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NORMAL, COHYPONORMAL AND NORMALOID WEIGHTED COMPOSITION OPERATORS ON THE HARDY AND WEIGHTED BERGMAN SPACES

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Abstract. If ψ is analytic on the open unit disk $\mathbb{D}$ and φ is an analytic self-map of $\mathbb{D}$, the weighted composition operator $C_{\psi,\varphi}$ is defined by $C_{\psi,\varphi}f(z) = \psi(z)f(\varphi(z))$, when f is analytic on $\mathbb{D}$. In this paper, we study normal, cohyponormal, hyponormal and normaloid weighted composition operators on the Hardy and weighted Bergman spaces. First, for some weighted Hardy spaces $H^2(\beta)$, we prove that if $C_{\psi,\varphi}$ is cohyponormal on $H^2(\beta)$, then ψ never vanishes on $\mathbb{D}$ and φ is univalent, when $\psi \not\equiv 0$ and φ is not a constant function. Moreover, for $\psi = K_a$, where $|a| < 1$, we investigate normal, cohyponormal and hyponormal weighted composition operators $C_{\psi,\varphi}$. After that, for φ which is a hyperbolic or parabolic automorphism, we characterize all normal weighted composition operators $C_{\psi,\varphi}$, when $\psi \not\equiv 0$ and ψ is analytic on $\overline{\mathbb{D}}$. Finally, we find all normal weighted composition operators which are bounded below.

1. Introduction

Let $H(\mathbb{D})$ denote the collection of all holomorphic functions on the open unit disk $\mathbb{D}$. A function f is called analytic on a closed set F if there exists an open set U such that f is analytic on U and $F \subseteq U$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$.

For f which is analytic on $\mathbb{D}$, we denote by $\hat{f}(n)$ the n-th coefficient of f in its Maclaurin series. The Hardy space H^2 is the collection of all such functions f for which

$$\|f\|_1^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.$$

The space $H^\infty(\mathbb{D})$, simply H^∞, consists of all functions that are analytic and bounded on $\mathbb{D}$. Recall that for $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) =$

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A^2_α, is the set of functions f analytic on the unit disk, satisfying the norm condition

$$\|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 w_\alpha(z) dA(z) < \infty,$$

where $w_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$ and dA is the normalized area measure. When $\alpha = 0$, this gives the Bergman space $A^2(\mathbb{D}) = A^2$.

Let e_w be the linear functional for evaluation at w, that is, $e_w(f) = f(w)$. Then for functional Hilbert spaces H, we let K_w denote the unique function in H which satisfies $\langle f, K_w\rangle = f(w)$ for every $f \in H$. In this case, the functional Hilbert space H is called a reproducing kernel Hilbert space. The weighted Bergman spaces A^2_α and the Hardy space H^2 are all reproducing kernel Hilbert spaces. Let $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. In H^2 and A^2_α, we have reproducing kernels $K_w(z) = (1 - \overline{w}z)^{-\gamma}$ with norm $(1 - |w|^2)^{-\gamma/2}$. Moreover, let k_w denote the normalized reproducing kernel.

Let φ be an analytic map of the open unit disk $\mathbb{D}$ into itself. We define the composition operator C_φ by $C_\varphi(f) = f \circ \varphi$, where f is analytic on $\mathbb{D}$. If ψ is in $H(\mathbb{D})$ and φ is an analytic map of the unit disk into itself, the weighted composition operator with symbols ψ and φ is the operator $C_{\psi,\varphi}$ which is defined by $C_{\psi,\varphi}(f) = \psi.(f \circ \varphi)$, where f is analytic on $\mathbb{D}$. If ψ is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator $C_{\psi,\varphi}$ is bounded on H^2 and A^2_α.

A linear-fractional self-map of $\mathbb{D}$ is a map of the form

$$\varphi(z) = \frac{az + b}{cz + d} \tag{1}$$

for some $a, b, c, d \in \mathbb{D}$ such that $ad - bc \neq 0$, with the property that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. We denote the set of those maps by LFT($\mathbb{D}$). It is well-known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the unit disk onto itself, are just the functions $\varphi(z) = \lambda(a - z)/(1 - \bar{a}z)$, where $|\lambda| = 1$ and $|a| < 1$ (see, e.g., [5]). We denote the class of automorphisms of $\mathbb{D}$ by Aut($\mathbb{D}$).

Let $d\theta$ be the arc-length measure on $\partial\mathbb{D}$. The space $L^2(\partial\mathbb{D})$ denotes the Lebesgue space of $\partial\mathbb{D}$ induced by $d\theta/(2\pi)$. Also $L^\infty(\partial\mathbb{D})$ is the space of all essentially bounded measurable functions on $\partial\mathbb{D}$. Suppose that $dA(z)$ is the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. For any $\alpha > -1$, let dA_α be the measure on $\mathbb{D}$ defined by

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

The Banach space $L^2(\mathbb{D}, dA_\alpha)$ denotes the space of Lebesgue measurable functions f on $\mathbb{D}$ with norm $\left(\int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z)\right)^{1/2}$. For each $b \in L^\infty(\partial\mathbb{D})$, we define the Toeplitz operator T_b on H^2 by $T_b(f) = P(bf)$, where P denotes the orthogonal projection of $L^2(\partial\mathbb{D})$ onto H^2. For each $\psi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator T_ψ on A^2_α by $T_\psi(f) = P_\alpha(\psi f)$, where P_α denotes the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto A^2_α. Since an orthogonal projection has norm 1, clearly T_ψ is bounded. If ψ is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator can be rewritten as $C_{\psi,\varphi} = T_\psi C_\varphi$.

If φ is as in Equation (1), then the adjoint of any linear-fractional composition operator C_φ, acting on H^2 and A^2_α, is given by $C^*_\varphi = T_g C_\sigma T^*_h$ (we call it Cowen's adjoint formula), where $\sigma(z) = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$ is a self-map of $\mathbb{D}$, $g(z) = (-\overline{b}z + \overline{d})^{-\gamma}$, $h(z) = (cz + d)^\gamma$, with $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α (see [8] and [17]). From now on, unless otherwise stated, we assume that σ, h and g are given as above.

