in \mathsf{H}; [h,a] \in \mathsf{H};\ \text{for all } h \in \mathsf{H}\}$.

The normalizer of H of an algebra with bracket A is $N_{\mathsf{A}}(\mathsf{H}) = N^l_{\mathsf{A}}(\mathsf{H}) \cap N^r_{\mathsf{A}}(\mathsf{H})$.

In general, $N_{\mathsf{A}}(\mathsf{H}), N^l_{\mathsf{A}}(\mathsf{H}), N^r_{\mathsf{A}}(\mathsf{H})$ are not subalgebras when H is a subset of A, even when H is a subalgebra of A. If H is a subalgebra of A, then $\mathsf{H} \subseteq N_{\mathsf{A}}(\mathsf{H})$.

Definition. It is said that an algebra with bracket A satisfies the normalizer condition if every proper subalgebra of A is properly contained in its normalizer.

Proposition 4.8. *If* A *is a nilpotent algebra with bracket, then* A *satisfies the normalizer condition.*

Proof. Let K be a proper subalgebra of A. Let $j \geq 1$ be the minimal integer such that $Z_j(\mathsf{A}) \nsubseteq \mathsf{K}$ (there always exists such a j thanks to Theorem 4.4). Then $[[\mathsf{K}, Z_j(\mathsf{A})]] \subseteq [[\mathsf{A}, Z_j(\mathsf{A})]] \subseteq Z_{j-1}(\mathsf{A}) \subseteq \mathsf{K}$ (by minimality of j). Thus $\mathsf{K} \subseteq \mathsf{K} + Z_j(\mathsf{A}) \subseteq N_{\mathsf{A}}(\mathsf{K})$. □

5. Homological characterization of nilpotency

Firstly we recall that the homology with trivial coefficients of an algebra with bracket A can be computed through the following complex [9]:

Let V a $\mathbb{K}$-vector space. Let be $\mathcal{R}_1(V) = V$ and $\mathcal{R}_n(V) = V^{\otimes n} \oplus V^{\otimes n}$ for $n \geq 2$. In order to distinguish elements from these tensor powers, we let $a_1 \otimes \cdots \otimes a_n$ be a typical element from the first component of $\mathcal{R}_n(V)$, while $a_1 \circ \cdots \circ a_n$ from the second component of $\mathcal{R}_n(V)$.

Let A be an AWB and consider $\mathbb{K}$ as a trivial representation over A, that is, $a \cdot k = k \cdot a = [a, k] = [k, a] = 0$ for all $a \in \mathsf{A}, k \in \mathbb{K}$. Then we define the complex

$$C_n^{\mathrm{AWB}}(\mathsf{A}) := R_{n+1}(\mathsf{A}),\ n \geq 0$$

with boundary maps given by

$$\begin{aligned} d_n(a_1 \otimes \cdots \otimes a_n) &= \sum_{i=1}^{n-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i \cdot a_{i+1} \otimes \cdots \otimes a_n \\ d_n(a_1 \circ \cdots \circ a_n) &= \sum_{i=1}^{n-1} a_1 \otimes \cdots \otimes [a_i, a_n] \otimes \cdots \otimes a_{n-1} \\ &\quad + \sum_{i=1}^{n-2} (-1)^i a_1 \circ \cdots \circ a_i \cdot a_{i+1} \circ \cdots \circ a_n. \end{aligned}$$

The homology of the complex $(C_n^{\mathrm{AWB}}(\mathsf{A}), d_n)$ is said to be the *homology with trivial coefficients* of the algebra with bracket A and we denote it by $\mathsf{H}_{\mathsf{n}}^{\mathsf{AWB}}(\mathsf{A}, \mathbb{K}), n \geq 0$ or briefly $\mathsf{H}_{\mathsf{n}}^{\mathsf{AWB}}(\mathsf{A})$.

A direct computation on the complex $(C_n^{\mathrm{AWB}}(\mathsf{A}), d)$ shows that $\mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \cong \mathsf{A}/[[\mathsf{A}, \mathsf{A}]] = \mathsf{A}_{\mathrm{ab}}$. Associated to an exact sequence $0 \to \mathsf{H} \to \mathsf{B} \xrightarrow{\pi} \mathsf{A} \to 0$ of AWB there exists [6, Theorem 2.13] the exact and natural five-term sequence

$$\mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \xrightarrow{\theta_{\mathsf{B}}} \mathsf{H}/[[\mathsf{H}, \mathsf{B}]] \to \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{B}) \to \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \to 0. \tag{4}$$

Hence the following isomorphism holds

$$\mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \cong (\mathsf{R} \cap [[\mathsf{F}, \mathsf{F}]])/[[\mathsf{R}, \mathsf{F}]] \tag{5}$$

for a free presentation $0 \to \mathsf{R} \to \mathsf{F} \xrightarrow{\rho} \mathsf{A} \to 0$ of the algebra with bracket A.

