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MONOTONICITY PROPERTIES OF THE GENERALIZED STRUVE FUNCTIONS

Rosihan M. Ali, Saiful R. Mondal, and Kottakkaran S. Nisar

Abstract. This paper introduces and studies a generalization of the classical Struve function of order p given by

$$ {}_a\mathtt{S}_{p,c}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma\big(ak+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1}. $$

Representation formulae are derived for ${}_a\mathtt{S}_{p,c}$. Further the function ${}_a\mathtt{S}_{p,c}$ is shown to be a solution of an $(a+1)$-order differential equation. Monotonicity and log-convexity properties for the generalized Struve function ${}_a\mathtt{S}_{p,c}$ are investigated, particulary for the case $c=-1$. As a consequence, Turán-type inequalities are established. For $a=2$ and $c=-1$, dominant and subordinant functions are obtained for the Struve function ${}_2\mathtt{S}_{p,-1}$.

1. Introduction

The Struve function of order p given by

$$ \mathtt{H}_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\big(k+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1} \tag{1.1} $$

is a particular solution of the non-homogeneous Bessel differential equation

$$ x^2y''(x)+xy'(x)+(x^2-p^2)y(x) = \frac{4\left(\frac{x}{2}\right)^{p+1}}{\sqrt{\pi}\,\Gamma\big(p+\frac{1}{2}\big)}. \tag{1.2} $$

Here Γ denote the gamma function. A solution of the non-homogeneous modified Bessel equation

$$ x^2y''(x)+xy'(x)-(x^2+p^2)y(x) = \frac{4\left(\frac{x}{2}\right)^{p+1}}{\sqrt{\pi}\,\Gamma\big(p+\frac{1}{2}\big)} \tag{1.3} $$

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yields the modified Struve function

$$\mathtt{L}_p(x) := -ie^{-\frac{ip\pi}{2}}\, \mathtt{H}_p(ix) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\big(k+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1}. \tag{1.4}$$

The Struve functions occur in various areas of physics and applied mathematics, for example, in water-wave and surface-wave problems [2, 13], as well as in problems on unsteady aerodynamics [23]. The Struve functions are also important in particle quantum dynamical studies of spin decoherence [22] and nanotubes [21].

The Struve function has gone through several generalizations and investigations, notably in [8, 9, 10, 14, 16, 24, 25, 26, 27, 28]. Recently, Orhan and Yagmur [20, 29] considered yet another generalization of the Struve function defined on the complex plane, and obtained sufficient conditions for it to be univalent, starlike, close-to-convex, and convex. Their generalization is given by the power series

$$W_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k \left(\frac{z}{2}\right)^{2k+p+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+p+\frac{b+2}{2}\right)}, \qquad p,b,c \in \mathbb{C}.$$

An association between this generalized Struve function and the Hardy space of analytic functions was investigated in [30].

Galué in a recent paper [12] introduced a generalization of the Bessel function of order p given by

$${}_a\mathtt{J}_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(ak+p+1)k!} \left(\frac{x}{2}\right)^{2k+p}, \quad x \in \mathbb{R}, \quad a \in \mathbb{N} = \{1,2,3,\ldots\}.$$

Recurrence relations for ${}_a\mathtt{J}_p$ and several identities involving their derivatives and integrals were derived. In [4], Baricz studied the Galué-type modified Bessel function

$${}_a\mathtt{I}_p(x) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(ak+p+1)k!} \left(\frac{x}{2}\right)^{2k+p}, \quad x \in \mathbb{R}, \quad a \in \mathbb{N}, \tag{1.5}$$

and obtained a Turàn-type inequality along with several other inequalities involving ${}_a\mathtt{I}_p$. It is evident that ${}_1\mathtt{J}_p := J_p$ and ${}_1\mathtt{I}_p := I_p$, where J_p and I_p are respectively the classical Bessel and modified Bessel functions.

In recent works, Baricz *et al.* [5, 6] studied monotonicity properties involving the ratio between two modified Bessel functions, as well as the ratio between a modified Bessel and a modified Struve function.

In the sequel, we define and give emphasis to the following generalization of the Struve function. For $a \in \mathbb{N}$ and $p, c, x \in \mathbb{R}$, let

$${}_a\mathtt{S}_{p,c}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma\big(ak+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1}. \tag{1.6}$$

Thus ${}_1\mathtt{S}_{p,1}(x) = \mathtt{H}_p(x)$ and ${}_1\mathtt{S}_{p,-1}(x) = \mathtt{L}_p(x)$, where $\mathtt{H}_p$ and $\mathtt{L}_p$ are given respectively by (1.1) and (1.4). In this light, the function ${}_a\mathtt{S}_{p,c}$ is called the Galué-type generalized Struve function of order p. It is also readily evident that the power series (1.6) is absolutely convergent for finite x.

In this paper, it is first shown that the works of Galué as well as Baricz *et al.* described earlier readily extend to the generalized Struve function ${}_a\mathtt{S}_{p,c}$ given by (1.6). In Section 2, representation formulae and a recurrence relation for ${}_a\mathtt{S}_{p,c}$ will be derived. The generalized Struve function of half-integer order will also be obtained in terms of the generalized hypergeometric function. More importantly, the function ${}_a\mathtt{S}_{p,c}$ is shown to be a solution of a certain differential equation of order $a+1$, which reduces to (1.2) and (1.3) in the case $a=1$ and for particular values of c.

Section 3 is devoted to the investigation of monotonicity and log-convexity properties involving the function ${}_aL_p(x) := {}_a\mathtt{S}_{p,-1}(x)$, as well as to the ratio between these two of different order. As a consequence, Turán-type inequalities are deduced. Dominant and subordinant functions for ${}_a\mathtt{S}_{p,c}$ are obtained for $a=2$ and $c=1$ or $c=-1$.

Monotonicity property is also studied for the function $\varphi_a : (0,\infty) \to \mathbb{R}$ given by

$$\varphi_a(x) := \frac{2^{a-1}x^{1-a}{}_aI_{p+a}(x)}{{}_aL_p(x)},$$

where ${}_aI_{p+a}$ is the Galué-type modified Bessel function given by (1.5). For $a=1$, Baricz and Pogány in [6] showed that φ_1 is decreasing for $p \in (-3/2,-1/2]$. We find the range over p for which φ_a is decreasing on $(0,\infty)$ for the cases $a=2$ and $a=3$, and additionally pose a conjecture for the remaining values of a.

2. Representations for the generalized Struve function

The Gauss multiplication theorem [1] for the gamma function states that

$$\Gamma(mz) = (2\pi)^{\frac{1-m}{2}} m^{mz-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left(z+\tfrac{j-1}{m}\right),\ z \neq 0, -\frac{1}{m}, \ldots, \tag{2.1}$$

$m \in \mathbb{N}$. Thus

$$\begin{aligned}\Gamma(ak+l) &= \Gamma\left(a\left(k+\tfrac{l}{a}\right)\right)\\ &= (2\pi)^{\frac{1-a}{2}} a^{ak+l-\frac{1}{2}} \prod_{j=1}^{a} \Gamma\left(k+\tfrac{l+j-1}{a}\right)\\ &= (2\pi)^{\frac{1-a}{2}} a^{ak+l-\frac{1}{2}} \prod_{j=1}^{a} \left(\tfrac{l+j-1}{a}\right)_k \Gamma\left(\tfrac{l+j-1}{a}\right),\end{aligned} \tag{2.2}$$

$l \neq -ak, -ak-1, -ak-2, \ldots$, and $k \in \mathbb{N}$. Here $(\alpha)_k$ denote the Pochhammer symbol defined by $(\alpha)_k = \alpha(\alpha+1)_{k-1}$ and $(\alpha)_0 = 1$. With $z = l/a$ and $m = a$, it follows from (2.1) that

$$\prod_{j=1}^{a} \Gamma\left(\tfrac{l+j-1}{a}\right) = \frac{\Gamma(l)}{(2\pi)^{\frac{1-a}{2}} a^{l-\frac{1}{2}}}.$$