A point ζ of $\overline{\mathbb{D}}$ is called a fixed point of a self-map φ of $\mathbb{D}$ if $\lim_{r\to 1} \varphi(r\zeta) = \zeta$. We will write $\varphi'(\zeta)$ for $\lim_{r\to 1} \varphi'(r\zeta)$. Each analytic self-map φ of $\mathbb{D}$ that is neither the identity nor an elliptic automorphism of $\mathbb{D}$ has a unique point w in $\overline{\mathbb{D}}$ that acts like an attractive fixed point in that $\varphi_n(z) \to w$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, where φ_n denotes φ composed with itself n times (φ_0 being the identity function). The point w, called the Denjoy-Wolff point of φ, is also characterized as follows:

- if $|w| < 1$, then $\varphi(w) = w$ and $|\varphi'(w)| < 1$;
- if $|w| = 1$, then $\varphi(w) = w$ and $0 < \varphi'(w) \le 1$.

More information about Denjoy-Wolff points can be found in [10, Chapter 2] or [23, Chapters 4 and 5].

A map $\varphi \in \mathrm{LFT}(\mathbb{D})$ is called parabolic if it has a single fixed point ζ in the Riemann sphere $\hat{\mathbb{C}}$ such that $\zeta \in \partial\mathbb{D}$. Let $\tau(z) = (1 + \overline{\zeta}z)/(1 - \overline{\zeta}z)$. The map τ takes the unit disk onto the right half-plane Π and takes ζ to ∞. The function $\phi = \tau \circ \varphi \circ \tau^{-1}$ is a linear-fractional self-map of Π that fixes only the point ∞, so it must have the form $\phi(z) = z + t$ for some complex number t, where $\mathrm{Re}(t) \ge 0$. Let us call t the translation number of either φ or ϕ. Note that if $\mathrm{Re}(t) = 0$, then $\varphi \in \mathrm{Aut}(\mathbb{D})$. Also if $\mathrm{Re}(t) > 0$, then $\varphi \notin \mathrm{Aut}(\mathbb{D})$. In [23, p. 3], J. H. Shapiro showed that among the linear-fractional transformations fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$. Let $\varphi \in \mathrm{LFT}(\mathbb{D})$ be parabolic with fixed point ζ and translation number t. Therefore,

$$\varphi(z) = \frac{(2-t)z + t\zeta}{2 + t - t\overline{\zeta}z}. \tag{2}$$

Recall that an operator T on a Hilbert space H is said to be normal if $TT^* = T^*T$ and essentially normal if $TT^* - T^*T$ is compact on H. Also T is unitary if $TT^* = T^*T = I$. The normal composition operators on A^2_α and H^2 have symbol $\varphi(z) = az$, where $|a| \le 1$ (see [10, Theorem 8.2]). An operator T on a Hilbert space H is said to be binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$. It is obvious that every normal operator is binormal. In [18], binormal composition operator C_φ was charactrized when φ is a linear-fractional self-map of $\mathbb{D}$. If $A^*A \ge AA^*$ or, equivalently, $\|Ah\| \ge \|A^*h\|$ for all vectors h, then A is said to be a hyponormal operator. An operator A is said to be cohyponormal if A^* is hyponormal. An operator A on a Hilbert space H is normal if and only if for all vectors $h \in H$, $\|Ah\| = \|A^*h\|$, so it is not hard to see that an operator A is normal if and only if A and A^* are hyponormal. Recall that an operator

T is said to be normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of T. Then we can see that all normal, hyponormal and cohyponormal operators are normaloid. The normal and unitary weighted composition operators on H^2 were investigated in [3] by Bourdon et al. After that, in [21], these results were extended to the bigger spaces containing the Hardy and weighted Bergman spaces. Recently, hyponormal and cohyponormal weighted composition operators have been investigated in [9] and [14]. In this paper, we work on normal, normaloid, cohyponormal and hyponormal weighted composition operators. In the second section, we extend [9, Theorem 3.2] to some weighted Hardy spaces. In the third section, for $\psi = K_a$, where $a \in \mathbb{D}$, we show that if $C_{\psi,\varphi}$ is normal, then $|\varphi(0)| = |a|$. In the fourth section, we state that for $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\psi \not\equiv 0$ which is analytic on $\overline{\mathbb{D}}$, if $C_{\psi,\varphi}$ is normal on H^2 or A^2_α and $\psi(\zeta) = 0$, then $\zeta \in \partial\mathbb{D}$, $\varphi(\zeta) = \zeta$ and ζ is not the Denjoy-Wolff point of φ. Also we prove that if $C_{\psi,\varphi}$ is normal on H^2 or A^2_α, then 1 is an eigenvalue of $C_{\psi,\varphi}$, when $\varphi \in \mathrm{Aut}(\mathbb{D})$, ψ is analytic on $\overline{\mathbb{D}}$ and $\psi \not\equiv 0$. Furthermore, for φ which is a parabolic or hyperbolic automorphism, we give a necessary and sufficient condition for $C_{\psi,\varphi}$ to be normal on H^2 and A^2_α, when ψ is analytic on $\overline{\mathbb{D}}$. Finally, we show that for a normal weighted composition operator $C_{\psi,\varphi}$ on a Hilbert space H which contains all the polynomials, $C_{\psi,\varphi}$ is Fredholm if and only if $C_{\psi,\varphi}$ has closed range.

2. Cohyponormal weighted composition operators

In this section, we provide the generalized result of [9, Theorem 3.2] on some weighted Hardy spaces. We first state the following well-known lemma which was proved in [13, p. 1211] and [19, p. 1524].

