

PRODUCTS OF DIFFERENTIATION AND COMPOSITION OPERATORS FROM THE BLOCH SPACE AND WEIGHTED DIRICHLET SPACES TO MORREY TYPE SPACES

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ABSTRACT. In this paper, we characterize the boundedness, compactness and essential norm of products of differentiation and composition operators from the Bloch space and weighted Dirichlet spaces to analytic Morrey type spaces.

1. Introduction

Let \mathbb{D} denote the open unit disk in the complex plane \mathbb{C} and $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all functions analytic on \mathbb{D} . For $a \in \mathbb{D}$, $g(z, a) = \log \frac{1}{|\sigma_a(z)|}$ is Green's function on \mathbb{D} , where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of \mathbb{D} . For a subarc $I \subseteq \partial\mathbb{D}$, let $S(I)$ be the Carleson box based on I with

$$S(I) = \left\{ z \in \mathbb{D} : 1 - I \leq |z| < 1, \frac{z}{|z|} \in I \right\},$$

where $|I| = \frac{1}{2\pi} \int_I |d\xi|$ is the normalized length of the subarc I of $\partial\mathbb{D}$. If $I = \partial\mathbb{D}$, let $S(I) = \mathbb{D}$. Let μ be a nonnegative Borel measure on \mathbb{D} . We say that μ is a Carleson measure on \mathbb{D} if

$$\|\mu\|^2 = \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty.$$

Here and henceforth $\sup_{I \subseteq \partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$.

As usual, H^∞ is the set of bounded analytic functions in \mathbb{D} . An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by \mathcal{B} , if (see [22])

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

Received March 28, 2016.

2010 *Mathematics Subject Classification.* Primary 47B33, 30D50.

Key words and phrases. composition operator, Bloch space, weighted Dirichlet space, analytic Morrey space.

This work was supported by the NNSF of China(No.11471143).

\mathcal{B} is a Banach space under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\beta}$. The little Bloch space, denoted by \mathcal{B}_0 , is the closed subspace of \mathcal{B} consisting of functions f with $\lim_{|z| \rightarrow 1} (1 - |z|^2)|f'(z)| = 0$.

For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space, denoted by A_{α}^p , is the set of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{A_{\alpha}^p}^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{\alpha} dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in \mathbb{D} such that $A(\mathbb{D}) = 1$. The weighted Dirichlet space \mathcal{D}_{α}^p consists of those $f \in H(\mathbb{D})$ such that $f' \in A_{\alpha}^p$. Hence, for $f \in \mathcal{D}_{\alpha}^p$ we have

$$\|f\|_{\mathcal{D}_{\alpha}^p}^p = |f(0)|^p + \|f'\|_{A_{\alpha}^p}^p < \infty.$$

It is well known that $A_{\alpha}^p = \mathcal{D}_{\alpha+p}^p$ (see, e.g., [22]).

For $0 < p < \infty$, the Hardy space H^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let $K : [0, \infty) \rightarrow [0, \infty)$ be a right-continuous and nondecreasing function. The analytic Morrey type space, denoted by H_K^2 , is the space of all analytic functions $f \in H^2$ on \mathbb{D} such that

$$\|f\|_{H_K^2}^2 = \sup_{I \subseteq \partial\mathbb{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f_I|^2 \frac{|d\zeta|}{2\pi} < \infty,$$

where

$$f_I = \frac{1}{|I|} \int_I f(\zeta) \frac{|d\zeta|}{2\pi}, I \subseteq \partial\mathbb{D}.$$

See [20] for more information of the Morrey type space H_K^2 . When $K(t) = t$, it gives the $BMOA$ space. It is well known that $f \in BMOA$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty.$$

When $K(t) = t^{\lambda}$ ($\lambda \in (0, 1)$), H_K^2 is the Morrey space $\mathcal{L}^{2,\lambda}$, which was studied by Wu and Xie in [18].

Let $S(\mathbb{D})$ denote the set of all analytic self-maps of \mathbb{D} . Let $\varphi \in S(\mathbb{D})$. Let \mathbb{Z} denote the set of nonnegative integer. For $f \in H(\mathbb{D})$, the composition operator C_{φ} on \mathbb{D} is defined by

$$C_{\varphi}(f) = f \circ \varphi.$$

The operator $C_{\varphi}D^n$ is defined by $C_{\varphi}D^n f = f^{(n)} \circ \varphi$, where $n \in \mathbb{Z}$. If $n = 0$, we get the composition operator C_{φ} . If $n = 1$, we get the operator $C_{\varphi}D$, which was studied in [4, 5, 6, 7, 8, 14, 15, 17, 24, 25]. In [13], Smith and Zhao characterized the boundedness and compactness of $C_{\varphi} : \mathcal{B} \rightarrow \mathcal{Q}_p$. In [19], Wulan characterized the boundedness and compactness of $C_{\varphi} : \mathcal{B} \rightarrow \mathcal{Q}_K$. In [9], Lindström etc. gave an asymptotic formula for the essential norm of the operator $C_{\varphi} : \mathcal{B} \rightarrow \mathcal{Q}_p$. In [12], Rättyä gave an asymptotic formula for

the essential norm of a composition operator $C_\varphi : \mathcal{D}_\alpha^p \rightarrow \mathcal{Q}_p$. Recall that the essential norm of a bounded linear operator $T : X \rightarrow Y$ is its distance to the set of compact operators K mapping X into Y , that is,

$$\|T\|_{e, X \rightarrow Y} = \inf\{\|T - K\|_{X \rightarrow Y} : K \text{ is compact}\},$$

where X and Y are Banach spaces, $\|\cdot\|_{X \rightarrow Y}$ is the operator norm.

In this paper, we study the boundedness, compactness and essential norm of products of differentiation and composition operators $C_\varphi D^n$ from the Bloch space and weighted Dirichlet spaces to analytic Morrey type spaces.