Together with (2.2), the latter identity gives

$$\Gamma(ak+l) = a^{ak}\, \Gamma(l) \prod_{j=1}^{a} \left(\tfrac{l+j-1}{a}\right)_k ,$$

and thus

$$\Gamma\left(p+\tfrac{3}{2}+ak\right) = a^{ak}\, \Gamma\big(p+\tfrac{3}{2}\big) \prod_{j=1}^{a} \big(\tfrac{2p+2j+1}{2a}\big)_k , \quad \text{and}$$

$$\Gamma\big(k+\tfrac{3}{2}\big) = \tfrac{\sqrt{\pi}}{2} \big(\tfrac{3}{2}\big)_k . \tag{2.3}$$

It is evident now from (1.6) and (2.3) that

$$\begin{aligned} &{}_a\mathtt{S}_{p,c}(x) \\ &= \frac{2x^{p+1}}{2^{p+1}\sqrt{\pi}\,\Gamma\big(p+\frac{3}{2}\big)} \sum_{k=0}^{\infty} \frac{1}{\big(\frac{3}{2}\big)_k \big(\frac{2p+3}{2a}\big)_k \big(\frac{2p+5}{2a}\big)_k \cdots \big(\frac{2p+2a+1}{2a}\big)_k} \left(-\frac{cx^2}{4a^a}\right)^k , \end{aligned}$$

which results in the following representation in terms of the generalized hypergeometric function (see [3]) given by

$${}_mF_n(x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_m)_k}{(b_1)_k (b_2)_k \cdots (b_n)_k k!} x^k.$$

Proposition 2.1. *Let $a \in \mathbb{N}$, and $p, c, x \in \mathbb{R}$. Then*

$${}_a\mathtt{S}_{p,c}(x) = \frac{x^{p+1}}{2^p \sqrt{\pi}\,\Gamma(p+\frac{3}{2})}\, {}_1F_{a+1}\left(1; \tfrac{3}{2}, \tfrac{2p+3}{2a}, \tfrac{2p+5}{2a}, \dots, \tfrac{2p+2a+1}{2a}; -\tfrac{cx^2}{4a^a}\right).$$

Next we look at the representation formulae for the generalized Struve function ${}_a\mathtt{S}_{p,c}$ of half-integer order. For $p = -1/2$, it follows from (2.3) that

$$\Gamma\left(ak+p+\tfrac{3}{2}\right) = \Gamma(ak+1) = a^{ak} k! \prod_{j=1}^{a-1} \big(\tfrac{j}{a}\big)_k \quad \text{and} \quad \Gamma\big(k+\tfrac{3}{2}\big) = \tfrac{\sqrt{\pi}}{2} \big(\tfrac{3}{2}\big)_k .$$

Proposition 2.1 now shows that

$${}_a\mathtt{S}_{-\frac{1}{2},c}(x) = \sqrt{\tfrac{2x}{\pi}}\; {}_0F_a\left(\, ; \tfrac{3}{2}, \tfrac{1}{a}, \tfrac{2}{a}, \dots, \tfrac{a-1}{a}; -\tfrac{cx^2}{4a^a}\right).$$

Similarly for $a \in \mathbb{N}$,

$${}_a\mathtt{S}_{a-\frac{3}{2},c}(x) = \frac{2\left(\frac{x}{2}\right)^{a-\frac{1}{2}}}{\sqrt{\pi}\Gamma(a)}\; {}_0F_a\left(\, ; \tfrac{3}{2}, \tfrac{a+1}{a}, \tfrac{a+2}{a}, \dots, \tfrac{a+(a-1)}{a}; -\tfrac{cx^2}{4a^a}\right).$$

In the case $a=1$, it follows from the formulae in [19, p. 291] that

$$ {}_1\mathtt{S}_{-\frac{1}{2},1}(x) = \sqrt{\tfrac{2x}{\pi}}\ {}_0F_1\left(\ ;\tfrac{3}{2};-\tfrac{x^2}{4}\right) = \sqrt{\tfrac{2}{\pi x}}\sin(x); $$

$$ {}_1\mathtt{S}_{-\frac{1}{2},-1}(x) = \sqrt{\tfrac{2x}{\pi}}\ {}_0F_1\left(\ ;\tfrac{3}{2};\tfrac{x^2}{4}\right) = \sqrt{\tfrac{2}{\pi x}}\sinh(x). $$

If $p=1/2$, Proposition 2.1 shows that

$$ {}_a\mathtt{S}_{\frac{1}{2},c}(x) = \sqrt{\tfrac{x^3}{2\pi}}\ {}_0F_a\left(\ ;\tfrac{3}{2},\tfrac{2}{a},\tfrac{3}{a},\dots,\tfrac{a-1}{a},\tfrac{a+1}{a};-\tfrac{cx^2}{4a^a}\right). $$

We next find an $(a+1)$-order differential equation satisfied by ${}_a\mathtt{S}_{p,c}$. Upon differentiation, (1.6) gives

$$
\begin{aligned}
&\frac{d}{dx}\left(x^{-p}{}_a\mathtt{S}_{p,c}(x)\right)\\
&= \sum_{k=0}^{\infty}\frac{(-c)^k\left(k+\frac{1}{2}\right)}{2^{2k+p}\Gamma\left(\frac{3}{2}+p+ak\right)\Gamma\left(k+\frac{3}{2}\right)}x^{2k}\\
&= \frac{1}{2^p\Gamma\left(\frac{3}{2}+p\right)\Gamma\left(\frac{1}{2}\right)} + \sum_{k=1}^{\infty}\frac{(-c)^k}{2^{2k+p}\Gamma\left(\frac{3}{2}+p+ak\right)\Gamma\left(k+\frac{1}{2}\right)}x^{2k}\\
&= \frac{1}{2^p\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)} + \sum_{k=0}^{\infty}\frac{(-c)^{k+1}}{2^{2k+p+2}\Gamma\left(\frac{3}{2}+p+a+ak\right)\Gamma\left(k+\frac{3}{2}\right)}x^{2k+2}\\
&= \frac{1}{2^p\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)}\\
&\qquad -cx^{1-a-p}\left(\tfrac{1}{2}\right)^{1-a}\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(\frac{3}{2}+p+a+ak\right)\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2k+p+a+1}
\end{aligned}
$$

$$ = \frac{1}{2^p\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)} - cx^{1-a-p}\left(\tfrac{1}{2}\right)^{1-a}{}_a\mathtt{S}_{p+a,c}(x). \tag{2.4} $$

Expanding the left side of (2.4) yields

$$ x\,{}_a\mathtt{S}'_{p,c}(x) = \frac{x^{p+1}}{2^p\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)} + p\,{}_a\mathtt{S}_{p,c}(x) - c\left(\frac{x}{2}\right)^{1-a}x\,{}_a\mathtt{S}_{p+a,c}(x). \tag{2.5} $$

Yet another form for $x\,{}_a\mathtt{S}'_{p,c}$ is obtained from

$$
\begin{aligned}
\frac{d}{dx}\left(x^{\frac{2p+1}{a}-p-1}{}_a\mathtt{S}_{p,c}(x)\right) &= \sum_{k=0}^{\infty}\frac{(-c)^k}{2^{2k+p+1}\Gamma\left(\frac{3}{2}+p+ak\right)\Gamma\left(k+\frac{3}{2}\right)}\frac{d}{dx}x^{2k+\frac{2p+1}{a}}\\
&= \frac{1}{a}\sum_{k=0}^{\infty}\frac{(-c)^k(\frac{1}{2}+p+ak)}{2^{2k+p}\Gamma\left(\frac{3}{2}+p+ak\right)\Gamma\left(k+\frac{3}{2}\right)}x^{2k+\frac{2p+1-a}{a}}\\
&= \frac{1}{a}\,x^{\frac{2p+1-a}{a}-p}\ {}_aS_{p-1,c}\left(x\right).
\end{aligned}
$$

Expanding the left-hand side of the above relation, it follows that

$$x\,{}_a\mathtt{S}'_{p,c}(x) = \frac{x}{a}\,{}_a\mathtt{S}_{p-1,c}(x) - \left(\frac{2p+1}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x). \tag{2.6}$$

Thus (2.5) and (2.6) lead to the following recurrence relation.