Lemma 2.1. *Let $C_{\psi,\varphi}$ be a bounded operator on H^2 and A^2_α. For each $w \in \mathbb{D}$, $C^*_{\psi,\varphi}K_w = \overline{\psi(w)}K_{\varphi(w)}$.*

Let H be a Hilbert space. The set of all bounded operators from H into itself is denoted by $B(H)$. Now assume that H is a Hilbert space of analytic functions on $\mathbb{D}$. For $f \in H$, let $[f]$ denote the smallest closed subspace of H which contains $\{z^n f\}_{n=0}^{\infty}$. If $S \in B(H)$ is the unilateral shift $Sf = zf$, then $[f]$ is the smallest closed subspace of H containing f which is invariant under S; moreover, if $[f] = H$, then the function f is called cyclic. Also for $\psi \in H$, we define a multiplication operator $M_\psi : H \to H$ that for each $f \in H$, $M_\psi(f) = \psi f$. In this section, we assume that M_ψ and S are bounded operators, but in general every multiplication operator is not bounded.

Theorem 2.2. *Assume that H is a Hilbert space of analytic functions on $\mathbb{D}$ and the polynomials are dense in H. Assume that $\psi \not\equiv 0$ and φ is not a constant function. If $C_{\psi,\varphi}$ is cohyponormal on H, then ψ is cyclic in H.*

Proof. Suppose that $C_{\psi,\varphi}$ is cohyponormal. By the Open Mapping Theorem, $\ker C_{\psi,\varphi} = \{0\}$. Then $\ker C^*_{\psi,\varphi} = \{0\}$. Since $\ker M^*_\psi \subseteq \ker C^*_{\psi,\varphi}$, we have ker $M_\psi^* = \{0\}$. [6, Theorem 2.19, p. 35] and [6, Corollary 2.10, p. 10] imply that $\overline{\operatorname{ran} M_\psi} = H$. Then ψH is dense in H. Because the polynomials are dense in H, it is easily seen that this is equivalent to saying that polynomial multiples of ψ are dense in H, that is, to ψ being a cyclic vector. □

Note that by [16, Corollary 1.5, p. 15], if $f \in H^2$ is cyclic, then it is an outer function. Then under the conditions of Theorem 2.2, if $C_{\psi,\varphi}$ is cohyponormal on H^2, then ψ is an outer function (see [9, Theorem 3.2]).

Lemma 2.3. *Let H be a reproducing kernel Hilbert space of analytic functions on $\mathbb{D}$. Assume that for each $w \in \mathbb{D}$, there is $g \in H$ such that $g(w) \neq 0$. Let ψ be cyclic in H. Then ψ never vanishes on $\mathbb{D}$.*

Proof. Since ψ is cyclic in H, $\{p\psi : p \text{ is a polynomial}\}$ is dense in H. Let $f \in H$. Then there is a sequence $\{p_n\}$ of polynomials such that $p_n\psi \to f$ as $n \to \infty$. Suppose that $\psi(w) = 0$ for some $w \in \mathbb{D}$. We can see that $f(w) = \langle f, K_w\rangle = \lim_{n\to\infty}\langle p_n\psi, K_w\rangle = 0$. Then for each $f \in H$, $f(w) = 0$ and it is a contradiction. □

Let H be a Hilbert space of analytic functions on the unit disk. If the monomials $1, z, z^2, \ldots$ are an orthogonal set of non-zero vectors with dense span in H, then H is called a weighted Hardy space. We will assume that the norm satisfies the normalization $\|1\| = 1$. The weight sequence for a weighted Hardy space H is defined to be $\beta(n) = \|z^n\|$. The weighted Hardy space with weight sequence $\beta(n)$ will be denoted by $H^2(\beta)$. The inner product on $H^2(\beta)$ is given by

$$\langle\sum_{j=0}^{\infty} a_j z^j, \sum_{j=0}^{\infty} c_j z^j\rangle = \sum_{j=0}^{\infty} a_j\overline{c_j}\beta(j)^2.$$

We require the following corollary, which is a generalization of [9, Theorem 3.2]. The proof which shows that φ is univalent of the following corollary relies on some ideas from [9, Theorem 3.2].

Corollary 2.4. *Let $H^2(\beta)$ be a weighted Hardy space. Suppose that* $\sup \beta(j+1)/\beta(j)$ *is finite. Assume that $\psi \not\equiv 0$ and φ is not a constant function. If $C_{\psi,\varphi}$ is cohyponormal on $H^2(\beta)$, then ψ never vanishes on $\mathbb{D}$ and φ is univalent.*

Proof. By Theorem 2.2, Lemma 2.3, [10, Proposition 2.7] and [10, Theorem 2.10], ψ never vanishes on $\mathbb{D}$. Assume that there are points w_1 and w_2 in $\mathbb{D}$ such that $\varphi(w_1) = \varphi(w_2)$ and $w_1 \neq w_2$. Hence

$$C^*_{\psi,\varphi}(\overline{\psi(w_2)}K_{w_1} - \overline{\psi(w_1)}K_{w_2}) = \overline{\psi(w_2)\psi(w_1)}K_{\varphi(w_1)} - \overline{\psi(w_1)\psi(w_2)}K_{\varphi(w_2)} \equiv 0.$$

We conclude that $0 \in \sigma_p(C^*_{\psi,\varphi})$. Therefore, by [7, Proposition 4.4, p. 47], $0 \in \sigma_p(C_{\psi,\varphi})$ and $C_{\psi,\varphi}(\overline{\psi(w_2)}K_{w_1} - \overline{\psi(w_1)}K_{w_2}) = 0$. Since ψ never vanishes on $\mathbb{D}$, $C_\varphi(\overline{\psi(w_2)}K_{w_1} - \overline{\psi(w_1)}K_{w_2}) = 0$. Setting $h = K_{w_2}/K_{w_1}$, we find

$$h \circ \varphi \equiv \overline{\left(\frac{\psi(w_2)}{\psi(w_1)}\right)}.$$

Since φ is not a constant function, $\varphi(\mathbb{D})$ is an open set by the Open Mapping Theorem. It follows that K_{w_2}/K_{w_1} is a constant function and it is a contradiction. $\square$

Suppose that T belongs to $B(H^2)$ or $B(A^2_\alpha)$. Through this paper, the spectrum of T, the essential spectrum of T and the point spectrum of T are denoted by $\sigma(T)$, $\sigma_e(T)$ and $\sigma_p(T)$, respectively.