Throughout this paper we need some constraints on K . Let φ_K be defined by

$$(1) \quad \varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

By [3], we may suppose that K is defined on $[0, 1]$ and extend its domain to $[0, \infty)$ by setting $K(t) = K(1)$ for $t > 1$, $K(t) \approx K(2t)$ and

$$(2) \quad \int_0^1 \frac{\varphi_K(s)}{s} ds < \infty \text{ and } \int_1^\infty \frac{\varphi_K(s)}{s^2} ds < \infty.$$

We shall also use the following standard notation: $f \lesssim g$ means that there is a constant C independent of the relevant variables such that $f \leq Cg$, and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$.

2. Characterization of the operator $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$

Lemma 2.1 ([20]). *Let K satisfy the conditions in (2). Then the following are equivalent:*

- (a) $f \in H_K^2$;
- (b) $\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f'(z)|^2 g(z, a) dA(z) < \infty$;
- (c) $\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty$;
- (d) $\sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z) < \infty$.

Remark 2.1. By Lemma 2.1, for $f \in H_K^2$, we have

$$\begin{aligned} \|f\|_{H_K^2}^2 &\approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f'(z)|^2 g(z, a) dA(z) \\ &\approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) \\ &\approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^2 (1 - |z|^2) dA(z). \end{aligned}$$

Lemma 2.2 ([22]). *For every positive integer n , $f \in \mathcal{B}$ if and only if $\sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f^{(n)}(z)| < \infty$. Moreover, the following asymptotic relationship holds*

$$\|f\|_{\mathcal{B}} \approx \sum_{k=0}^{n-1} |f^{(k)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f^{(n)}(z)|.$$

The following lemma is widely known, but we can not find a proof for it. Here we give a complete proof.

Lemma 2.3. *A sequence $\{f_j\}$ in \mathcal{B}_0 converges weakly to 0 in \mathcal{B} if and only if $\sup_j \|f_j\|_{\mathcal{B}} < \infty$ and $f_j \rightarrow 0$ pointwise in \mathbb{D} .*

Proof. From Theorem 5.15 in [22], we see that the dual space of \mathcal{B}_0 is A^1 , the Bergman space. Proposition 1.2 in [2] dictates that f_j converges weakly to 0 in \mathcal{B}_0 if and only if $\sup_j \|f_j\|_{\mathcal{B}} < \infty$ and $f_j \rightarrow 0$ pointwise in \mathbb{D} . Now consider the sequence $\{f_j\}$ as belonging to \mathcal{B} . It is easy to see that weak convergence in \mathcal{B} is equivalent to weak convergence in \mathcal{B}_0 . In one direction, restrict an arbitrary functional on \mathcal{B} to a functional on \mathcal{B}_0 ; in the other direction, use the Hahn-Banach theorem to extend an arbitrary functional on \mathcal{B}_0 to a functional on \mathcal{B} . \square

Theorem 2.1. *Let φ be an analytic self-map of \mathbb{D} , $n \in \mathbb{Z}$ and K satisfy the conditions in (2). Then the following statements are equivalent:*

- (a) $C_{\varphi} D^n : \mathcal{B} \rightarrow H_K^2$ is bounded;
- (b)

$$\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) < \infty;$$

- (c)

$$\sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |\sigma_a(z)|^2) dA(z) < \infty;$$

- (d)

$$\sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) < \infty.$$

Proof. (a) \Rightarrow (b). Assume that $C_{\varphi} D^n : \mathcal{B} \rightarrow H_K^2$ is bounded, we have

$$\|C_{\varphi} D^n f\|_{H_K^2} \leq \|C_{\varphi} D^n\| \|f\|_{\mathcal{B}}$$

for all $f \in \mathcal{B}$. By [23] we may choose two Bloch functions f_1 and f_2 satisfying

$$\frac{1}{(1 - |z|^2)^{n+1}} \approx |f_1^{(n+1)}(z)| + |f_2^{(n+1)}(z)|, \quad z \in \mathbb{D}.$$

So that

$$\frac{|\varphi'(z)|}{(1 - |\varphi(z)|^2)^{n+1}} \approx |(f_1^{(n)} \circ \varphi)'(z)| + |(f_2^{(n)} \circ \varphi)'(z)|.$$

By elementary inequality $(a + b)^2 \leq 2(a^2 + b^2)$, we get

$$\frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} \lesssim 2|(f_1^{(n)} \circ \varphi)'(z)|^2 + 2|(f_2^{(n)} \circ \varphi)'(z)|^2,$$

which implies that

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) \\ & \lesssim \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \left(|(f_1^{(n)} \circ \varphi)'(z)|^2 + |(f_2^{(n)} \circ \varphi)'(z)|^2 \right) g(z, a) dA(z) \\ & \lesssim \|C_\varphi D^n\|^2 (\|f_1\|_{\mathcal{B}}^2 + \|f_2\|_{\mathcal{B}}^2) < \infty, \end{aligned}$$

as desired.

(b) \Rightarrow (a). Let $f \in \mathcal{B}$. By the assumption, Remark 2.1 and Lemma 2.2, we have

$$\begin{aligned} \|C_\varphi D^n f\|_{H_K^2}^2 & \approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(f^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \\ & = \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{|f^{(n+1)}(\varphi(z))|^2 (1 - |\varphi(z)|^2)^{2(n+1)}}{(1 - |\varphi(z)|^2)^{2(n+1)}} |\varphi'(z)|^2 g(z, a) dA(z) \\ & \lesssim \|f\|_{\mathcal{B}}^2 \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) \\ & < \infty. \end{aligned}$$

Thus $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is bounded.