Proposition 2.2. *Let $a \in \mathbb{N}$, and $p, c, x \in \mathbb{R}$. Then*

$$\frac{x}{a}\,{}_a\mathtt{S}_{p-1,c}(x) + c\left(\frac{x}{2}\right)^{1-a} x\,{}_a\mathtt{S}_{p+a,c}(x) = \frac{x^{p+1}}{2^p\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)} + \left(\tfrac{2p+1}{a} - 1\right){}_a\mathtt{S}_{p,c}(x).$$

Using the recurrence relations (2.5) and (2.6), the next result derives the differential equation satisfied by ${}_a\mathtt{S}_{p,c}$.

Theorem 2.1. *Let $a \in \mathbb{N}$, and the operator D be given by $D := x(d/dx)$. For each $k = 1, \ldots, a$, the generalized Struve function ${}_a\mathtt{S}_{p,c}$ satisfies the differential equation*

$$\begin{aligned}
&(D-p)\prod_{j=1}^{k}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x) + \frac{cx^{k+2-a}}{a^k 2^{1-a}}\,{}_a\mathtt{S}_{p-k+a,c}(x)\\
&= \frac{2^{k+1}\left(\frac{x}{2}\right)^{p+1}}{a^k\sqrt{\pi}\,\Gamma\left(p + \frac{3-2k}{2}\right)}.
\end{aligned} \tag{2.7}$$

In particular, the generalized Struve function ${}_a\mathtt{S}_{p,c}$ is a solution of the differential equation

$$(D-p)\prod_{j=1}^{a}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right)y(x) + \frac{cx^2}{a^a 2^{1-a}}y(x) = \frac{2^{a+1}\left(\frac{x}{2}\right)^{p+1}}{a^a\sqrt{\pi}\,\Gamma\left(p + \frac{3-2a}{2}\right)}. \tag{2.8}$$

Proof. The proof is by induction. In terms of the differential operator D, the identity (2.6) takes the form

$$\left(D + \tfrac{2p+1}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x) = \frac{x}{a}\,{}_a\mathtt{S}_{p-1,c}(x). \tag{2.9}$$

Now the identity (2.5) gives

$$\begin{aligned}
D(x\,{}_a\mathtt{S}_{p-1,c}(x)) &= x^2\,{}_a\mathtt{S}'_{p-1,c}(x) + x\,{}_a\mathtt{S}_{p-1,c}(x)\\
&= \frac{x^{p+1}}{2^{p-1}\sqrt{\pi}\Gamma\left(p+\frac{1}{2}\right)} + px\,{}_a\mathtt{S}_{p-1,c}(x) - c\left(\tfrac{x}{2}\right)^{1-a}x^2\,{}_a\mathtt{S}_{p-1+a,c}(x)\\
&= \frac{x^{p+1}}{2^{p-1}\sqrt{\pi}\Gamma\left(p+\frac{1}{2}\right)} + pa\left(D + \tfrac{2p+1}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x)\\
&\quad - c\left(\tfrac{x}{2}\right)^{1-a}x^2\,{}_a\mathtt{S}_{p-1+a,c}(x).
\end{aligned}$$

Applying the operator D to both sides of (2.9), the latter equation leads to

$$D\left(D + \tfrac{2p+1}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x)$$

$$
\begin{aligned}
&= \frac{1}{a} D(x {}_a\mathtt{S}_{p-1,c}(x)) \\
&= \frac{x^{p+1}}{a 2^{p-1}\sqrt{\pi}\Gamma\left(p+\frac{1}{2}\right)} + p\left(D + \tfrac{2p+1}{a} - p - 1\right) {}_a\mathtt{S}_{p,c}(x) \\
&\quad - \frac{c}{a 2^{1-a}} x^{3-a} {}_a\mathtt{S}_{p-1+a,c}(x),
\end{aligned}
$$

whence

$$
\begin{aligned}
&(D-p)\left(D + \tfrac{2p+1}{a} - p - 1\right) {}_a\mathtt{S}_{p,c}(x) \\
&= \frac{x^{p+1}}{a 2^{p-1}\sqrt{\pi}\Gamma\left(p+\frac{1}{2}\right)} - \frac{c}{a 2^{1-a}} x^{3-a} {}_a\mathtt{S}_{p-1+a,c}(x).
\end{aligned}
$$

This establishes (2.7) for $k = 1$.

Assuming (2.7) holds $k = n$, we will complete the inductive step by showing it also holds for $k = n + 1$. It follows from (2.6) that

$$
\begin{aligned}
&D(x^{n-a+2} {}_a\mathtt{S}_{p-n+a,c}(x)) \\
&= x^{n-a+3} {}_a\mathtt{S}'_{p-n+a,c}(x) + (n-a+2) x^{n-a+2} {}_a\mathtt{S}_{p-n+a,c}(x) \\
&= \frac{x^{n+3-a}}{a} {}_a\mathtt{S}_{p-n-1+a,c}(x) - \left(\tfrac{2(p-n)+1}{a} - p - 1\right) x^{n-a+2} {}_a\mathtt{S}_{p-n+a,c}(x).
\end{aligned}
$$

Applying the operator D to both sides of (2.7) for $k = n$, the above equation shows that

$$
\begin{aligned}
&D(D-p)\prod_{j=1}^{n}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right) {}_a\mathtt{S}_{p,c}(x) \\
&= \frac{(p+1)2^{n+1}\left(\frac{x}{2}\right)^{p+1}}{a^n\sqrt{\pi}\Gamma\left(p+\frac{3-2n}{2}\right)} - \frac{c}{a^{n+1}2^{1-a}} x^{n+3-a} {}_a\mathtt{S}_{p-n-1+a,c}(x) \\
&\quad + \frac{c}{a^n 2^{1-a}}\left(\tfrac{2(p-n)+1}{a} - p - 1\right) x^{n-a+2} {}_a\mathtt{S}_{p-n+a,c}(x).
\end{aligned}
$$

The induction formula allows us to rewrite the final term above in the form

$$
\begin{aligned}
&D(D-p)\prod_{j=1}^{n}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right) {}_a\mathtt{S}_{p,c}(x) \\
&= \frac{(p+1)2^{n+1}\left(\frac{x}{2}\right)^{p+1}}{a^n\sqrt{\pi}\Gamma\left(p+\frac{3-2n}{2}\right)} - \frac{c}{a^{n+1}2^{1-a}} x^{n+3-a} {}_a\mathtt{S}_{p-n-1+a,c}(x) \\
&\quad + \left(\frac{2(p-n)+1}{a} - p - 1\right)\left[\frac{2^{n+1}\left(\frac{x}{2}\right)^{p+1}}{a^n\sqrt{\pi}\Gamma\left(p+\frac{3-2n}{2}\right)}\right. \\
&\quad \left. -(D-p)\prod_{j=1}^{n}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right) {}_a\mathtt{S}_{p,c}(x)\right].
\end{aligned}
$$

Thus

$$\left(D + \tfrac{2(p-n)+1}{a} - p - 1\right)(D-p)\prod_{j=1}^{n}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x)$$
$$= \frac{2^{n+1}\left(\frac{x}{2}\right)^{p+1}}{a^n\sqrt{\pi}\Gamma\left(p+\frac{3-2n}{2}\right)}\frac{2(p-n)+1}{a} - \frac{c}{a^{n+1}2^{1-a}}x^{n+3-a}{}_a\mathtt{S}_{p-n-1+a,c}(x),$$

that is,

$$(D-p)\prod_{j=1}^{n+1}\left(D + \tfrac{2p+3-2j}{a} - p - 1\right){}_a\mathtt{S}_{p,c}(x) + \frac{cx^{n+3-a}}{a^{n+1}2^{1-a}}{}_a\mathtt{S}_{p-n-1+a,c}(x)$$
$$= \frac{2^{n+2}\left(\frac{x}{2}\right)^{p+1}}{a^{n+1}\sqrt{\pi}\,\Gamma\left(p+\frac{3-2(n+1)}{2}\right)}. \qquad \square$$

Remark 2.1. The differential equation (2.8) reduces respectively to (1.2) and (1.3) for $a = 1, c = 1$ and $a = 1, c = -1$. For $a = 2$, (2.8) reduces to

$$4x^3y'''(x) + 4(1-p)x^2y''(x) - (1-4p)xy'(x) + (2cx^2 - 3p)y(x) = \frac{8\left(\frac{x}{2}\right)^{p+1}}{\sqrt{\pi}\Gamma\left(p-\frac{1}{2}\right)}.$$