Remark 2.5. Suppose that $C_{\psi,\varphi}$ is cohyponormal on H^2 or A^2_α and $\psi \not\equiv 0$. First, assume that φ is not a constant function. Since H^2 and A^2_α are weighted Hardy spaces, Theorem 2.2 and Corollary 2.4 imply that ψ is cyclic and ψ never vanishes on $\mathbb{D}$. Now suppose that $\varphi \equiv c$, where c is a constant number and $|c| < 1$. Assume that there are points w_1 and w_2 in $\mathbb{D}$ such that $\psi(w_1) = 0$ and $\psi(w_2) \neq 0$. From Lemma 2.1, we observe that $C^*_{\psi,\varphi}(K_{w_1}) = \overline{\psi(w_1)}K_c \equiv 0$. Since $C_{\psi,\varphi}$ is cohyponormal, we have $C_{\psi,\varphi}(K_{w_1}) = 0$. Since $\psi \cdot K_{w_1} \circ \varphi \equiv 0$, we obtain that $\psi(w_2)K_{w_1}(c) = 0$ and so $\psi(w_2) = 0$. It is a contradiction. Hence we conclude that ψ never vanishes on $\mathbb{D}$.

3. Normaloid weighted composition operators

Let α be a complex number of modulus 1 and φ be an analytic self-map of $\mathbb{D}$. Since $\mathrm{Re}\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right)$ is a positive harmonic function on $\mathbb{D}$, this function is the poisson integral of a finite positive Borel measure μ_α on $\partial\mathbb{D}$. Let us write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of the singular parts μ^s_α of the measures μ_α as $|\alpha| = 1$. In the next lemma and proposition, the set of points which φ makes contact with $\partial\mathbb{D}$ is $\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) \in \partial\mathbb{D}\}$.

Lemma 3.1 ([14, Lemma 3.2])**.** *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points which φ makes contact with $\partial\mathbb{D}$ is finite. Assume that there are a positive integer n and $\zeta \in \partial\mathbb{D}$ such that $E(\varphi_n) = \{\zeta\}$, where ζ is the Denjoy-Wolff point of φ. Let $\psi \in H^\infty$ be continuous at ζ. Then*

$$r_\gamma(C_{\psi,\varphi}) = |\psi(\zeta)|\varphi'(\zeta)^{-\gamma/2}.$$

Proposition 3.2. *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points which φ makes contact with $\partial\mathbb{D}$ is finite. Assume that there are a positive integer n and $\zeta \in \partial\mathbb{D}$ such that $E(\varphi_n) = \{\zeta\}$, where ζ is the Denjoy-Wolff point of φ. Suppose that $\psi = K_a$ for some $a \in \mathbb{D}$. Let $C_{\psi,\varphi}$ be normaloid on H^2 or A^2_α. Then*

$$\frac{(1 - |\varphi(a)|^2)(1 + |a|)}{1 - |a|} \geq \varphi'(\zeta).$$

In particular, if $\varphi'(\zeta) = 1$, then $2|a| \geq |\varphi(a)|^2(1 + |a|)$.

Proof. Assume that $C_{\psi,\varphi}$ is normaloid. Let $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. By Lemmas 2.1 and 3.1, we can see that

$$\begin{aligned}\left|\frac{1}{1-\overline{a}\zeta}\right|^{2\gamma}\varphi'(\zeta)^{-\gamma} &= \|C_{\psi,\varphi}\|^2 \\ &\geq \langle C^*_{\psi,\varphi}k_a, C^*_{\psi,\varphi}k_a\rangle \\ &= (1-|a|^2)^\gamma\langle C^*_{\psi,\varphi}K_a, C^*_{\psi,\varphi}K_a\rangle \\ &= \frac{1}{(1-|a|^2)^{2\gamma}}\left(\frac{1-|a|^2}{1-|\varphi(a)|^2}\right)^\gamma.\end{aligned}$$

Then

$$\frac{1}{(1-|a|)^{2\gamma}}\varphi'(\zeta)^{-\gamma} \geq \frac{1}{(1-|a|^2)^{2\gamma}}\left(\frac{1-|a|^2}{1-|\varphi(a)|^2}\right)^\gamma,$$

so the result follows. Now suppose that $\varphi'(\zeta) = 1$. In this case, after some computation, we can see that $2|a| \geq |\varphi(a)|^2 + |a||\varphi(a)|^2$. □

Corollary 3.3. *Let φ satisfy the hypotheses of Proposition 3.2. If C_φ is normaloid on H^2 or A^2_α, then $1 - |\varphi(0)|^2 \geq \varphi'(\zeta)$. Moreover, if $\varphi'(\zeta) = 1$, then C_φ is not normaloid.*

Proof. Let $\psi \equiv K_0$. By Proposition 3.2, we can see that if C_φ is normaloid, then $1 - |\varphi(0)|^2 \geq \varphi'(\zeta)$. Now assume that $\varphi'(\zeta) = 1$. Suppose that C_φ is normaloid. Then $1 - |\varphi(0)|^2 \geq 1$. Hence $\varphi(0) = 0$ and it is a contradiction. □

In Proposition 3.4, we only prove the third part. Proofs of the other parts is similar to part (c) and follows from the definitions of hyponormal and cohyponormal operators.

Proposition 3.4. *Let φ be an analytic self-map of $\mathbb{D}$ and $\psi = K_a$ for some $a \in \mathbb{D}$. The following statements hold on H^2 or A^2_α.*

(a) *If $C_{\psi,\varphi}$ is cohyponormal, then $|\varphi(0)| \geq |a|$.*
(b) *If $C_{\psi,\varphi}$ is hyponormal, then $|\varphi(0)| \leq |a|$.*
(c) *If $C_{\psi,\varphi}$ is normal, then $|\varphi(0)| = |a|$.*

Proof. (c) Let $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. Since $K_0 \equiv 1$, it follows from Lemma 2.1 that

$$\begin{aligned}\frac{1}{(1-|\varphi(0)|^2)^\gamma} &= \langle\overline{\psi(0)}K_{\varphi(0)}, \overline{\psi(0)}K_{\varphi(0)}\rangle \\ &= \langle C_{\psi,\varphi}C^*_{\psi,\varphi}K_0, K_0\rangle \\ &= \langle C^*_{\psi,\varphi}C_{\psi,\varphi}K_0, K_0\rangle \\ &= \langle\psi,\psi\rangle \\ &= \frac{1}{(1-|a|^2)^\gamma},\end{aligned}$$

which implies that $|\varphi(0)| = |a|$. □

Let $\varphi \in \mathrm{LFT}(\mathbb{D})$. It is easy to see that φ must belong to one of the following three disjoint classes:

- Automorphism of $\mathbb{D}$.
- Non-automorphism of $\mathbb{D}$ with $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$.
- Non-automorphism of $\mathbb{D}$ with $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$.