(a) \Rightarrow (d). Assume that $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is bounded. Fix an arc $I \subset \partial \mathbb{D}$ and consider the test function

$$f_{m_2, \theta}(z) = \sum_{k=1}^{\infty} \frac{2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (e^{i\theta})^{2^k} z^{2^k + 2^{m_2}}$$

for $m_2 \in \mathbb{N}$ such that $2^{m_2} - n \geq 0$ and $\theta \in [0, 2\pi)$. It is easy to check that $\|f_{m_2, \theta}\|_{\mathcal{B}} < \infty$. By Fubini's theorem we have

$$\begin{aligned} \infty & > \int_0^{2\pi} \|C_\varphi D^n f_{m_2, \theta}\|_{H_K^2}^2 \frac{d\theta}{2\pi} \\ & \geq \frac{1}{K(|I|)} \int_{S(I)} |\varphi'(z)|^2 (1 - |z|^2) \left\{ \int_0^{2\pi} |f_{m_2, \theta}^{(n+1)}(\varphi(z))|^2 \frac{d\theta}{2\pi} \right\} dA(z) \end{aligned}$$

for all $m_2 \in \mathbb{N}$. Parseval's formula gives

$$\begin{aligned} \int_0^{2\pi} |f_{m_2, \theta}^{(n+1)}(\varphi(z))|^2 \frac{d\theta}{2\pi} & = |\varphi(z)|^{2(2^{m_2} - n)} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} 2^{k(n+1)} e^{2^k i\theta} (\varphi(z))^{2^k - 1} \right|^2 \frac{d\theta}{2\pi} \\ & = |\varphi(z)|^{2^{m_2+1} - 2n} \sum_{k=1}^{\infty} 2^{2k(n+1)} |\varphi(z)|^{2(2^k - 1)}. \end{aligned}$$

By the formula (3.8) in [10], it is obvious that when $|\varphi(z)| > \frac{1}{\sqrt{2}}$, we have

$$\sum_{k=1}^{\infty} 2^{2k(n+1)} |\varphi(z)|^{2(2^k-1)} \gtrsim \frac{1}{(1-|\varphi(z)|^2)^{2(n+1)}}.$$

Hence we obtain

$$(3) \quad \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > \frac{1}{\sqrt{2}}\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} (1-|z|^2) dA(z) < \infty$$

for any $I \subset \partial\mathbb{D}$. Since $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is bounded, applying the operator $C_\varphi D^n$ to z^{n+1} , we obtain $\varphi \in H_K^2$. Thus

$$(4) \quad \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| \leq \frac{1}{\sqrt{2}}\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} (1-|z|^2) dA(z) < \infty$$

for any $I \subset \partial\mathbb{D}$. Inequalities (3) and (4) show that

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} (1-|z|^2) dA(z) < \infty.$$

The proof of (d) \Rightarrow (a) is similar to (b) \Rightarrow (a), (a) \Rightarrow (c) is similar to (a) \Rightarrow (b) and (c) \Rightarrow (a) is similar to (b) \Rightarrow (a). Hence the proof of these are omitted. The proof is completed. \square

Theorem 2.2. Let φ be an analytic self-map of \mathbb{D} , $n \in \mathbb{Z}$ and K satisfy the conditions in (2). Suppose that $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is bounded. Then

$$\|C_\varphi D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \approx \sqrt{A} \approx \sqrt{B} \approx \sqrt{U}.$$

Here

$$\begin{aligned} A &= \limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z), \\ B &= \limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} (1-|\sigma_a(z)|^2) dA(z), \\ U &= \limsup_{r \rightarrow 1} \sup_{I \subset \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} (1-|z|^2) dA(z). \end{aligned}$$

Proof. First we prove that

$$\|C_\varphi D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \gtrsim \sqrt{A}.$$

Let $\{\lambda_{m_1}\} \subset (1/2, 1)$ such that $\lambda_{m_1} \rightarrow 1$ as $m_1 \rightarrow \infty$. Define

$$\begin{aligned} & f_{m_1, m_2, \theta}(z) \\ &= \frac{z^{2^{m_2}}}{\lambda_{m_1}} \sum_{k=1}^{\infty} \frac{2^{k(n+1)}}{(2^k + 2^{m_2})(2^k + 2^{m_2} - 1) \cdots (2^k + 2^{m_2} - n)} (\lambda_{m_1} e^{i\theta})^{2^k} z^{2^k} \\ &= \frac{1}{\lambda_{m_1}} \sum_{k=1}^{\infty} \frac{2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1} e^{i\theta})^{2^k} z^{2^k + 2^{m_2}} \end{aligned}$$

for $m_1, m_2 \in \mathbb{N}$ such that $2^{m_2} - n \geq 0$ and $\theta \in [0, 2\pi)$. Since

$$\begin{aligned} 0 &\leq \lim_{k \rightarrow \infty} \left| \frac{2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1} e^{i\theta})^{2^k} \right| \\ &= \lim_{k \rightarrow \infty} \frac{2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1})^{2^k} \\ &\leq \lim_{k \rightarrow \infty} (\lambda_{m_1})^{2^k} = 0, \end{aligned}$$

the function $f_{m_1, m_2, \theta}$ belongs to \mathcal{B}_0 by Theorem 1 of [21]. Moreover,

$$\begin{aligned} &\sup_{k \in \mathbb{N}} \left| \frac{2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} (\lambda_{m_1} e^{i\theta})^{2^k} \right| \\ &= \sup_{k \in \mathbb{N}} \frac{(\lambda_{m_1})^{2^k} 2^k}{2^k + 2^{m_2}} \frac{2^k}{2^k + 2^{m_2} - 1} \cdots \frac{2^k}{2^k + 2^{m_2} - n} \leq 1. \end{aligned}$$

The proof of Theorem 1 in [21] shows that there exists a positive constant M such that $\|f_{m_1, m_2, \theta}\|_{\mathcal{B}} \leq M$ for all $m_1, m_2 \in \mathbb{N}$ such that $2^{m_2} - n \geq 0$ and $\theta \in [0, 2\pi)$. Define $g_{m_1, m_2, \theta} = f_{m_1, m_2, \theta}/M$. Then the sequence $\{g_{m_1, m_2, \theta}\}_{m_2=1}^{\infty}$ is contained in the closed unit ball of \mathcal{B}_0 . Moreover, $g_{m_1, m_2, \theta}$ tends to zero uniformly on compact subsets of \mathbb{D} for every m_1 and θ as $m_2 \rightarrow \infty$, and therefore $g_{m_1, m_2, \theta}$ tends to zero weakly as $m_2 \rightarrow \infty$ by Lemma 2.3. It follows that for any compact operator $T : \mathcal{B} \rightarrow H_K^2$,

$$\begin{aligned} &\|C_{\varphi} D^n - T\|_{\mathcal{B} \rightarrow H_K^2} \\ &\geq \limsup_{m_2 \rightarrow \infty} \sup_{m_1, \theta} \|(C_{\varphi} D^n - T)(g_{m_1, m_2, \theta})\|_{H_K^2} \\ &\geq \limsup_{m_2 \rightarrow \infty} \sup_{m_1, \theta} \|C_{\varphi} D^n(g_{m_1, m_2, \theta})\|_{H_K^2} - \limsup_{m_2 \rightarrow \infty} \sup_{m_1, \theta} \|T(g_{m_1, m_2, \theta})\|_{H_K^2} \\ &= \limsup_{m_2 \rightarrow \infty} \sup_{m_1, \theta} \|C_{\varphi} D^n(g_{m_1, m_2, \theta})\|_{H_K^2}. \end{aligned}$$