Definition. Let A be a nilpotent algebra with bracket of class n. An extension $0 \to \mathsf{H} \to \mathsf{B} \xrightarrow{\pi} \mathsf{A} \to 0$ is said to be of class n if B is nilpotent of class n.

Theorem 5.1. *A central extension $0 \to \mathsf{H} \to \mathsf{B} \xrightarrow{\pi} \mathsf{A} \to 0$ is of class n if and only if θ_{B} vanishes over $\mathsf{Ker}(\tau)$, where $\tau : \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}/\mathsf{A}^{[n]})$ is induced by the canonical projection $\mathsf{A} \twoheadrightarrow \mathsf{A}/\mathsf{A}^{[n]}$.*

Proof. Consider the following diagrams of free presentations:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[dl] & \\
 & & \mathsf{R} \arrow[dl] \arrow[d] & & \\
0 \arrow[r] & \mathsf{S} \arrow[r] \arrow[d] & \mathsf{F} \arrow[d, "\rho"] \arrow[dr, "\pi\circ\rho"] & & \\
0 \arrow[r] & \mathsf{H} \arrow[r] \arrow[d] & \mathsf{B} \arrow[r, "\pi"] \arrow[d] & \mathsf{A} \arrow[r] \arrow[dr] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[dl] & \\
 & & \mathsf{S} \arrow[dl] \arrow[d] & & \\
0 \arrow[r] & \mathsf{T} \arrow[r] \arrow[d] & \mathsf{F} \arrow[d, "\pi\circ\rho"] \arrow[dr, "pr\circ\pi\circ\rho"] & & \\
0 \arrow[r] & \mathsf{A}^{[n]} \arrow[r] \arrow[d] & \mathsf{A} \arrow[r, "pr"] \arrow[d] & \mathsf{A}/\mathsf{A}^{[n]} \arrow[r] \arrow[dr] & 0 \\
 & 0 & 0 & & 0
\end{tikzcd}$$

then $\theta_{\mathsf{B}}(x + [[\mathsf{S}, \mathsf{F}]]) = \rho(x)$ and $\mathsf{Ker}(\tau) = \frac{[[\mathsf{T},\mathsf{F}]]}{[[\mathsf{S},\mathsf{F}]]}$.

Assume that B is nilpotent of class n and consider $x + [[\mathsf{S}, \mathsf{F}]] \in \mathsf{Ker}(\tau)$. Then $\theta_{\mathsf{B}}(x + [[\mathsf{S}, \mathsf{F}]]) = \rho(x) = 0$ since $\rho(x) \in [[\rho(\mathsf{T}), \rho(\mathsf{F})]] \subseteq [[\mathsf{B}^{[n]} + \mathsf{H}, \mathsf{B}]] = \mathsf{B}^{[n+1]} = 0$. For the last inclusion is necessary to have in mind that $\pi \circ \rho(\mathsf{T}) \subseteq \mathsf{A}^{[n]} = \pi(\mathsf{B}^{[n]})$ and consequently $\rho(\mathsf{T}) \subseteq \mathsf{B}^{[n]} + \mathsf{H}$.

Conversely, $\mathsf{B}^{[n+1]} = [[\mathsf{B}^{[n]}, \mathsf{B}]] = [[\rho(\mathsf{F}^{[n]}), \rho(\mathsf{F})]] \subseteq \rho[[\mathsf{T}, \mathsf{F}]] = 0$ since $[[\mathsf{T}, \mathsf{F}]] \subseteq \mathsf{R}$ because θ_{B} vanishes over $\mathsf{Ker}(\tau)$. For the last inclusion is necessary to have in mind that $\pi \circ \rho(\mathsf{F}^{[n]}) \subseteq \mathsf{A}^{[n]}$, hence $\mathsf{F}^{[n]} \subseteq \mathsf{T}$. $\square$

Definition. Let H be a two-sided ideal of an algebra with bracket A. We call lower central series determined by H to the sequence of two-sided ideals defined inductively by

$$\gamma_1(\mathsf{H}, \mathsf{A}) = \mathsf{H}; \quad \gamma_{k+1}(\mathsf{H}, \mathsf{A}) = [[\gamma_k(\mathsf{H}, \mathsf{A}), \mathsf{A}]], \ k \geq 1.$$