Let $\varphi \in \mathrm{LFT}(\mathbb{D})$ such that $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$. Then by [10, Theorem 2.48], φ has a fixed point $p \in \mathbb{D}$. Suppose that $\varphi \in \mathrm{LFT}(\mathbb{D})$ such that $\overline{\varphi(\mathbb{D})} \subseteq \mathbb{D}$ or φ is the identity or an automorphism of $\mathbb{D}$ with a fixed point in $\mathbb{D}$. All normal weighted composition operators $C_{\psi,\varphi}$ of these types were found (see [3, Theorem 10], [9, Theorem 3.7] and [21, Theorem 4.3]). Also suppose $\varphi \in \mathrm{Aut}(\mathbb{D})$ which has no fixed point in $\mathbb{D}$ and $\psi = K_a$ for some $a \in \mathbb{D}$; all normal weighted composition operators $C_{\psi,\varphi}$ on H^2 and A^2_α are characterized in Theorem 4.5. Bourdon et al. in [3, Proposition 12] obtained a condition that reveals what is required for normality of a weighted composition operator $C_{\psi,\varphi}$ on H^2, where φ is a linear-fractional and $\psi = K_{\sigma(0)}$ (also by the similar proof, an analogue result holds on A^2_α). In the following corollary, for $\psi = K_a$ and $\varphi \in \mathrm{LFT}(\mathbb{D})$, where $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$ and $a \in \mathbb{D}$, we investigate normal weighted composition operators $C_{\psi,\varphi}$ on H^2 and A^2_α.

Corollary 3.5. *Suppose that $\varphi \in LFT(\mathbb{D})$ is not an automorphism and $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Assume that $\psi = K_a$ for some $a \in \mathbb{D}$. If $C_{\psi,\varphi}$ is normal on H^2 or A^2_α, then φ is a parabolic non-automorphism and $|a| = |t/(2+t)| = |\sigma(0)|$, where t is the translation number of φ.*

Proof. Let $C_{\psi,\varphi}$ be normal on H^2 or A^2_α. Then $C_{\psi,\varphi}$ is essentially normal. Since ψ never vanishes on $\partial\mathbb{D}$, we conclude from [13, Theorem 2.6] and [13, Remark 2.7] that φ is a parabolic non-automorphism and the result follows from Proposition 3.4 and Equation (2). □

4. Normal weighted composition operators

Each disk automorphism φ has two fixed points on the sphere, counting multiplicity. The automorphisms are classified according to the location of their fixed points: elliptic if one fixed point is in $\mathbb{D}$ and a second fixed point is in the complement of the closed disk, hyperbolic if both fixed points are in $\partial\mathbb{D}$, and parabolic if there is one fixed point in $\partial\mathbb{D}$ of multiplicity two (see [10] and [23]). Let φ be an automorphism of $\mathbb{D}$. In [13] and [15], the present authors investigated essentially normal weighted composition operator $C_{\psi,\varphi}$, when $\psi \in A(\mathbb{D})$ and $\psi(z) \neq 0$ for each $z \in \overline{\mathbb{D}}$. In this section, we just assume that ψ is analytic on $\overline{\mathbb{D}}$ and we attempt to find all normal weighted composition operators $C_{\psi,\varphi}$. Also we will show that ψ never vanishes on $\overline{\mathbb{D}}$.

Lemma 4.1. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $f \in A(\mathbb{D})$. Then $T^*_f C_\varphi - C_\varphi T^*_{f\circ\varphi^{-1}}$ is compact on H^2 and A^2_α.*

Proof. We know that $\sigma = \varphi^{-1}$. It is not hard to see that

$$C_\varphi^* T_f = T_g C_\sigma T_h^* T_f = C_\sigma T_{g\circ\varphi} T_h^* T_f$$

and

$$T_{f\circ\varphi^{-1}} C_\varphi^* = T_{f\circ\varphi^{-1}} C_\sigma T_{g\circ\varphi} T_h^* = C_\sigma T_f T_{g\circ\varphi} T_h^*.$$

Since $C_\varphi C_\sigma = I$, by [11, Proposition 7.22] and [22, Corollary 1(c)], $C_\varphi^* T_f - T_{f\circ\varphi^{-1}} C_\varphi^*$ is compact and the result follows. □

In this section, we assume that $\varphi(z) = \lambda(a-z)/(1-\overline{a}z)$ and $w(z) = (1-\overline{a}z)^\gamma \psi(z)$, where $a \in \mathbb{D}$, $|\lambda| = 1$, $\psi \in A(\mathbb{D})$ and $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A_α^2. We will use the notation $A \equiv B$ to indicate that the difference of the two bounded operators A and B is compact.