Therefore, from the definition of the essential norm, we get

$$\begin{aligned} \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 &= \inf_T \|C_{\varphi} D^n - T\|_{\mathcal{B} \rightarrow H_K^2}^2 \\ &\gtrsim \frac{1}{M^2} \limsup_{m_2 \rightarrow \infty} \sup_{m_1, \theta} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_{m_1, m_2, \theta}^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z). \end{aligned}$$

Given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$M^2 \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 + \varepsilon \geq \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_{m_1, m_2, \theta}^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z)$$

for all a, θ and m_1 when $m_2 \geq N$. Let $a \in \mathbb{D}$ be fixed. Integrating with respect to θ , using Fubini's theorem and Parseval's formula, we obtain

$$\begin{aligned} &2\pi(M^2 \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 + \varepsilon) \\ &\geq \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \int_0^{2\pi} |f_{m_1, m_2, \theta}^{(n+1)}(\varphi(z))|^2 d\theta |\varphi'(z)|^2 g(z, a) dA(z) \end{aligned}$$

$$\begin{aligned}
&= \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\varphi(z)|^{2(2^{m_2}-n)} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} 2^{k(n+1)} e^{2^k i \theta} (\lambda_{m_1} \varphi(z))^{2^k-1} \right|^2 d\theta \\
&\quad \times |\varphi'(z)|^2 g(z, a) dA(z) \\
&= \frac{1-|a|^2}{K(1-|a|^2)} \\
&\quad \times \int_{\mathbb{D}} |\varphi(z)|^{2^{m_2+1}-2n} \left(\sum_{k=1}^{\infty} 2^{2k(n+1)} |\lambda_{m_1} \varphi(z)|^{2(2^k-1)} \right) |\varphi'(z)|^2 g(z, a) dA(z).
\end{aligned}$$

By the formula (3.8) in [10], there exists a positive constant C such that

$$\sum_{k=1}^{\infty} 2^{2k(n+1)} |\lambda_{m_1} \varphi(z)|^{2(2^k-1)} \geq \frac{C}{(1-|\lambda_{m_1} \varphi(z)|^2)^{2(n+1)}}$$

for all $z \in \mathbb{D}$ with $|\varphi(z)| > 1/2$. Thus by Fatou's Lemma, we get

$$\begin{aligned}
&2\pi(M^2 \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 + \varepsilon) \\
&\gtrsim \liminf_{m_1 \rightarrow \infty} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\varphi(z)|^{2^{m_2+1}-2n} \frac{|\varphi'(z)|^2}{(1-|\lambda_{m_1} \varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) \\
&\gtrsim \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\varphi(z)|^{2^{m_2+1}-2n} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) \\
&\gtrsim \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\varphi(z)|^{2^{m_2+1}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z).
\end{aligned}$$

Since $a \in \mathbb{D}$ was arbitrary, we obtain that

$$\begin{aligned}
&2\pi(M^2 \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 + \varepsilon) \\
&\gtrsim \frac{1}{e} \limsup_{m_2 \rightarrow \infty} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\{|\varphi(z)| > 1-2^{-(m_2+1)}\}} \frac{|\varphi'(z)|^2 g(z, a)}{(1-|\varphi(z)|^2)^{2(n+1)}} dA(z) \\
&= \frac{1}{e} \limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2 g(z, a)}{(1-|\varphi(z)|^2)^{2(n+1)}} dA(z)
\end{aligned}$$

for all $\varepsilon > 0$. Therefore

$$\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \gtrsim \sqrt{A}.$$

A similar argument in the proof above shows that

$$\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \gtrsim \sqrt{B}, \quad \|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \gtrsim \sqrt{U}.$$

Next we prove that

$$\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \lesssim \sqrt{A}.$$

For $j \in \mathbb{N}$, define $K_j(f) = K_{\psi_j}(f)$, where $\psi_j(z) = \frac{jz}{j+1}$, i.e., $K_j f(z) = f(\frac{j}{j+1}z)$, $z \in \mathbb{D}$. Since the operator K_j is compact on \mathcal{B} for all $j \in \mathbb{N}$ (see [9]), and $C_{\varphi} D^n : \mathcal{B} \rightarrow H_K^2$ is bounded, it follows that

$$\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 \leq \|C_{\varphi} D^n - C_{\varphi} D^n K_j\|_{\mathcal{B} \rightarrow H_K^2}^2 = \|C_{\varphi} D^n (Id - K_j)\|_{\mathcal{B} \rightarrow H_K^2}^2$$

$$\begin{aligned}
 &\approx \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 g(z, a) dA(z) \\
 &\leq \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| \leq r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 g(z, a) dA(z) \\
 &\quad + \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 g(z, a) dA(z) \\
 &= I_1 + I_2
 \end{aligned}$$

for all $r \in (0, 1)$ and $j \in \mathbb{N}$, where $Id(f) = f$ and

$$I_1 = \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| \leq r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 g(z, a) dA(z)$$

and

$$I_2 = \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 g(z, a) dA(z).$$

Since $C_{\varphi} D^n : \mathcal{B} \rightarrow H_K^2$ is bounded, from the proof of Theorem 2.1 we see that $\varphi \in H_K^2$, and hence

$$\tilde{K} = \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\varphi'(z)|^2 g(z, a) dA(z) < \infty.$$

Since $f - f \circ \psi_j$ and its derivative tend to zero uniformly in a compact subset of \mathbb{D} as $j \rightarrow \infty$, it follows that

$$I_1 \leq \tilde{K} \limsup_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{\{|\varphi(z)| \leq r\}} |(f - f \circ \psi_j)^{(n+1)}(\varphi(z))|^2 = 0.$$