Thus its particular solution is ${}_2\mathtt{S}_{p,c}$, which from Proposition 2.1 can be expressed in the form

$${}_2\mathtt{S}_{p,c}(x) = \sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(2k+p+\frac{3}{2}\right)\Gamma\left(k+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{2k+p+1}$$
$$= \frac{x^{p+1}}{2^p\sqrt{\pi}\,\Gamma(p+\frac{3}{2})}\,{}_1F_3\left(1;\tfrac{3}{2},\tfrac{2p+3}{4},\tfrac{2p+5}{4};-\tfrac{cx^2}{16}\right).$$

We close this section by establishing integral representations for ${}_a\mathtt{S}_{p,c}$ for $c = 1$ and $c = -1$. For $a \in \mathbb{N}$, the identity (2.1) yields

(2.10)
$$\Gamma\left(ak + \tfrac{3}{2} + p\right) = \Gamma\left(a\left(k + \tfrac{p+\frac{3}{2}}{a}\right)\right) = (2\pi)^{\frac{1-a}{2}}a^{p+ak+1}\prod_{j=1}^{a}\Gamma\left(k + \tfrac{p+j+\frac{1}{2}}{a}\right).$$

Now the beta function $B(x,y)$ [1, 3] is given by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{2.11}$$

for $\operatorname{Re} x > 0, \operatorname{Re} y > 0$. Replacing x by $(k+1)$ and y by $((2p+2j+1)/2a - 1)$ in (2.11) leads to

(2.12)
$$\frac{1}{\Gamma\left(k+\frac{p+j+\frac{1}{2}}{a}\right)} = \frac{2}{\Gamma(k+1)\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)}\int_0^1 t^{2k+1}(1-t^2)^{\frac{2p+2j+1-4a}{2a}}dt,$$

where $\operatorname{Re} p > (2a-3)/2$.

The Legendre duplication formula (see [1, 3])

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \tag{2.13}$$

shows that

$$\Gamma(k+1)\,\Gamma\left(k+\tfrac{3}{2}\right) = 2^{-2k-1}\sqrt{\pi}\Gamma(2k+2) = \frac{\sqrt{\pi}(2k+1)!}{2^{2k+1}}.$$

Thus from (2.10) and (2.12), the latter identity shows that the generalized Struve function ${}_a\mathtt{H}_p := {}_a\mathtt{S}_{p,1}$ can be expressed as

$$\begin{aligned}
{}_a\mathtt{S}_{p,1}(x) &= \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{x}{2}\right)^{2k+p+1}}{(2k+1)!}\frac{2^{2k+1}}{\sqrt{\pi}}\frac{2}{(2\pi)^{\frac{1-a}{2}}a^{p+ak+1}} \\
&\quad\times \prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)}\int_0^1 t^{2k+1}(1-t^2)^{\frac{2p+2j+1-4a}{2a}}dt \\
&= \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-\frac{a}{2}}}\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)} \\
&\quad\times \int_0^1 \sum_{k=0}^{\infty}\frac{(-1)^k(xt)^{2k+1}}{(2k+1)!(a^{\frac{a}{2}})^{2k+1}}(1-t^2)^{\frac{2p+2j+1-4a}{2a}}dt \\
&= \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-\frac{a}{2}}}\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)} \\
&\quad\times \int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}}\sin\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt.
\end{aligned} \tag{2.14}$$

Substituting $t = \sin\phi$ yields

$$\begin{aligned}
{}_a\mathtt{H}_p(x) &= {}_a\mathtt{S}_{p,1}(x) \\
&= \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-\frac{a}{2}}}\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)} \\
&\quad\times \int_0^{\frac{\pi}{2}} (\cos\phi)^{\frac{2p+2j+1-3a}{a}}\sin\left(\frac{x\sin\phi}{a^{\frac{a}{2}}}\right)d\phi.
\end{aligned}$$

Similarly, the integral representation for the generalized modified Struve function ${}_aL_p := {}_a\mathtt{S}_{p,-1}$ takes the form

$$\begin{aligned}
{}_aL_p(x) &:= {}_a\mathtt{S}_{p,-1}(x) \\
&= \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-\frac{a}{2}}}\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)} \\
&\quad\times \int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}}\sinh\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt
\end{aligned} \tag{2.15}$$

$$= \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}} a^{p+1-\frac{a}{2}}} \prod_{j=1}^{a} \frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)}$$
$$\times \int_0^{\frac{\pi}{2}} (\cos\phi)^{\frac{2p+2j+1-3a}{a}} \sinh\left(\frac{x\sin\phi}{a^{\frac{a}{2}}}\right) d\phi.$$

3. Monotonicity properties of the generalized modified Struve

Recall that the generalized modified Struve function ${}_aL_p$ of order p is

$$(3.1) \qquad {}_aL_p(x) := {}_a\mathtt{S}_{p,-1}(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(ak+p+\frac{3}{2})\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k+p+1}.$$

For $a \in \mathbb{N}$, consider the function $\varphi_a : (0,\infty) \to \mathbb{R}$ given by

$$(3.2) \qquad \varphi_a(x) := \frac{2^{a-1}x^{1-a}{}_aI_{p+a}(x)}{{}_aL_p(x)},$$

where ${}_aI_{p+a}$ is the Galué-type modified Bessel function given by (1.5). For $a=1$, Baricz and Pogány in [6] found that φ_1 is increasing for $p \geq -1/2$, and decreasing for $p \in (-3/2,-1/2]$. We investigate the latter problem of finding the range of p over which φ_a is decreasing on $(0,\infty)$. The following result of Biernacki and Krzyż [7] will be required.

Lemma 3.1 ([7])**.** *Consider the power series* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k x^k$*, where* $a_k \in \mathbb{R}$ *and* $b_k > 0$ *for all* k*. Further suppose that both series converge on* $|x| < r$*. If the sequence* $\{a_k/b_k\}_{k\geq 0}$ *is increasing* (*or decreasing*)*, then the function* $x \mapsto f(x)/g(x)$ *is also increasing* (*or decreasing*) *on* $(0,r)$*.*

The above lemma still holds when both f and g are even, or both are odd functions. Here is the main result associated with $a=2$ and $a=3$.

Theorem 3.1. *Let* φ_a *be given by* (3.2)*.*

(i) *If* $a=2$ *and* $p \in (-3/2, p_2]$*, where* $p_2 \approx 4.7081$ *is the positive root of the equation* $4p^2 - 8p - 51 = 0$*, then* φ_a *is decreasing on* $(0,\infty)$*.*
(ii) *If* $a=3$ *and* $p \in (-3/2, p_3]$*, where* $p_3 \approx 14.8115$ *is the positive root of the equation* $8p^3 - 60p^2 - 758p - 1605 = 0$*, then* φ_a *is decreasing on* $(0,\infty)$*.*

Proof. It follows from (1.5) that

$$2^{a-1}x^{1-a}{}_aI_{p+a}(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(ak+p+a+1)} \left(\frac{x}{2}\right)^{2k+p+1}$$
$$:= \sum_{k=0}^{\infty} \alpha_{p,a,k} \left(\frac{x}{2}\right)^{2k+p+1},$$

and from (1.6) that

$$ {}_aL_p(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\big(k+\frac{3}{2}\big)\Gamma\big(ak+p+\frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1} := \sum_{k=0}^{\infty} \beta_{p,a,k} \left(\frac{x}{2}\right)^{2k+p+1}. $$

Note that both series converge for all x.

The assertions above will be deduced from Lemma 3.1 by showing that $\{\alpha_{p,a,k}/\beta_{p,a,k}\}$ is decreasing. The assumption $p > -3/2$ is necessary to ensure that $\beta_{p,a,k} > 0$ for all k. Let

$$ r_k = \frac{\alpha_{p,a,k}}{\beta_{p,a,k}} = \frac{\Gamma\big(k+\frac{3}{2}\big)\Gamma\big(ak+p+\frac{3}{2}\big)}{k!\Gamma(ak+p+a+1)}. $$

Then

$$ \begin{aligned} \frac{r_{k+1}}{r_k} &= \frac{\Gamma\big(k+1+\frac{3}{2}\big)\Gamma\big(ak+a+p+\frac{3}{2}\big)}{(k+1)!\Gamma(ak+p+2a+1)} \times \frac{k!\,\Gamma(ak+p+a+1)}{\Gamma\big(k+\frac{3}{2}\big)\Gamma\big(ak+p+\frac{3}{2}\big)} \\ &= \frac{k+\frac{3}{2}}{k+1} \prod_{j=1}^{a} \frac{ak+a+p-j+\frac{3}{2}}{ak+p-j+2a+1}. \end{aligned} \tag{3.3} $$

We next show that $r_{k+1}/r_k < 1$ for $a = 2$ and $a = 3$.