Proposition 4.2. *Let $\psi \in A(\mathbb{D})$ and $\varphi \in \mathrm{Aut}(\mathbb{D})$. If $C_{\psi,\varphi}$ is hyponormal on H^2 or A_α^2, then for each $\zeta \in \partial\mathbb{D}$, $|w(\zeta)| - |w(\varphi(\zeta))| \geq 0$. Moreover, if $C_{\psi,\varphi}$ is cohyponormal on H^2 or A_α^2, then for each $\zeta \in \partial\mathbb{D}$, $|w(\varphi(\zeta))| - |w(\zeta)| \geq 0$.*

Proof. Suppose that $C_{\psi,\varphi}$ is hyponormal. By [11, Proposition 7.22], [22, Corollary 1(c)] and the preceding lemma, we can see that

$$\begin{aligned} C_{\psi,\varphi} C_{\psi,\varphi}^* &= T_\psi C_\varphi T_g C_\sigma T_h^* T_\psi^* \\ &= T_\psi C_\varphi C_\sigma T_{g\circ\varphi} T_h^* T_\psi^* \\ &\equiv T_{|\psi|^2 \cdot \overline{h} \cdot g\circ\varphi} \end{aligned}$$

and

$$\begin{aligned} C_{\psi,\varphi}^* C_{\psi,\varphi} &= T_g C_\sigma T_h^* T_\psi^* T_\psi C_\varphi \\ &= T_g C_\sigma T_{h\psi}^* C_\varphi T_{\psi\circ\varphi^{-1}} \\ &\equiv T_g C_\sigma C_\varphi T_{(h\psi)\circ\varphi^{-1}}^* T_{\psi\circ\varphi^{-1}} \\ &\equiv T_{|\psi\circ\varphi^{-1}|^2 \cdot g \cdot \overline{h}\circ\varphi^{-1}}. \end{aligned}$$

Let $\varphi(a) = 0$ for $a \in \mathbb{D}$. After some computation, we see that $\overline{h(z)} g(\varphi(z)) = |1-\overline{a}z|^{2\gamma}/(1-|a|^2)^\gamma$, where $\gamma = 1$ for H^2 and $\gamma = \alpha+2$ for A_α^2. Hence by [12, Corollary 2.6] and [4, Corollary 1.3], we have

$$\begin{aligned} \sigma_e(C_{\psi,\varphi}^* C_{\psi,\varphi} - C_{\psi,\varphi} C_{\psi,\varphi}^*) &= \frac{1}{(1-|a|^2)^\gamma} \sigma_e(T_{|w\circ\varphi^{-1}|^2 - |w|^2}) \\ &= \left\{ \frac{|w(\varphi^{-1}(\zeta))|^2 - |w(\zeta)|^2}{(1-|a|^2)^\gamma} : \zeta \in \partial\mathbb{D} \right\}. \end{aligned}$$

Since $C_{\psi,\varphi}^* C_{\psi,\varphi} \geq C_{\psi,\varphi} C_{\psi,\varphi}^*$, $|w(\varphi^{-1}(\zeta))|^2 - |w(\zeta)|^2 \geq 0$ for each $\zeta \in \partial\mathbb{D}$ and so $|w(\zeta)|^2 - |w(\varphi(\zeta))|^2 \geq 0$ for any $\zeta \in \partial\mathbb{D}$. Therefore, the conclusion follows. The idea of the proof of the result for cohyponormal operator $C_{\psi,\varphi}$ is similar to hyponormal operator, so it is left for the reader. □

Suppose ψ is a non-constant analytic function on $\overline{\mathbb{D}}$ and ψ never vanishes on $\mathbb{D}$. By [5, Exercise 1, p. 129], we see that $|\psi|$ assumes its minimum value on $\partial\mathbb{D}$. Assume that $C_{\psi,\varphi}$ is essentially normal on H^2 or A^2_α. Now let $\psi(\zeta) = 0$ for some $\zeta \in \partial\mathbb{D}$. In the following proposition, we show that $\varphi(\zeta) = \zeta$ and ζ is not the Denjoy-Wolff point of φ.

Proposition 4.3. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$. Suppose that ψ is analytic on $\overline{\mathbb{D}}$ and ψ never vanishes on $\mathbb{D}$. Let $C_{\psi,\varphi}$ be essentially normal on H^2 or A^2_α. If $\psi(\zeta) = 0$ for some $\zeta \in \partial\mathbb{D}$, then $\varphi(\zeta) = \zeta$ and ζ is not the Denjoy-Wolff point of φ.*

Proof. Assume that there is $\zeta \in \partial\mathbb{D}$ such that $\psi(\zeta) = 0$. According to [13, Theorem 3.2] and [15, Theorem 3.3], the map ψ is zero on $B = \{\zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots\}$. It is trivial that $B \subset \partial\mathbb{D}$. Since ψ is an analytic function on $\overline{\mathbb{D}}$ never vanishing on $\mathbb{D}$, we get that B is a finite set. Suppose that B contains N elements. If $\varphi(\zeta) \neq \zeta$, then one can write $B = \{\zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots, \varphi_{N-1}(\zeta)\}$. Observe that $\varphi_N(b) = b$ for each $b \in B$, which ensures that $N \leq 2$. If $N = 1$, then it is straightforward to see that $\varphi(\zeta) = \zeta$ but this is a contradiction. If $N = 2$, i.e., $B = \{\zeta, \varphi(\zeta)\}$, then B is precisely the set of all fixed points of φ_2. Since $\varphi_2(\zeta) \neq \varphi(\zeta)$, the point $\varphi(\zeta)$ cannot be fixed by φ. Since all fixed points of φ belong to B, we obtain that ζ must be a fixed point of φ, which is a contradiction. Thus $\varphi(\zeta) = \zeta$.

It remains to show that ζ is not the Denjoy-Wolff point of φ. Assume that ζ is the Denjoy-Wolff point of φ. Since $|w| = |w \circ \varphi|$ on $\partial\mathbb{D}$ from [13, Theorem 3.2] and [15, Theorem 3.3], choose a constant $\lambda \in \partial\mathbb{D}$ such that $w \circ \varphi = \lambda w$. Denjoy-Wolff Theorem implies that

$$0 = |w(\zeta)| = \lim_{n\to\infty} |w(\varphi_n(z))| = \lim_{n\to\infty} |\lambda^n w(z)| = |w(z)|$$

for each $z \in \mathbb{D}$. Then $\psi \equiv 0$ on $\mathbb{D}$, which is a contradiction. Hence, ζ is not the Denjoy-Wolff point of φ. □

If $C_{\psi,\varphi}$ is normal on H^2 or A^2_α, then by Remark 2.5, ψ never vanishes on $\mathbb{D}$. In Proposition 4.3, we saw that for $\varphi \in \mathrm{Aut}(\mathbb{D})$ which is not a hyperbolic automorphism and ψ that is analytic on $\overline{\mathbb{D}}$, if $C_{\psi,\varphi}$ is normal on H^2 or A^2_α, then ψ never vanishes on $\overline{\mathbb{D}}$.