Now we estimate I_2 . Since

$$(5) \quad \|f - f \circ \psi_j\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}} + \|f \circ \psi_j\|_{\mathcal{B}} \leq 2\|f\|_{\mathcal{B}} \leq 2,$$

by Lemma 2.2 we get

$$I_2 \leq \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z).$$

Consequently,

$$\begin{aligned}
 &\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 \\
 &\leq \limsup_{j \rightarrow \infty} \|C_{\varphi} D^n - C_{\varphi} D^n K_j\|_{\mathcal{B} \rightarrow H_K^2}^2 \leq \limsup_{j \rightarrow \infty} I_1 + \limsup_{j \rightarrow \infty} I_2 \\
 &\lesssim \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z)
 \end{aligned}$$

for all $r \in (0, 1)$. Thus $\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \lesssim \sqrt{A}$. A similar argument shows that $\|C_{\varphi} D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 \lesssim \sqrt{B}$.

Finally we prove that $\|C_\varphi D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \lesssim \sqrt{U}$. By Lemma 2.1 we have

$$\begin{aligned} & \|C_\varphi D^n (Id - K_j) f\|_{H_K^2}^2 \\ & \approx \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \\ & \leq \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| \leq r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \\ & \quad + \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \end{aligned}$$

for any $r \in (0, 1)$. By Lemma 2.2 and (5) we obtain

$$\begin{aligned} & \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \\ & \lesssim \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) \end{aligned}$$

for any $r \in (0, 1)$ and $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq 1$. Now we only need to prove that

$$\sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| \leq r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 |\varphi'(z)|^2 (1 - |z|^2) dA(z) \rightarrow 0$$

as $j \rightarrow \infty$. We put $v = \varphi(z)$ and denote the radial segment by $[\frac{j}{j+1}v, v]$. We obtain that

$$\left| f^{(n+1)}(v) - f^{(n+1)}\left(\frac{j}{j+1}v\right) \right| \leq \frac{1}{j+1} |v| |f^{(n+2)}(\xi(v))|$$

for some $\xi(v) \in [\frac{j}{j+1}v, v]$. An application of Cauchy's estimate on the circle with center at $\xi(v)$ and radius $R \in (0, 1 - r)$ shows that

$$|f^{(n+2)}(\xi(v))| \leq \frac{(n+2)!}{R^{n+2}} \max_{|\zeta|=R+r} |f(\zeta)|.$$

From the last two inequalities and the fact that $\varphi \in H_K^2$, we get

$$\begin{aligned} & \sup_{\|f\|_{\mathcal{B}} \leq 1} \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| \leq r\}} \left| (f - f \circ \psi_j)^{(n+1)}(\varphi(z)) \right|^2 \\ & \quad \times |\varphi'(z)|^2 (1 - |z|^2) dA(z) \\ & \lesssim \frac{((n+2)!)^2 r^2}{R^{2n+4} (j+1)^2} \left(\log \frac{1}{1 - (R+r)} \right)^2 \|\varphi\|_{H_K^2}^2 \rightarrow 0 \end{aligned}$$

as $j \rightarrow \infty$. Thus, we have

$$\begin{aligned} & \|C_\varphi D^n\|_{e, \mathcal{B} \rightarrow H_K^2}^2 \leq \liminf_{j \rightarrow \infty} \|C_\varphi D^n - C_\varphi D^n K_j\|_{\mathcal{B} \rightarrow H_K^2}^2 \\ & = \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{B}} \leq 1} \|C_\varphi D^n (Id - K_j) f\|_{H_K^2}^2 \\ & \lesssim \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) \end{aligned}$$

for any $r \in (0, 1)$. Letting $r \rightarrow 1$, we obtain that $\|C_\varphi D^n\|_{e, \mathcal{B} \rightarrow H_K^2} \lesssim \sqrt{U}$. The proof is completed. \square

From Theorem 2.2, we immediately get the following result.

Theorem 2.3. *Let φ be an analytic self-map of \mathbb{D} , $n \in \mathbb{Z}$ and K satisfy the conditions in (2). Suppose that $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is bounded. Then the following statements are equivalent.*

- (a) $C_\varphi D^n : \mathcal{B} \rightarrow H_K^2$ is compact;
- (b)

$$\limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} g(z, a) dA(z) = 0;$$

- (c)

$$\limsup_{r \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |\sigma_a(z)|^2) dA(z) = 0;$$

- (d)

$$\limsup_{r \rightarrow 1} \sup_{I \subset \partial \mathbb{D}} \frac{1}{K(|I|)} \int_{S(I) \cap \{|\varphi(z)| > r\}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^{2(n+1)}} (1 - |z|^2) dA(z) = 0.$$

3. Characterization of the operator $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$

In this section, we study the boundedness, compactness and the essential norm of the operator $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$. Hence, we first state some lemmas which will be used in the proofs of the main results in this section. The following result is Luecking's characterization of Carleson measure in terms of functions in the Dirichlet type spaces (see [11]). In comparison with the original result, $f^{(n)}$ has been replaced by $f^{(n+1)}$ since this appears to be convenient for the purposes of the paper.

Lemma 3.1 ([11]). *Let μ be a positive measure on \mathbb{D} , $0 < p \leq 2$ and $-1 < \alpha < \infty$. Then μ is a bounded $\frac{2(2+\alpha)}{p} + 2n$ -Carleson measure if and only if there is a positive constant C , depending only on α, p and n such that*

$$\int_{\mathbb{D}} |f^{(n+1)}(z)|^2 d\mu(z) \leq C \|f\|_{\mathcal{D}_\alpha^p}^2$$

for all $f \in \mathcal{D}_\alpha^p$. Moreover, if μ is a bounded $\frac{2(2+\alpha)}{p} + 2n$ -Carleson measure, then $C = C_1 C_2$, where $C_1 > 0$ depends only on α, p and n and

$$C_2 = \sup_I \frac{\mu(S(I))}{|I|^{\frac{2(2+\alpha)}{p} + 2n}}.$$

It is well-known that the bounded t -Carleson measure can be characterized by a global integral condition (see [1]), namely,

$$(6) \quad \sup_I \frac{\mu(S(I))}{|I|^t} \approx \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(z)|^t d\mu(z), \quad 0 < t < \infty.$$

The following lemma is a partial boundary version of this result.