(i) Let $a = 2$. Then from (3.3), it follows that

$$ \frac{r_{k+1}}{r_k} = \frac{\big(k+\frac{3}{2}\big)\big(2k+p+\frac{5}{2}\big)\big(2k+p+\frac{3}{2}\big)}{(k+1)(2k+p+4)(2k+p+3)}. $$

Thus

$$ \begin{aligned} &\big(k+\tfrac{3}{2}\big)\big(2k+p+\tfrac{5}{2}\big)\big(2k+p+\tfrac{3}{2}\big) - (k+1)(2k+p+4)(2k+p+3) \\ =\; & -4k^2 + k\big(-p-\tfrac{41}{4}\big) + \tfrac{p^2}{2} - p - \tfrac{51}{8} \le 0, \end{aligned} $$

provided $4p^2 - 8p - 51 \le 0$, which holds for $p \in (-3/2, p_2]$.

(ii) For $a = 3$, it follows from (3.3) that

$$ \frac{r_{k+1}}{r_k} = \frac{\big(k+\frac{3}{2}\big)\big(3k+p+\frac{7}{2}\big)\big(3k+p+\frac{5}{2}\big)\big(3k+p+\frac{3}{2}\big)}{(k+1)(3k+p+6)(3k+p+5)(3k+p+4)}. $$

Now $r_{k+1} < r_k$ if the expression

$$ \begin{aligned} &-54k^3 + k^2\big(-\tfrac{63p}{2} - \tfrac{405}{2}\big) + k\big(-3p^2 - \tfrac{315p}{4} - 249\big) \\ &\quad + \frac{p^3}{2} - \frac{15p^2}{4} - \frac{379p}{8} - \frac{1605}{16} \le 0, \end{aligned} $$

which holds for all $p \in (-3/2, p_3)$, with p_3 the positive root of the equation $8p^3 - 60p^2 - 758p - 1605 = 0$. □

The result of Baricz and Pogány in [6] for the case $a = 1$ and the results obtained in the preceding theorem for the cases $a = 2$ and $a = 3$ support the following conjecture.

Conjecture 1. *Let $p > -3/2$, $a \in \mathbb{N}$ be fixed, and the function φ_a be given by (3.2). Then φ_a is decreasing on $(0, \infty)$ if $p \le p_0$, where p_0 is the smallest root in $(-3/2, \infty)$ of*

$$3 \prod_{j=1}^{a} \left(a + p - j + \tfrac{3}{2}\right) = 2 \prod_{j=1}^{a} (2a + p - j + 1).$$

We now turn to looking at monotonicity properties involving the modified Struve function as well as the ratio between two modified Struve functions of different order. On certain occasions, we shall also be interested in the normalized function

$$\begin{aligned} {}_a\mathcal{L}_p(x) &= \left(\tfrac{2}{x}\right)^p \Gamma\left(p + \tfrac{3}{2}\right) {}_a\mathtt{L}_p(x) \\ &= \Gamma\left(p + \tfrac{3}{2}\right) \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(k + \frac{3}{2}\right)\Gamma\left(ak + p + \frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2k+1}. \end{aligned} \tag{3.4}$$

Theorem 3.2. *Let $a, b \in \mathbb{N}$.*

(a) *If $q \ge p > -3/2$ and $a \le b$, then $x \mapsto 2^{p-q} x^{q-p} {}_a\mathtt{L}_p(x)/{}_b\mathtt{L}_q(x)$ is increasing on $(0, \infty)$.*
(b) *The function $p \mapsto {}_a\mathcal{L}_p(x)$ given by (3.4) is decreasing and log-convex on $(-3/2, \infty)$ for each fixed $x > 0$.*
(c) *The function $p \mapsto {}_a\mathtt{L}_{p+a}(x)/{}_a\mathtt{L}_p(x)$ is decreasing on $(-3/2, \infty)$ for each fixed $x > 0$.*
(d) *The function $x \mapsto x {}_a\mathtt{L}'_p(x)/{}_a\mathtt{L}_p(x)$ is increasing on $(0, \infty)$ for each fixed $p > -3/2$.*
(e) *The function*

$$x \mapsto a \prod_{j=1}^{a} \frac{1}{\Gamma\left(\frac{2p+2j+1}{2a}\right)} + \left(\frac{x}{2}\right)^{1-p-a} \sqrt{\pi} (2\pi)^{\frac{1-a}{2}} a^{p+2} {}_a\mathtt{L}_{p+a}(x)$$

is log-convex on $(0, \infty)$ for each fixed $p > -3/2$.

Proof. (a) From (1.6) it is evident that

$$\frac{x^{q-p} {}_a\mathtt{L}_p(x)}{2^{q-p} {}_b\mathtt{L}_q(x)} = \frac{\sum_{k=0}^{\infty} \alpha_{k,p,a} \left(\frac{x}{2}\right)^{2k}}{\sum_{k=0}^{\infty} \alpha_{k,q,b} \left(\frac{x}{2}\right)^{2k}},$$

where

$$\alpha_{k,p,a} = \frac{1}{\Gamma\left(k + \frac{3}{2}\right)\Gamma\left(ak + p + \frac{3}{2}\right)} \quad \text{and} \quad \alpha_{k,q,b} = \frac{1}{\Gamma\left(k + \frac{3}{2}\right)\Gamma\left(bk + q + \frac{3}{2}\right)}.$$

Write $w_k = \alpha_{k,p,a}/\alpha_{k,q,b}$; then

$$\frac{w_{k+1}}{w_k} = \frac{\Gamma\left(ak + p + \frac{3}{2}\right)\Gamma\left(bk + b + q + \frac{3}{2}\right)}{\Gamma\left(bk + q + \frac{3}{2}\right)\Gamma\left(ak + a + p + \frac{3}{2}\right)} = \frac{\left(bk + q + \frac{3}{2}\right)_b}{\left(ak + p + \frac{3}{2}\right)_a} \ge 1$$

when $b \ge a$ and $q \ge p$. The result now follows from Lemma 3.1.

(b) Let $a \in \mathbb{N}$, and $q \geq p > -3/2$. Then $(q+3/2)_{ak} \geq (p+3/2)_{ak}$ for all $k \in \{0,1,2,\ldots\}$. Thus

$$\gamma_{k,q,a} := \Gamma\left(q+\frac{3}{2}\right)\alpha_{k,q,a} = \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\left(q+\frac{3}{2}\right)_{ak}} \leq \frac{1}{\Gamma\left(k+\frac{3}{2}\right)\left(p+\frac{3}{2}\right)_{ak}} := \gamma_{k,p,a}.$$

Since

$${}_a\mathcal{L}_q(x) = \sum_{k=0}^{\infty}\Gamma\left(q+\frac{3}{2}\right)\alpha_{k,q,a}\left(\frac{x}{2}\right)^{2k+1} = \sum_{k=0}^{\infty}\gamma_{k,q,a}\left(\frac{x}{2}\right)^{2k+1},$$

we deduce that

$${}_a\mathcal{L}_q(x) \leq {}_a\mathcal{L}_p(x)$$

for each fixed $x > 0$. Thus $p \mapsto {}_a\mathcal{L}_p$ is decreasing for $p > -3/2$.

To show log-convexity of ${}_a\mathcal{L}_p$, it is sufficient to show that $\gamma_{k,p,a}$ is log-convex for all $k \in \{0,1,2,3,\ldots\}$. The result will then follow from the fact that sums of log-convex functions are also log-convex.