Proposition 4.4. *Let $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\varphi(a) = 0$ for some $a \in \mathbb{D}$. Assume that ψ is analytic on $\overline{\mathbb{D}}$ and $\psi \not\equiv 0$. If $C_{\psi,\varphi}$ is normal on H^2 or A^2_α, then w is an eigenvector for the operator C_φ and the corresponding C_φ-eigenvalue for w is 1.*

Proof. Suppose that $C_{\psi,\varphi}$ is normal. Proposition 4.2 implies that $|w| = |w \circ \varphi|$ on $\partial\mathbb{D}$. Since ψ never vanishes on $\mathbb{D}$, by [5, Exercise 6, p. 130], we conclude that $C_\varphi(w) = \lambda w$, where $|\lambda| = 1$. If φ is an elliptic automorphism with a fixed point $t \in \mathbb{D}$, then $w(t) = w(\varphi(t)) = \lambda w(t)$. Since ψ never vanishes on $\mathbb{D}$, $\lambda = 1$. Now suppose that φ is a parabolic or hyperbolic automorphism with Denjoy-Wolff point ζ. Then

$$w(\zeta) = \lim_{r\to 1} w(\varphi(r\zeta)) = \lambda \lim_{r\to 1} w(r\zeta) = \lambda w(\zeta).$$

By Proposition 4.3, $\psi(\zeta) \neq 0$ and so $\lambda = 1$. Therefore, we conclude that $C_\varphi(w) = w$. □

Let φ be an elliptic automorphism or the identity. As we stated before Corollary 3.5, all normal weighted composition operators $C_{\psi,\varphi}$ of these types were found. Also we must say that in this case ψ never vanishes on $\overline{\mathbb{D}}$. In the next theorem, for φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, which is in $\mathrm{Aut}(\mathbb{D})$, we show that constant multiples of $K_{\sigma(0)}$ are the only examples for ψ that $C_{\psi,\varphi}$ are normal, where ψ is analytic on $\overline{\mathbb{D}}$. It is interesting that again ψ never vanishes on $\overline{\mathbb{D}}$ and these weighted composition operators are actually a constant multiples of unitary weighted composition operators (see [3, Theorem 6] and [21, Corollary 3.6]).

Theorem 4.5. *Assume that φ, not the identity and not an elliptic automorphism of $\mathbb{D}$, is in* $\mathrm{Aut}(\mathbb{D})$. *Suppose that ψ is analytic on $\overline{\mathbb{D}}$ and $\psi \not\equiv 0$. Then $C_{\psi,\varphi}$ is normal on H^2 or A^2_α if and only if $\psi = \psi(0)K_{\sigma(0)}$; hence ψ never vanishes on $\overline{\mathbb{D}}$.*

Proof. Suppose that $C_{\psi,\varphi}$ is normal and $\varphi(a) = 0$. By Proposition 4.4, $w \circ \varphi = w$ on $\mathbb{D}$. Then for each integer n, $\psi(z)(1 - \overline{a}z)^\gamma = \psi(\varphi_n(z))(1 - \overline{a}\varphi_n(z))^\gamma$ on $\mathbb{D}$, where $\gamma = 1$ for H^2 and $\gamma = \alpha + 2$ for A^2_α. Let ζ be the Denjoy-Wolff point of φ. Applying Denjoy-Wolff Theorem, we have

$$\psi(\zeta)(1 - \overline{a}\zeta)^\gamma = \lim_{n\to\infty} \psi(\varphi_n(z))(1 - \overline{a}\varphi_n(z))^\gamma = \psi(z)(1 - \overline{a}z)^\gamma$$

for each $z \in \mathbb{D}$. Hence $\psi(z) = \frac{\psi(\zeta)(1-\overline{a}\zeta)^\gamma}{(1-\overline{a}z)^\gamma}$ and so $\psi = \psi(0)K_{\sigma(0)}$.

Conversely, it is not hard to see that the fact that for a constant number c, $C_{c\psi,\varphi}$ is normal implies that $C_{\psi,\varphi}$ is also. Then without loss of generality, we assume that $\psi = K_{\sigma(0)}$. Observe that $g \circ \varphi = \frac{1}{(1-|a|^2)^\gamma K_a}$ and $h\psi \equiv 1$. Since $\sigma = \varphi^{-1}$, we obtain from Cowen's adjoint formula that

$$C^*_{\psi,\varphi}C_{\psi,\varphi} = T_g C_\sigma T^*_{h\psi} T_\psi C_\varphi = T_g C_\sigma C_\varphi T_{\psi\circ\sigma} = T_{g\cdot\psi\circ\varphi^{-1}}$$

and

$$C_{\psi,\varphi}C^*_{\psi,\varphi} = T_\psi C_\varphi T_g C_\sigma T^*_{h\psi} = T_\psi C_\varphi C_\sigma T_{g\circ\varphi} = T_{\psi\cdot g\circ\varphi}.$$

After some computation, one can see that $\psi \cdot g \circ \varphi = g \cdot \psi \circ \varphi^{-1} = \frac{1}{(1-|a|^2)^\gamma}$. Hence $C_{\psi,\varphi}$ is normal. □

Lemma 4.6. *Assume that $\varphi \in$ $\mathrm{Aut}(\mathbb{D})$. Suppose that for some $\zeta \in \partial\mathbb{D}$, $\{\varphi_n(\zeta) : n$ is a positive integer$\}$ is a finite set. If φ is parabolic or hyperbolic, then ζ is a fixed point of φ.*

Proof. Let $\{\zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots\}$ be a finite set. Then there is an integer N such that $\varphi_N(\zeta) = \zeta$, so ζ is a fixed point of φ_N. It is not hard to see that φ is parabolic or hyperbolic if and only if φ_N is parabolic or hyperbolic, respectively and fixed points of φ and φ_N are the same. Hence ζ is the fixed point of φ. □

Let $\psi \in A(\mathbb{D})$. If $C_{\psi,\varphi}$ is cohyponormal on H^2 or A^2_α, then ψ never vanishes on $\mathbb{D}$ or $\psi \equiv 0$ (see Remark 2.5). Therefore, by Maximum Modulus Theorem and [5, Exercise 1, p. 129], there are $\zeta_1, \zeta_2 \in \partial\mathbb{D}$, with $|w(\zeta_1)| \le |w(z)|$ and $|w(z)| \le |w(\zeta_2)|$ for all $z \in \mathbb{D}$. In the following theorem, we assume that ζ_1 and ζ_2 are given as above.