Obviously, if $\mathsf{H} = \mathsf{A}$, then $\gamma_k(\mathsf{A}, \mathsf{A}) = \mathsf{A}^{[k]}$, denoted as $\gamma_k(\mathsf{A})$ for short. On the other hand, $\gamma_{k+1}(\mathsf{H}, \mathsf{A}) \trianglelefteq \gamma_k(\mathsf{H}, \mathsf{A})$ and $\gamma_{k+1}(\mathsf{H}, \mathsf{A})/\gamma_k(\mathsf{H}, \mathsf{A})$ is an abelian algebra with bracket. If $\varphi : \mathsf{A} \to \mathsf{B}$ is a homomorphism of algebras with bracket such that $\varphi(\mathsf{H}) \subseteq \mathsf{K}$, where H is a two-sided ideal of A and K a two-sided ideal of B, then $\varphi(\gamma_k(\mathsf{H}, \mathsf{A})) \subseteq \gamma_k(\mathsf{K}, \mathsf{A}), k \geq 1$.

Theorem 5.2. *Let $\varphi : \mathsf{A} \to \mathsf{B}$ be a homomorphism of algebras with bracket such that $\varphi(\mathsf{H}) \subseteq \mathsf{K}$, where H is a two-sided ideal of A and K a two-sided ideal of B, and the following conditions hold*:

i) $\varphi : \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{B})$ *is an isomorphism.*
ii) $\varphi : \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B})$ *is an epimorphism.*
iii) $\bar{\varphi} : \mathsf{A}/\mathsf{H} \to \mathsf{B}/\mathsf{K}$ *is an isomorphism.*

Then $\varphi_k : \mathsf{A}/\gamma_k(\mathsf{H}) \to \mathsf{B}/\gamma_k(\mathsf{K}), \ k \geq 1$, is an isomorphism.

Proof. By induction on k. For $k = 1$ we use condition iii). For $k \geq 2$ we apply sequence (4), which is natural, to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \gamma_{k-1}(\mathsf{H}) \arrow[r] \arrow[d] & \mathsf{A} \arrow[r] \arrow[d, "\varphi"'] & \mathsf{A}/\gamma_{k-1}(\mathsf{H}) \arrow[r] \arrow[d, "\varphi_{k-1}"', "\wr"] & 0 \\
0 \arrow[r] & \gamma_{k-1}(\mathsf{K}) \arrow[r] & \mathsf{B} \arrow[r] & \mathsf{B}/\gamma_{k-1}(\mathsf{K}) \arrow[r] & 0
\end{tikzcd}$$

and we get the following commutative diagram:

$$\begin{tikzcd}
\mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \arrow[r] \arrow[d] & \mathsf{H}_1^{\mathsf{AWB}}(\frac{\mathsf{A}}{\gamma_{k-1}(\mathsf{H})}) \arrow[r] \arrow[d] & \frac{\gamma_{k-1}(\mathsf{H})}{[[\gamma_{k-1}(\mathsf{H}),\mathsf{A}]]} \arrow[r] \arrow[d] & \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \arrow[r] \arrow[d] & \mathsf{H}_0^{\mathsf{AWB}}(\frac{\mathsf{A}}{\gamma_{k-1}(\mathsf{H})}) \arrow[r] \arrow[d] & 0 \\
\mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B}) \arrow[r] & \mathsf{H}_1^{\mathsf{AWB}}(\frac{\mathsf{B}}{\gamma_{k-1}(\mathsf{K})}) \arrow[r] & \frac{\gamma_{k-1}(\mathsf{K})}{[[\gamma_{k-1}(\mathsf{K}),\mathsf{B}]]} \arrow[r] & \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{B}) \arrow[r] & \mathsf{H}_0^{\mathsf{AWB}}(\frac{\mathsf{B}}{\gamma_{k-1}(\mathsf{K})}) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, which holds in a semi-abelian category [4, 19], we get $\frac{\gamma_{k-1}(\mathsf{H})}{[[\gamma_{k-1}(\mathsf{H}),\mathsf{A}]]} \cong \frac{\gamma_{k-1}(\mathsf{K})}{[[\gamma_{k-1}(\mathsf{K}),\mathsf{B}]]}$, i.e., $\frac{\gamma_{k-1}(\mathsf{H})}{\gamma_k(\mathsf{H})} \cong \frac{\gamma_{k-1}(\mathsf{K})}{\gamma_k(\mathsf{K})}$.