Denote the digamma function by $\Psi(p) = \Gamma'(p)/\Gamma(p)$. Then evidently

$$\frac{\partial^2}{\partial p^2}(\log(\gamma_{k,p,a})) = \Psi'\left(p+\tfrac{3}{2}\right) - \Psi'\left(ak+p+\tfrac{3}{2}\right).$$

From [1, p. 260], Ψ' has the explicit form

$$\Psi'(t) = \sum_{n=0}^{\infty}\frac{1}{(t+n)^2}, \quad t \in \mathbb{R}\setminus\{0,-1,-2,\ldots\}.$$

This implies that

$$\frac{\partial^2}{\partial p^2}(\log(\gamma_{k,p,a})) = \sum_{n=0}^{\infty}\frac{ak(ak+2p+3+2n)}{\left(p+\frac{3}{2}+n\right)^2\left(ak+p+\frac{3}{2}+n\right)^2} \geq 0$$

for all $k \in \{0,1,2,\ldots\}$ and $p > -3/2$. Thus $p \mapsto \gamma_{k,p,a}$ is log-convex on $(-3/2,\infty)$ for all $a \in \mathbb{N}$, and consequently ${}_a\mathcal{L}_p$ is log-convex for each fixed $x > 0$.

(c) Let $q \geq p > -3/2$. From part (a), it follows that

$$\left(\frac{2^p x^{-p}{}_a\mathtt{L}_p(x)}{2^q x^{-q}{}_a\mathtt{L}_q(x)}\right)' \geq 0$$

when $x > 0$, and this is equivalent to the inequality

$$\left(x^{-p}{}_a\mathtt{L}_p(x)\right)'\left(x^{-q}{}_a\mathtt{L}_q(x)\right) - \left(x^{-p}{}_a\mathtt{L}_p(x)\right)\left(x^{-q}{}_a\mathtt{L}_q(x)\right)' \geq 0. \tag{3.5}$$

For $c = -1$, the identity (2.4) reduces to

$$\left(x^{-p}{}_a\mathtt{L}_p(x)\right)' = \frac{1}{2^p\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+p\right)} + x^{1-a-p}\left(\tfrac{1}{2}\right)^{1-a}{}_a\mathtt{L}_{p+a}(x).$$

It now follows from (3.5) that

$$\begin{aligned}
&x^{-p-q}\left(\frac{x}{2}\right)^{1-a}\left({}_a\mathtt{L}_{p+a}(x)\ {}_a\mathtt{L}_q(x) - {}_a\mathtt{L}_p(x)\ {}_a\mathtt{L}_{q+a}(x)\right)\\
\geq\ & \frac{2^{-q}x^{-p}{}_a\mathtt{L}_p(x)}{\sqrt{\pi}\Gamma\left(\frac{3}{2}+q\right)} - \frac{2^{-p}x^{-q}{}_a\mathtt{L}_q(x)}{\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)}\\
=\ & \frac{2^{-p-q}}{\sqrt{\pi}\Gamma\left(\frac{3}{2}+p\right)\Gamma\left(\frac{3}{2}+q\right)}\left({}_a\mathcal{L}_p(x) - {}_a\mathcal{L}_q(x)\right).
\end{aligned}$$

In view of result (b), the final term above is non-negative, whence ${}_a\mathtt{L}_{p+a}/{}_a\mathtt{L}_p$ is decreasing for $p > -3/2$.

(d) Let $\beta_{k,p,a} := (2k+p+1)\alpha_{k,p,a}$. Then the quotient $x_a\mathtt{L}'_p/{}_a\mathtt{L}_p$ can be written as

$$\frac{x_a\mathtt{L}'_p(x)}{{}_a\mathtt{L}_p(x)} = \frac{\sum_{k=0}^{\infty}\beta_{k,p,a}\left(\frac{x}{2}\right)^{2k}}{\sum_{k=0}^{\infty}\alpha_{k,p,a}\left(\frac{x}{2}\right)^{2k}}.$$

Clearly, the sequence $\{\beta_{k,p,a}/\alpha_{k,p,a}\}_{k\geq 0} = \{2k+p+1\}_{k\geq 0}$ is increasing, and hence Lemma 3.1 shows that the function $x \mapsto x_a\mathtt{L}'_p/{}_a\mathtt{L}_p$ is increasing on $(0,\infty)$.

(e) From (2.15), the integral representation for ${}_a\mathtt{L}_p$ is

$$\begin{aligned}
{}_a\mathtt{L}_p(x) =\ & \frac{2\left(\frac{x}{2}\right)^p}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-\frac{a}{2}}}\\
&\times\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)}\int_0^1(1-t^2)^{\frac{2p+2j+1-4a}{2a}}\sinh\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt.
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
&\int_0^1(1-t^2)^{\frac{2p+2j+1-4a}{2a}}\sinh\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt\\
=\ & -\frac{a^{\frac{a}{2}}}{x} + \frac{a^{\frac{a}{2}}(2p+2j+1-4a)}{ax}\int_0^1(1-t^2)^{\frac{2p+2j+1-6a}{2a}}t\cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt,
\end{aligned}$$

which implies

$$\begin{aligned}
{}_a\mathtt{L}_p(x) =\ & -\frac{\left(\frac{x}{2}\right)^{p-1}}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+1-a}}\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)} + \frac{(x/2)^{p-1}}{\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+2-a}}\\
&\times\prod_{j=1}^{a}\frac{(2p+2j+1-4a)}{\Gamma\left(\frac{2p+2j+1-2a}{2a}\right)}\int_0^1(1-t^2)^{\frac{2p+2j+1-6a}{2a}}t\cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right)dt.
\end{aligned}$$

Replacing p by $p+a$, it follows that

$$a\prod_{j=1}^{a}\frac{1}{\Gamma\left(\frac{2p+2j+1}{2a}\right)} + \left(\frac{x}{2}\right)^{1-p-a}\sqrt{\pi}(2\pi)^{\frac{1-a}{2}}a^{p+2}{}_a\mathtt{L}_{p+a}$$

$$= \prod_{j=1}^{a} \frac{(2p+2j+1-2a)}{\Gamma\left(\frac{2p+2j+1}{2a}\right)} \int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right) dt.$$

To prove the assertion, we next show that

$$\int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right) dt \tag{3.6}$$

is log-convex. It is known that hyperbolic functions are log-convex. Thus for $\alpha \in [0,1]$ and $x, y > 0$,

$$\int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left((\alpha x + (1-\alpha)y)\frac{t}{a^{\frac{a}{2}}}\right) dt$$
$$\leq \int_0^1 \left((1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right)\right)^{\alpha} \left((1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{yt}{a^{\frac{a}{2}}}\right)\right)^{1-\alpha} dt.$$

Applying the well-known Hölder-Rogers inequality for integrals yields

$$\int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left((\alpha x + (1-\alpha)y)\frac{t}{a^{\frac{a}{2}}}\right) dt$$
$$\leq \left(\int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{xt}{a^{\frac{a}{2}}}\right) dt\right)^{\alpha} \left(\int_0^1 (1-t^2)^{\frac{2p+2j+1-4a}{2a}} t \cosh\left(\frac{yt}{a^{\frac{a}{2}}}\right) dt\right)^{1-\alpha}.$$

Thus the integral in (3.6) is log-convex, and consequently

$$x \mapsto a \prod_{j=1}^{a} \frac{1}{\Gamma\left(\frac{2p+2j+1}{2a}\right)} + \left(\frac{x}{2}\right)^{1-p-a} \sqrt{\pi}(2\pi)^{\frac{1-a}{2}} a^{p+2} {}_a\mathtt{L}_{p+a}(x)$$

is also log-convex. ☐

The results in Theorem 3.2 are also proved in [6] for the case $a = 1$.