Theorem 4.7. *Suppose that $\psi \in A(\mathbb{D})$. Let $\{\eta \in \partial\mathbb{D} : |w(\eta)| = |w(\zeta_1)|\}$ and $\{\eta \in \partial\mathbb{D} : |w(\eta)| = |w(\zeta_2)|\}$ be finite sets. Suppose that $C_{\psi,\varphi}$ is cohyponormal on H^2 or A^2_α. The following statements hold.*

(a) *If φ is a parabolic automorphism, then $|w|$ is a constant function on $\partial\mathbb{D}$. Moreover, if $\psi \not\equiv 0$, then $C_{\psi,\varphi}$ is normal and $\psi = \psi(0)K_{\sigma(0)}$.*

(b) *If φ is a hyperbolic automorphism, then ζ_1 and ζ_2 are the fixed points of φ.*

Proof. (a) Suppose that φ is a parabolic automorphism. Since $C_{\psi,\varphi}$ is cohyponormal, by Proposition 4.2, for each positive integer n, we have

$$|w(\varphi_n(\zeta_2))| \ge |w(\varphi_{n-1}(\zeta_2))| \ge |w(\varphi_{n-2}(\zeta_2))| \ge \cdots \ge |w(\zeta_2)|$$

and

$$|w(\zeta_1)| \ge |w(\varphi^{-1}(\zeta_1))| \ge |w(\varphi_2^{-1}(\zeta_1))| \ge \cdots \ge |w(\varphi_n^{-1}(\zeta_1))|.$$

It is not hard to see that φ^{-1} is parabolic and φ and φ^{-1} have the same fixed point. Then by Lemma 4.6 and the statement which was stated before Theorem 4.7, we can see that $\zeta_1 = \zeta_2$. Then $|w|$ is constant on $\partial\mathbb{D}$. Now assume that $\psi \not\equiv 0$. We know that ψ never vanishes on $\mathbb{D}$ (see Corollary 2.4), so by [5, Exercise 2, p. 129], w is a constant function. Then $\psi = \psi(0)K_a$ and by Theorem 4.5, $C_{\psi,\varphi}$ is normal.

(b) Suppose that φ is a hyperbolic automorphism. By the proof of part (a) and Lemma 4.6, we can see that ζ_1 and ζ_2 are the fixed points of φ. □

For $\psi \in H^\infty$ and φ which is an elliptic automorphism of $\mathbb{D}$, cohyponormality and normality of a weighted composition operator $C_{\psi,\varphi}$ on H^2 are equivalent (see [9, Proposition 3.17]). In the previous theorem, we showed that if φ is a parabolic automorphism and ψ and w satisfy the hypotheses of this theorem, then $C_{\psi,\varphi}$ is cohyponormal on H^2 or A^2_α if and only if $C_{\psi,\varphi}$ is normal and we saw that $\psi = \psi(0)K_{\sigma(0)}$.

Recall that a bounded linear operator T between two Banach spaces is Fredholm if it is invertible modulo compact operators. We say that an operator $A \in B(H)$ is bounded below if there is a constant $c > 0$ such that $c\|h\| \le \|A(h)\|$ for all $h \in H$. Moreover, we know that a normal operator N on a Hilbert space H is bounded below if and only if N is invertible (see [6, Exercise 15, p. 36]). By this fact, the statements (a) and (c) in Theorem 4.8 are equivalent.

Theorem 4.8. *Suppose that $C_{\psi,\varphi}$ is a normal operator on a Hilbert space H of analytic functions on $\mathbb{D}$. Assume that all the polynomials belong to H. The following statements are equivalent.*

(a) *The operator $C_{\psi,\varphi}$ is bounded below.*
(b) *The operator $C_{\psi,\varphi}$ is Fredholm.*
(c) *The operator $C_{\psi,\varphi}$ is invertible.*

Proof. (b) implies (c). Suppose that $C_{\psi,\varphi}$ is Fredholm. Since by [6, Corollary 2.4, p. 352], dim ker $C_{\psi,\varphi} < \infty$, it is not hard to see that $\psi \not\equiv 0$. We claim that φ is not a constant function. Assume that $\varphi \equiv c$, where $|c| < 1$. It is not hard to see that for each n, $z^n(z-c) \in \ker C_{\psi,\varphi}$, so dim ker $C_{\psi,\varphi} = \infty$ and it is a contradiction. By the Open Mapping Theorem, we can see that $0 \notin \sigma_p(C_{\psi,\varphi})$. Assume that $C_{\psi,\varphi}$ is not invertible. Then by [6, Proposition 4.6, p. 359], $0 \in \sigma_p(C_{\psi,\varphi})$ and it is a contradiction.

(c) implies (b). This is clear. □

Assume $\psi \not\equiv 0$ and φ is not a constant function. By the Open Mapping Theorem, it is clear that ker $C_{\psi,\varphi} = (0)$. Then $C_{\psi,\varphi}$ has closed range if and only if $C_{\psi,\varphi}$ is bounded below. We know that $C_{\psi,\varphi}$ on H^2 or A^2_α is invertible if and only if $\varphi \in \mathrm{Aut}(\mathbb{D})$ and $\psi \in H^\infty$ is bounded away from zero on $\mathbb{D}$ (see [2, Theorem 3.4]). As we stated before if φ is an elliptic automorphism or the identity, all normal weighted composition operators were found; moreover others were characterized in Theorem 4.8. Then closed range weighted composition operators on H^2 and A^2_α which are normal were found.

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