Lemma 3.2 ([12]). *Let $0 < r < 1, 1 \leq t < \infty$ and let μ be a positive Borel measure on \mathbb{D} . Then*

$$\sup_I \frac{\mu(S(I) \setminus \Delta(0, r))}{|I|^t} \lesssim \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma'_b(z)|^t d\mu(z),$$

where $\Delta(0, r) := \{z : |z| < r\}$.

Lemma 3.3 ([2]). *Let g and u be positive measurable functions on \mathbb{D} , and let $\varphi \in S(\mathbb{D})$. Then*

$$\int_{\mathbb{D}} (g \circ \varphi)(z) |\varphi'(z)|^2 u(z) dA(z) = \int_{\mathbb{D}} g(w) U(\varphi, w) dA(z),$$

where $U(\varphi, w) = \sum_{z \in \varphi^{-1}\{w\}} u(z)$ for $w \in \mathbb{D} \setminus \{\varphi(0)\}$.

For an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in \mathbb{D} , define

$$T_j f(z) = \sum_{k=0}^j a_k z^k, \quad R_j f(z) = \sum_{k=j+1}^{\infty} a_k z^k.$$

Lemma 3.4. *Let $1 < p < \infty, n \in \mathbb{Z}$ and $-1 < \alpha < \infty$. For each $w \in \mathbb{D}$, positive integer j and $f \in \mathcal{D}_{\alpha}^p$,*

$$\left| (R_j f(w))^{(n+1)} \right| \lesssim \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \|f\|_{\mathcal{D}_{\alpha}^p} \sum_{k=j}^{\infty} \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} |w|^k,$$

where Γ denotes the Gamma function.

Proof. Since $R_j f \in \mathcal{D}_{\alpha}^p$, then $R_{j-1} f' = (R_j f)' \in A_{\alpha}^p$, we have

$$(R_j f)'(w) = \int_{\mathbb{D}} (R_j f)'(z) K_z(w) dA_{\alpha}(z),$$

where $K_w(z)$ is the Bergman Kernel function. Thus

$$(R_j f)^{(n+1)}(w) = \int_{\mathbb{D}} R_{j-1} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \frac{\bar{z}^n}{(1 - \bar{z}w)^{\alpha+2+n}} dA_{\alpha}(z).$$

The orthogonality of monomials z^{γ} with respect to dA_{α} shows

$$\begin{aligned} & \int_{\mathbb{D}} R_{j-1} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \frac{\bar{z}^n}{(1 - \bar{z}w)^{\alpha+2+n}} dA_{\alpha}(z) \\ &= \int_{\mathbb{D}} f'(z) \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} R_{j-1} \left(\frac{\bar{z}^n}{(1 - \bar{z}w)^{\alpha+2+n}} \right) dA_{\alpha}(z). \end{aligned}$$

By Hölder inequality, we get

$$\begin{aligned}
 & \left| (R_j f)^{(n+1)}(w) \right| \\
 & \leq \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \int_{\mathbb{D}} |f'(z)| \left| R_{j-1} \left(\frac{\bar{z}^n}{(1 - \bar{z}w)^{\alpha+2+n}} \right) \right| dA_{\alpha}(z) \\
 & \approx \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \int_{\mathbb{D}} |f'(z)| \sum_{k=j-1}^{\infty} \frac{\Gamma(k + n + \alpha + 2)}{k! \Gamma(n + \alpha + 2)} w^k \bar{z}^{k+n} dA_{\alpha}(z) \\
 & \leq \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \|f\|_{\mathcal{D}_{\alpha}^p} \left(\int_{\mathbb{D}} \left(\sum_{k=j-1}^{\infty} \frac{\Gamma(k + n + \alpha + 2)}{k! \Gamma(n + \alpha + 2)} |w|^k |z|^{k+n} \right)^q dA_{\alpha}(z) \right)^{\frac{1}{q}} \\
 & \lesssim \frac{\Gamma(\alpha + 2 + n)}{\Gamma(\alpha + 2)} \|f\|_{\mathcal{D}_{\alpha}^p} \sum_{k=j-1}^{\infty} \frac{\Gamma(k + n + \alpha + 2)}{k! \Gamma(n + \alpha + 2)} |w|^k.
 \end{aligned}$$

□

Theorem 3.1. *Let φ be an analytic self-map of \mathbb{D} , $1 < p \leq 2$, $-1 < \alpha < \infty$, $n \in \mathbb{Z}$. Assume that K satisfy the conditions in (2). Then the following statements are equivalent.*

- (a) $C_{\varphi} D^n : \mathcal{D}_{\alpha}^p \rightarrow H_K^2$ is bounded;
- (b)

$$\sup_{a, b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z) < \infty;$$

- (c)

$$\sup_{a, b \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) < \infty.$$

Proof. The proof of (a) \Leftrightarrow (c) is similar to (a) \Leftrightarrow (b). Hence we only prove (a) \Leftrightarrow (b).

(b) \Rightarrow (a). Let $f \in \mathcal{D}_{\alpha}^p$. By Lemma 3.1, (6) and Lemma 3.3 we have

$$\begin{aligned}
 \|C_{\varphi} D^n f\|_{H_K^2}^2 & \approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(f^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \\
 & = \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f^{(n+1)}(w)|^2 d\mu_a(w) \\
 & \leq \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_I \frac{\int_{S(I)} d\mu_a(w)}{|I|^{\frac{2(2+\alpha)}{p} + 2n}} \|f\|_{\mathcal{D}_{\alpha}^p}^2 \\
 & \approx \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(w)|^{\frac{2(2+\alpha)}{p} + 2n} d\mu_a(w) \|f\|_{\mathcal{D}_{\alpha}^p}^2 \\
 & = \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z) \|f\|_{\mathcal{D}_{\alpha}^p}^2
 \end{aligned}$$

$$\begin{aligned}
&= \sup_{a,b \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p}+2n} |\varphi'(z)|^2 g(z,a) dA(z) \|f\|_{\mathcal{D}_\alpha^p}^2 \\
&< \infty,
\end{aligned}$$

where $d\mu_a(w) = \sum_{z \in \varphi^{-1}\{w\}} g(z,a) dA(w)$. Thus $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$ is bounded.