Remark 3.1. Theorem 3.2 has interesting consequences, among which is the Turán-type inequality for the normalized Galué-type modified Struve function ${}_a\mathcal{L}_p$ given by (3.4). From the definition of log-convexity, it follows from Theorem 3.2(b) that

$${}_a\mathcal{L}_{\alpha p_1+(1-\alpha)p_2}(x) \leq \left({}_a\mathcal{L}_{p_1}(x)\right)^{\alpha} \left({}_a\mathcal{L}_{p_2}(x)\right)^{1-\alpha},$$

where $\alpha \in [0,1]$, $p_1, p_2 > -3/2$, and $x > 0$. Choosing $\alpha = 1/2$, $p_1 = p - a$ and $p_2 = p + a$, the above inequality yields

$${}_a\mathcal{L}_p^2(x) - {}_a\mathcal{L}_{p+a}(x) {}_a\mathcal{L}_{p-a}(x) \leq 0,$$

which is equivalent to

$$\Gamma^2\big(p+\tfrac{3}{2}\big) {}_a\mathtt{L}_p^2(x) - \Gamma\big(p+a+\tfrac{3}{2}\big)\Gamma\big(p-a+\tfrac{3}{2}\big) {}_a\mathtt{L}_{p+a}(x) {}_a\mathtt{L}_{p-a}(x) \leq 0.$$

This reduces to a Turán-type inequality for ${}_a\mathtt{L}_p$:

$${}_a\mathtt{L}_p^2(x) \leq \frac{\Gamma(p+a+\frac{3}{2})\Gamma(p-a+\frac{3}{2})}{\big(\Gamma(p+\frac{3}{2})\big)^2} {}_a\mathtt{L}_{p+a}(x) {}_a\mathtt{L}_{p-a}(x).$$

We turn now to another function of interest given by

$$\begin{aligned}\mathcal{T}_{a,p}(x) &:= \left(\tfrac{2}{x}\right)\, {}_a\mathcal{L}_p(x) = \left(\tfrac{2}{x}\right)^{p+1}\Gamma\big(p+\tfrac{3}{2}\big)\,{}_a\mathrm{L}_p(x)\\ &= \Gamma\big(p+\tfrac{3}{2}\big)\sum_{k=0}^{\infty}\frac{1}{\Gamma\big(k+\frac{3}{2}\big)\Gamma\big(ak+p+\frac{3}{2}\big)}\left(\frac{x}{2}\right)^{2k}. \qquad (3.7)\end{aligned}$$

Monotonicity properties involving the ratio of $\mathcal{T}_{a,p}$ is given in the following result, which lead to a Turán-type inequality for $\mathcal{T}_{a,p}$.

Theorem 3.3. *Let $a \in \mathbb{N}$ and $p > -3/2$.*

(a) *The function $p \mapsto \mathcal{T}_{a,p+1}(x)/\mathcal{T}_{a,p}(x)$ is increasing, that is, for $q \geq p > -3/2$, the inequality*

$$\mathcal{T}_{a,q+1}(x)\mathcal{T}_{a,p}(x) \geq \mathcal{T}_{a,q}(x)\mathcal{T}_{a,p+1}(x) \qquad (3.8)$$

holds for each fixed $x \in \mathbb{R}$.

(b) *If $q \geq p > -3/2$, then the function*

$$x \mapsto \big(q+\tfrac{3}{2}\big)_a \log(x\mathcal{T}_{a,q}(x)) - \big(p+\tfrac{3}{2}\big)_a \log(x\mathcal{T}_{a,p}(x))$$

is increasing on $(0,\infty)$.

Proof. (a) Let $\gamma_{a,p}$ be given by $\gamma_{a,p}(x^2) = \mathcal{T}_{a,p}(x)$. The first assertion is equivalent to showing

$$\gamma_{a,q+1}(x)\gamma_{a,p}(x) \geq \gamma_{a,q}(x)\gamma_{a,p+1}(x) \qquad (3.9)$$

for $x \geq 0$. Write

$$\gamma_{a,p}(x) := \sum_{n=0}^{\infty} b_n(p)x^n = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)^n\Gamma\big(p+\frac{3}{2}\big)}{\Gamma\big(an+p+\frac{3}{2}\big)\Gamma\big(n+\frac{3}{2}\big)}x^n.$$

Thus (3.9) is equivalent to showing

$$\left(\sum_{n=0}^{\infty} b_n(q+1)x^n\right)\left(\sum_{n=0}^{\infty} b_n(p)x^n\right) \geq \left(\sum_{n=0}^{\infty} b_n(q)x^n\right)\left(\sum_{n=0}^{\infty} b_n(p+1)x^n\right),$$

which holds provided

$$b_i(q+1)b_j(p) + b_j(q+1)b_i(p) \geq b_j(q)b_i(p+1) + b_i(q)b_j(p+1) \qquad (3.10)$$

for all $i, j \in \mathbb{N}$.

Let

$$\begin{aligned}\beta_1 &= \Gamma(ai+q+5/2)\Gamma(aj+p+5/2) \text{ and}\\ \beta_2 &= \Gamma(aj+q+5/2)\Gamma(ai+p+5/2).\end{aligned}$$

Then

$$\begin{aligned}& b_i(q+1)b_j(p) + b_j(q+1)b_i(p) \qquad (3.11)\\ &= \frac{\left(\frac{1}{4}\right)^{i+j}\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(q+\frac{5}{2}\right)}{\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}\left[\frac{aj+p+\frac{3}{2}}{\beta_1} + \frac{ai+q+\frac{3}{2}}{\beta_2}\right].\end{aligned}$$

Similarly,

$$
\begin{aligned}
(3.12) \qquad & b_j(q)b_i(p+1)+b_i(q)b_j(p+1) \\
&= \frac{\left(\frac{1}{4}\right)^{i+j}\Gamma\left(q+\frac{3}{2}\right)\Gamma\left(p+\frac{5}{2}\right)}{\Gamma\left(i+\frac{3}{2}\right)\Gamma\left(j+\frac{3}{2}\right)}\left[\frac{ai+q+\frac{3}{2}}{\beta_1}+\frac{aj+p+\frac{3}{2}}{\beta_2}\right].
\end{aligned}
$$

With $i \geq j$, the relations (3.11) and (3.12) show that inequality (3.10) is equivalent to

$$
\begin{aligned}
&\left(q+\tfrac{3}{2}\right)\left(\beta_2\left(aj+p+\tfrac{3}{2}\right)+\beta_1\left(ai+p+\tfrac{3}{2}\right)\right) \\
\geq\ &\left(p+\tfrac{3}{2}\right)\left(\beta_1\left(aj+q+\tfrac{3}{2}\right)+\beta_2\left(ai+q+\tfrac{3}{2}\right)\right).
\end{aligned}
$$

This can be further simplified to

$$
\beta_2 aj\left(q+\tfrac{3}{2}\right)+\beta_1 ai \geq \left(p+\tfrac{3}{2}\right)\beta_1 aj+\beta_2 ai.
$$

Since $q \geq p$, the latter inequality holds true when $(\beta_1-\beta_2)(i-j)\geq 0$, that is, provided $\beta_1 \geq \beta_2$.

Now let

$$
\phi_i := \frac{\Gamma\left(ai+q+\frac{5}{2}\right)}{\Gamma\left(ai+p+\frac{5}{2}\right)} = \frac{\Gamma\left(q+\frac{5}{2}\right)\left(q+\frac{5}{2}\right)_{ai}}{\Gamma\left(p+\frac{5}{2}\right)\left(p+\frac{5}{2}\right)_{ai}}.
$$

Then

$$
\begin{aligned}
\phi_{i+1}-\phi_i &= \frac{\Gamma\left(q+\frac{5}{2}\right)}{\Gamma\left(p+\frac{5}{2}\right)}\left(\frac{\left(q+\frac{5}{2}\right)_{ai+a}}{\left(p+\frac{5}{2}\right)_{ai+a}}-\frac{\left(q+\frac{5}{2}\right)_{ai}}{\left(p+\frac{5}{2}\right)_{ai}}\right) \\
&= \frac{\left(q+\frac{5}{2}\right)_{ai}\Gamma\left(q+\frac{5}{2}\right)}{\left(p+\frac{5}{2}\right)_{ai}\Gamma\left(p+\frac{5}{2}\right)}\left(\frac{(q+ai+\frac{5}{2})_a}{(p+ai+\frac{5}{2})_a}-1\right) \geq 0,
\end{aligned}
$$

that is, ϕ_i is increasing for $i \in \mathbb{N}$. Thus $\phi_i \geq \phi_j$ for $i \geq j$, and consequently $\beta_1 \geq \beta_2$. This validates inequality (3.9).