Then the Short Five Lemma applied to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{\gamma_{k-1}(\mathsf{H})}{\gamma_k(\mathsf{H})} \arrow[r] \arrow[d] & \frac{\mathsf{A}}{\gamma_k(\mathsf{H})} \arrow[r] \arrow[d] & \frac{\mathsf{A}}{\gamma_{k-1}(\mathsf{H})} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{\gamma_{k-1}(\mathsf{K})}{\gamma_k(\mathsf{K})} \arrow[r] & \frac{\mathsf{B}}{\gamma_k(\mathsf{K})} \arrow[r] & \frac{\mathsf{B}}{\gamma_{k-1}(\mathsf{K})} \arrow[r] & 0
\end{tikzcd}$$

and the induction complete the proof. $\square$

Corollary 5.3. *Let* $\varphi : \mathsf{A} \to \mathsf{B}$ *be a homomorphism of algebras with bracket such that* $\varphi_{ab} : \mathsf{A}_{\mathrm{ab}} \to \mathsf{B}_{\mathrm{ab}}$ *is an isomorphism and* $\bar{\varphi} : \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B})$ *is an epimorphism. If* A *and* B *are nilpotent algebras with bracket, then* φ *is an isomorphism.*

Proof. Take $\mathsf{H} = \mathsf{A}$ and $\mathsf{K} = \mathsf{B}$ in Theorem 5.2. The assertion follows keeping in mind that $\mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \cong \mathsf{A}_{\mathrm{ab}}$ and there exists $k \geq 1$ such that $\gamma_k(\mathsf{A}) = \gamma_k(\mathsf{B}) = 0$. $\square$

Corollary 5.4. *Let* A *be a nilpotent algebra with bracket such that the canonical epimorphism* $\mathsf{A} \to \mathsf{A}_{\mathrm{Poiss}}$ (*see Example 2.1* i)) *induces an epimorphism* $\mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}_{\mathrm{Poiss}})$*, then* $\mathsf{A} \cong \mathsf{A}_{\mathrm{Poiss}}$*, that is, the unique nilpotent algebras with bracket whose Poissonization is nilpotent are the Poisson algebras.*

Proof. Apply Corollary 5.3 to the canonical epimorphism $\mathsf{A} \to \mathsf{A}_{\mathrm{Poiss}}$. $\square$

Lemma 5.5. *Let* H *be a two-sided ideal of an algebra with bracket* A*. The lower central series determined by* H *vanishes if and only if there exists* $i \geq 0$ *such that* $\mathsf{H} \subseteq \mathcal{Z}_i(\mathsf{A})$.

Proof. It is enough to use the following obvious equivalence:

$$\gamma_i(\mathsf{H}, \mathsf{A}) \subseteq \mathcal{Z}_k(\mathsf{A}) \Leftrightarrow \gamma_{i+1}(\mathsf{H}, \mathsf{A}) \subseteq \mathcal{Z}_{k-1}(\mathsf{A}) \qquad \square$$

Theorem 5.6. *Let* $\mathsf{H} \subseteq \mathcal{Z}_i(\mathsf{A})$ *and* $\mathsf{K} \subseteq \mathcal{Z}_j(\mathsf{B})$ *be,* $i, j \geq 0$*. Let* $\varphi : \mathsf{A} \to \mathsf{B}$ *be a homomorphism of algebras with bracket such that* $\varphi(\mathsf{H}) \subseteq \mathsf{K}$ *and the following conditions hold*:

i) $\varphi : \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{B})$ *is an isomorphism.*
ii) $\varphi : \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B})$ *is an epimorphism.*
iii) $\bar{\varphi} : \mathsf{A}/\mathsf{H} \to \mathsf{B}/\mathsf{K}$ *is an isomorphism.*

then $\varphi : \mathsf{A} \to \mathsf{B}$ *is an isomorphism.*

Proof. Apply Theorem 5.2 and Lemma 5.5. $\square$

Corollary 5.7. *Let* $\varphi : \mathsf{A} \to \mathsf{B}$ *be a homomorphism of algebras with bracket such that* $\varphi(\mathcal{Z}_i(\mathsf{A})) \subseteq \mathcal{Z}_i(\mathsf{B})$ *for any* $i \geq 0$ *and the following conditions hold*:

i) *The induced homomorphism* $\varphi : \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_0^{\mathsf{AWB}}(\mathsf{B})$ *is an isomorphism.*
ii) *The induced homomorphism* $\varphi : \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{B})$ *is an epimorphism.*
iii) $\bar{\varphi} : \mathsf{A}/\mathcal{Z}_i(\mathsf{A}) \to \mathsf{B}/\mathcal{Z}_i(\mathsf{B})$ *is an isomorphism.*

then $\varphi : \mathsf{A} \to \mathsf{B}$ *is an isomorphism.*

Proof. Apply Theorem 5.6 to the case $\mathsf{H} = \mathcal{Z}_i(\mathsf{A})$ and $\mathsf{K} = \mathcal{Z}_i(\mathsf{B})$. $\square$