(a) \Rightarrow (b). Assume that $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$ is bounded. Let $b \in \mathbb{D}$. Set

$$f_b(z) = \int_0^z \left(\frac{1-|b|^2}{(1-\bar{b}w)^2} \right)^{\frac{\alpha+2}{p}} dw, \quad z \in \mathbb{D}.$$

Then $\|f_b\|_{\mathcal{D}_\alpha^p} = 1$ for all $b \in \mathbb{D}$. Let $\zeta \in \partial\mathbb{D}$ be the center of arc $I \subset \partial\mathbb{D}$ and $b = (1-|I|)\zeta \in \mathbb{D}$. Then

$$f_b^{(n+1)}(z) = \frac{\Gamma(\frac{4+2\alpha}{p}+n)}{\Gamma(\frac{4+2\alpha}{p})} \frac{(1-|b|^2)^{\frac{\alpha+2}{p}} \bar{b}^n}{(1-\bar{b}z)^{\frac{2(2+\alpha)}{p}+n}}$$

and $|f_b^{(n+1)}(z)| \gtrsim \frac{1}{(1-|b|)^{\frac{2(2+\alpha)}{p}+2n}}, z \in S(I)$. Thus

$$\begin{aligned}
\infty &> \|C_\varphi D^n\|^2 \|f_b\|_{\mathcal{D}_\alpha^p}^2 \geq \|C_\varphi D^n f_b\|_{H_K^2}^2 \\
&\approx \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(f_b^{(n)} \circ \varphi)'(z)|^2 g(z,a) dA(z) \\
&= \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z,a) dA(z) \\
&= \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(w)|^2 d\mu_a(w) \\
&= \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} \left| \frac{\Gamma(\frac{4+2\alpha}{p}+n)}{\Gamma(\frac{4+2\alpha}{p})} \frac{(1-|b|^2)^{\frac{\alpha+2}{p}} \bar{b}^n}{(1-\bar{b}w)^{\frac{2(2+\alpha)}{p}+n}} \right|^2 d\mu_a(w) \\
&\gtrsim \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{S(I)} \frac{1}{(1-|b|)^{\frac{2(2+\alpha)}{p}+2n}} d\mu_a(w)
\end{aligned}$$

for all $b \in \mathbb{D}$. By (6) and Lemma 3.3 we have

$$\begin{aligned}
\infty &> \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \sup_I \frac{\int_{S(I)} d\mu_a(w)}{|I|^{\frac{2(2+\alpha)}{p}+2n}} \\
&\approx \sup_{a \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \sup_{b \in \mathbb{D}} \int_{\mathbb{D}} |\sigma'_b(w)|^{\frac{2(2+\alpha)}{p}+2n} d\mu_a(w) \\
&= \sup_{a,b \in \mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p}+2n} |\varphi'(z)|^2 g(z,a) dA(z).
\end{aligned}$$

This completes the proof of this theorem. \square

Theorem 3.2. Let φ be an analytic self-map of \mathbb{D} , $1 < p \leq 2$, $-1 < \alpha < \infty$, $n \in \mathbb{Z}$. Assume that K satisfy the conditions in (2). Suppose that $C_\varphi D^n :$

$\mathcal{D}_\alpha^p \rightarrow H_K^2$ is bounded. Then

$$\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \approx \sqrt{P} \approx \sqrt{Q}.$$

Here

$$P = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z),$$

$$Q = \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z).$$

Proof. We only need to prove that $\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \approx \sqrt{P}$. Since the proof for $\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \approx \sqrt{Q}$ is similar.

First we prove that $\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \gtrsim \sqrt{P}$. Let $b \in \mathbb{D}$. Set

$$f_b(z) = \int_0^z \left(\frac{1 - |b|^2}{(1 - \bar{b}w)^2} \right)^{\frac{\alpha+2}{p}} dw, \quad z \in \mathbb{D}.$$

We have $\|f_b\|_{\mathcal{D}_\alpha^p} = 1$ and $f_b \rightarrow 0$ uniformly on compact subsets of \mathbb{D} as $|b| \rightarrow 1$. Since \mathcal{D}_α^p is reflexive, we see that $f_b \rightarrow 0$ weakly in \mathcal{D}_α^p as $|b| \rightarrow 1$. Thus $\|J(f_b)\|_{H_K^2} \rightarrow 0$ as $|b| \rightarrow 1$ for every compact operator $J : \mathcal{D}_\alpha^p \rightarrow H_K^2$. Hence

$$\begin{aligned} \|C_\varphi D^n - J\|_{\mathcal{D}_\alpha^p \rightarrow H_K^2}^2 &\geq \limsup_{|b| \rightarrow 1} \|C_\varphi D^n(f_b) - J(f_b)\|_{H_K^2}^2 \\ &\geq \limsup_{|b| \rightarrow 1} \|C_\varphi D^n(f_b)\|_{H_K^2}^2 - \limsup_{|b| \rightarrow 1} \|J(f_b)\|_{H_K^2}^2 \\ &= \limsup_{|b| \rightarrow 1} \|C_\varphi D^n(f_b)\|_{H_K^2}^2 \end{aligned}$$

for every compact operator $J : \mathcal{D}_\alpha^p \rightarrow H_K^2$. By (3) and Lemma 3.3 we have

$$\begin{aligned} &\limsup_{|b| \rightarrow 1} \|C_\varphi D^n(f_b)\|_{H_K^2}^2 \\ &\approx \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(f_b^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \\ &= \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(\varphi(z))|^2 |\varphi'(z)|^2 g(z, a) dA(z) \\ &= \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |f_b^{(n+1)}(w)|^2 d\mu_a(w) \\ &= \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} \left| \frac{\Gamma(\frac{4+2\alpha}{p} + n)}{\Gamma(\frac{4+2\alpha}{p})} \frac{(1 - |b|^2)^{\frac{\alpha+2}{p}} \bar{b}^n}{(1 - \bar{b}w)^{\frac{2(2+\alpha)}{p} + n}} \right|^2 d\mu_a(w) \\ &\gtrsim \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{S(I)} \frac{1}{(1 - |b|)^{\frac{2(2+\alpha)}{p} + 2n}} d\mu_a(w) \\ &\approx \limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(w)|^{\frac{2(2+\alpha)}{p} + 2n} d\mu_a(w) \end{aligned}$$

$$= P.$$

Therefore, from the definition of the essential norm, we obtain

$$\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2}^2 = \inf_J \|C_\varphi D^n - J\|_{\mathcal{D}_\alpha^p \rightarrow H_K^2}^2 \gtrsim P.$$