From now on, we consider the free presentation $0 \to \mathsf{R} \to \mathsf{F} \xrightarrow{\rho} \mathsf{A} \to 0$ of the algebra with bracket A. Then the *c-nilpotent multiplier* of A, $c \geq 0$, is defined to be the abelian AWB

$$\mathcal{M}^{(c)}(\mathsf{A}) := \frac{\mathsf{R} \cap \gamma_{c+1}(\mathsf{F})}{\gamma_{c+1}(\mathsf{R}, \mathsf{F})}.$$

From (5), we have $\mathcal{M}^{(1)}(\mathsf{A}) \cong \mathsf{H}_1^{\mathsf{AWB}}(\mathsf{A})$.

Since AWB is a category of Ω-groups, but not a category of interest (see [8]), and following Proposition 4.3.2 in [14] we can conclude that the collection of all nilpotent objects of class $\leq k$ in AWB form a variety. Now following [1, 2, 3, 11, 17] can be showed that $\mathcal{M}^{(c)}(\mathsf{A})$ is a Baer-invariant, which means that its definition does not depend on the choice of the free presentation. Furthermore, from [10] can be derived that $\gamma_{c+1}^*(\mathsf{A}) = \frac{\gamma_{c+1}(\mathsf{F})}{\gamma_{c+1}(\mathsf{R}, \mathsf{F})}$ is also a Baer-invariant.

Proposition 5.8. *Let* $\mathsf{A} \in \mathsf{AWB}$ *and* $c \geq 1$*. Then*

i) $\gamma_{c+1}^*(\mathsf{A}) = 0$ *if and only if* A *is nilpotent of class* c *and* $\mathcal{M}^{(c)}(\mathsf{A}) = 0$.
ii) *If* $\gamma_{c+1}^*(\mathsf{A}) = 0$*, then* $\gamma_{c+1}^*(\mathsf{A}/\mathsf{K}) = 0$ *for any two-sided ideal* K *of* A.

Proof. i) $\gamma_{c+1}^*(\mathsf{A}) = 0$ implies that $\gamma_{c+1}(\mathsf{F}) \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{R})$, thus

$$\mathsf{A}^{[c+1]} = [[\gamma_c(\mathsf{A}), \mathsf{A}]] = \gamma_{c+1}(\mathsf{F})/\mathsf{R} \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{R})/\mathsf{R} \subseteq 0.$$

Moreover $\mathcal{M}^{(c)}(\mathsf{A}) = \frac{\mathsf{R} \cap \gamma_{c+1}(\mathsf{F})}{\gamma_{c+1}(\mathsf{F}, \mathsf{R})} \subseteq \frac{\mathsf{R} \cap \gamma_{c+1}(\mathsf{F}, \mathsf{R})}{\gamma_{c+1}(\mathsf{F}, \mathsf{R})} = 0.$

Conversely, is $\mathcal{M}^{(c)}(\mathsf{A}) = 0$, then $\mathsf{R} \cap \gamma_{c+1}(\mathsf{F}) \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{R})$, since $\mathsf{A}^{[c+1]} = 0$ implies that $\gamma_{c+1}(\mathsf{F})/\mathsf{R} = 0$, thus $\gamma_{c+1}(\mathsf{F}) \subseteq \mathsf{R}$. Hence $\gamma_{c+1}(\mathsf{F}) \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{R})$ and, consequently, $\gamma^*_{c+1}(\mathsf{A}) = 0$.

ii) Let K be a two-sided ideal of A and consider the free presentation $0 \to \mathsf{S} \to \mathsf{F} \stackrel{\tau \circ \rho}{\to} \mathsf{A}/\mathsf{K} \to 0$, where $\tau : \mathsf{A} \to \mathsf{A}/\mathsf{K}$ is the canonical projection. Since $\mathsf{R} \subseteq \mathsf{S}$, then $\gamma^*_{c+1}(\mathsf{A}) = 0$ implies that $\gamma_{c+1}(\mathsf{F}) \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{R}) \subseteq \gamma_{c+1}(\mathsf{F}, \mathsf{S})$ which ends the proof. $\square$

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