Next we prove that $\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \lesssim \sqrt{P}$. For an analytic function $f(z) = \sum_{k=0}^\infty a_k z^k$ in \mathbb{D} , since T_j is compact on \mathcal{D}_α^p , we have

$$\begin{aligned} \|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} &= \|C_\varphi D^n(T_j + R_j)\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \\ &\leq \|C_\varphi D^n T_j\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} + \|C_\varphi D^n R_j\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \\ &= \|C_\varphi D^n R_j\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \\ &\leq \|C_\varphi D^n R_j\|_{\mathcal{D}_\alpha^p \rightarrow H_K^2}. \end{aligned}$$

Hence

$$\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \leq \liminf_{j \rightarrow \infty} \|C_\varphi D^n R_j\|_{\mathcal{D}_\alpha^p \rightarrow H_K^2}.$$

Therefore, by Lemma 3.3, we get

$$\begin{aligned} &\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2}^2 \leq \liminf_{j \rightarrow \infty} \|C_\varphi D^n R_j\|_{\mathcal{D}_\alpha^p \rightarrow H_K^2}^2 \\ &\leq \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \|C_\varphi D^n(R_j f)\|_{H_K^2}^2 \\ &\approx \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |((R_j f)^{(n)} \circ \varphi)'(z)|^2 g(z, a) dA(z) \\ (7) \quad &= \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w). \end{aligned}$$

Let $r \in (0, 1)$. For each $a \in \mathbb{D}$ and $f \in \mathcal{D}_\alpha^p$, by Lemma 3.4 we have

$$\begin{aligned} &\int_{|w| \leq r} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w) \\ &\lesssim \|f\|_{\mathcal{D}_\alpha^p}^2 \int_{|w| \leq r} \left(\sum_{k=j}^\infty \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} |w|^k \right)^2 d\mu_a(w) \\ &\leq \|f\|_{\mathcal{D}_\alpha^p}^2 \left(\sum_{k=j}^\infty \frac{\Gamma(k + \alpha + 2 + n)}{k! \Gamma(\alpha + 2 + n)} r^k \right)^2 \int_{|w| \leq r} d\mu_a(w). \end{aligned}$$

Since $\varphi \in H_K^2$, by Lemma 3.3 we have

$$\begin{aligned} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|w| \leq r} d\mu_a(w) &= \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|\varphi(z)| \leq r} |\varphi'(z)|^2 g(z, a) dA(z) \\ &\approx \|C_\varphi D^n(z^{n+1})\|_{H_K^2}^2 < \infty. \end{aligned}$$

It is well known that $\sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha+2+n)}{k!\Gamma(\alpha+2+n)} r^k \approx \frac{1}{(1-r)^{\alpha+n+2}}$ for any $r \in (0, 1)$. Hence

$$(8) \quad \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|w| \leq r} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w) = 0.$$

We now estimate $\int_{|w| > r} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w)$. By Lemmas 3.1, 3.2 and 3.3 we obtain

$$\begin{aligned} & \int_{|w| > r} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w) \\ & \lesssim \|R_j f\|_{\mathcal{D}_\alpha^p}^2 \sup_I \frac{\int_{S(I) \setminus \Delta(0, r)} d\mu_a(w)}{|I|^{\frac{2(2+\alpha)}{p} + 2n}} \\ & \lesssim \|R_j f\|_{\mathcal{D}_\alpha^p}^2 \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma'_b(w)|^{\frac{2(2+\alpha)}{p} + 2n} d\mu_a(w) \\ (9) \quad & = \|R_j f\|_{\mathcal{D}_\alpha^p}^2 \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z). \end{aligned}$$

Using (7), (8) and (9), for any $r \in (0, 1)$ we get

$$\begin{aligned} & \|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2}^2 \\ & \leq \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{|w| > r} |(R_j f)^{(n+1)}(w)|^2 d\mu_a(w) \\ & \lesssim \liminf_{j \rightarrow \infty} \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \|R_j f\|_{\mathcal{D}_\alpha^p}^2 \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} \\ & \quad \times |\varphi'(z)|^2 g(z, a) dA(z) \\ & \leq \sup_{\|f\|_{\mathcal{D}_\alpha^p} \leq 1} \|f\|_{\mathcal{D}_\alpha^p}^2 \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \sup_{|b| \geq r} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z). \end{aligned}$$

Taking the limit as $r \rightarrow 1$, we get the desired result. Thus we have

$$\|C_\varphi D^n\|_{e, \mathcal{D}_\alpha^p \rightarrow H_K^2} \approx \sqrt{P}.$$

The proof is complete. \square

From Theorem 3.2, we immediately get the following result.

Theorem 3.3. *Let φ be an analytic self-map of \mathbb{D} , $1 < p \leq 2$, $-1 < \alpha < \infty$, $n \in \mathbb{Z}$. Assume that K satisfy the conditions in (2). Suppose that $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$ is bounded. Then the following statements are equivalent.*

- (a) $C_\varphi D^n : \mathcal{D}_\alpha^p \rightarrow H_K^2$ is compact;
- (b)

$$\limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 g(z, a) dA(z) = 0;$$

(c)

$$\limsup_{|b| \rightarrow 1} \sup_{a \in \mathbb{D}} \frac{1 - |a|^2}{K(1 - |a|^2)} \int_{\mathbb{D}} |\sigma'_b(\varphi(z))|^{\frac{2(2+\alpha)}{p} + 2n} |\varphi'(z)|^2 (1 - |\sigma_a(z)|^2) dA(z) = 0.$$

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