(b) We next show that for $q \geq p > -3/2$, the function $\psi : (0,\infty) \to \mathbb{R}$ given by

$$
(3.13) \qquad \psi(x) := \left(q+\tfrac{3}{2}\right)_a \log(x\mathcal{T}_{a,q}(x)) - \left(p+\tfrac{3}{2}\right)_a \log(x\mathcal{T}_{a,p}(x))
$$

is increasing on $(0,\infty)$. A computation yields

$$
x\mathcal{T}_{a,p}{}'(x)+\mathcal{T}_{a,p}(x) = \frac{x^2}{2\left(p+\frac{3}{2}\right)_a}\mathcal{T}_{a,p+a}(x)+\frac{2}{\sqrt{\pi}}.
$$

Thus

$$
\begin{aligned}
\psi'(x) &= \left(q+\tfrac{3}{2}\right)_a\left(\frac{x\mathcal{T}_{a,q}{}'(x)+\mathcal{T}_{a,q}(x)}{x\mathcal{T}_{a,q}(x)}\right)-\left(p+\tfrac{3}{2}\right)_a\left(\frac{x\mathcal{T}_{a,p}{}'(x)+\mathcal{T}_{a,p}(x)}{x\mathcal{T}_{a,p}(x)}\right) \\
&= \frac{x}{2}\left(\frac{\mathcal{T}_{a,q+a}(x)}{\mathcal{T}_{a,q}(x)}-\frac{\mathcal{T}_{a,p+a}(x)}{\mathcal{T}_{a,p}(x)}\right)+\frac{2\left(q+\frac{3}{2}\right)_a}{x\mathcal{T}_{a,p}(x)\sqrt{\pi}}\left(\frac{\mathcal{T}_{a,p}(x)}{\mathcal{T}_{a,q}(x)}-\frac{(p+\frac{3}{2})_a}{\left(q+\frac{3}{2}\right)_a}\right).
\end{aligned}
$$

Since $q \geq p$, it follows that $(q+3/2)_a \geq (p+3/2)_a$. Further the inequality (3.9) gives

$$\frac{\mathcal{T}_{a,q}(x)}{\mathcal{T}_{a,p}(x)} \leq \frac{\mathcal{T}_{a,q+1}(x)}{\mathcal{T}_{a,p+1}(x)} \leq \frac{\mathcal{T}_{a,q+2}(x)}{\mathcal{T}_{a,p+2}(x)} \leq \cdots \leq \frac{\mathcal{T}_{a,q+a}(x)}{\mathcal{T}_{a,p+a}(x)}.$$

Thus $\mathcal{T}_{a,q+a}(x)\mathcal{T}_{a,p}(x) \geq \mathcal{T}_{a,p+a}(x)\mathcal{T}_{a,q}(x)$.

From (3.7), write

$$\mathcal{T}_{a,p}(x) = \sum_{n=0}^{\infty} \alpha_n(p) x^n := \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)^n \Gamma(p+\frac{3}{2})}{\Gamma(an+p+\frac{3}{2})\Gamma(n+\frac{3}{2})} x^{2n},$$

and let

$$H(x) = \frac{\mathcal{T}_{a,p}(x)}{\mathcal{T}_{a,q}(x)} - \frac{(p+\frac{3}{2})_a}{\left(q+\frac{3}{2}\right)_a}.$$

A computation yields

$$\frac{\alpha_n(p)}{\alpha_n(q)} = \frac{\left(q+\frac{3}{2}\right)_{an}}{\left(p+\frac{3}{2}\right)_{an}}.$$

Since

$$\frac{\alpha_{n+1}(p)}{\alpha_{n+1}(q)} - \frac{\alpha_n(p)}{\alpha_n(q)} = \frac{\left(q+\frac{3}{2}\right)_{an}}{(p+\frac{3}{2})_{an}} \left(\frac{(q+an+\frac{3}{2})_a}{(p+an+\frac{3}{2})_a} - 1 \right) \geq 0$$

for $q \geq p$, the sequence $\{\alpha_n(p)/\alpha_n(q)\}$ is increasing. Lemma 3.1 shows that $\mathcal{T}_{a,p}(x)/\mathcal{T}_{a,q}(x)$ is increasing for $x > 0$. Evidently

$$\frac{\mathcal{T}_{a,p}(x)}{\mathcal{T}_{a,q}(x)} \to \frac{2/\sqrt{\pi}}{2/\sqrt{\pi}} = 1$$

as $x \to 0$, and thus $H(x) > 0$ for all $x > 0$. Consequently $\psi'(x) > 0$ for $x > 0$, and ψ is increasing on $(0, \infty)$. This completes the proof. $\square$

Remark 3.2. Inequality (3.8) leads to a generalization of the Turán-type inequality

$$\mathcal{T}^2_{a,p+1}(x) \leq \mathcal{T}_{a,p}(x)\ \mathcal{T}_{a,p+2}(x).$$

Example 3.1. This examples illustrates Theorem 3.3(a). The formulae from [19, p. 291] show that

$$_1\mathtt{L}_{-\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}}\sinh(x) \quad \text{and} \quad {}_1\mathtt{L}_{\frac{1}{2}}(x) = \sqrt{\tfrac{2}{\pi x}}(\cosh(x) - 1).$$

With $p = -1/2$, the increasing property of the function $p \mapsto \mathcal{T}_{a,p+1}(x)/\mathcal{T}_{a,p}(x)$ yields

$$\frac{\mathcal{T}_{1,p+1}(x)}{\mathcal{T}_{1,p}(x)} \geq \frac{\mathcal{T}_{1,\frac{1}{2}}(x)}{\mathcal{T}_{1,-\frac{1}{2}}(x)} \implies \frac{(2p+3)}{x}\frac{_1\mathtt{L}_{p+1}(x)}{_1\mathtt{L}_p(x)} \geq \tfrac{2}{x}(\coth(x) - \operatorname{csch}(x)).$$

Thus for $p \geq -1/2$ and $x \in (0, \infty)$,

$$\frac{_1\mathtt{L}_{p+1}(x)}{_1\mathtt{L}_p(x)} \geq \frac{2}{(2p+3)}(\coth(x) - \operatorname{csch}(x)).$$

Now Theorem 3.2(c) shows that $p \mapsto {}_1\mathtt{L}_{p+1}(x)/{}_1\mathtt{L}_p(x)$ is decreasing for each fixed $x > 0$. Thus

$$\frac{{}_1\mathtt{L}_{p+1}(x)}{{}_1\mathtt{L}_p(x)} \le \frac{{}_1\mathtt{L}_{\frac{1}{2}}(x)}{{}_1\mathtt{L}_{-\frac{1}{2}}(x)} = \coth(x) - \operatorname{csch}(x),$$

and whence, the ratio of the modified generalized functions satisfies

$$\frac{2}{(2p+3)}(\coth(x) - \operatorname{csch}(x)) \le \frac{{}_1\mathtt{L}_{p+1}(x)}{{}_1\mathtt{L}_p(x)} \le \coth(x) - \operatorname{csch}(x)$$

for $p \ge -1/2$ and $x \in (0, \infty)$.

Example 3.2. This second example illustrates Theorem 3.3(b). Choose $a = 1$, $p = -1/2$, and $q = 1/2$. Then the function

$$\psi(x) = 2\log(x\mathcal{T}_{1,1/2}(x)) - \log(x\mathcal{T}_{1,-1/2}(x))$$

is increasing on $(0, \infty)$. Let $x_0 \approx 0.8841$ be the non-negative root of

$$\sqrt{\pi}x^2 \sinh(x) = 8(\cosh(x) - 1)^2.$$

Thus $\psi(x_0) = 0$, and whence $\psi(x) \ge 0$ in $[x_0, \infty)$.

Consequently,

$$x\mathcal{T}_{1,-\frac{1}{2}}(x) \le \left(x\mathcal{T}_{1,\frac{1}{2}}(x)\right)^2,$$

that is,

$$\sqrt{2x}\,{}_1\mathtt{L}_{-\frac{1}{2}}(x) \le x^2 \left(2^{\frac{3}{2}} x^{-\frac{3}{2}}\,{}_1\mathtt{L}_{\frac{1}{2}}(x)\right)^2$$
$$\Longrightarrow \sqrt{2x}\sqrt{\frac{2}{\pi x}}\sinh(x) \le \frac{16x^{-1}}{\pi x}(\cosh(x) - 1)^2.$$

Thus

$$\sqrt{\pi}x^2 \sinh(x) \le 8(\cosh(x) - 1)^2, \quad x \in [x_0, \infty).$$

In closing, we deal with the special case $a = 2$. In this case, ${}_2\mathtt{S}_{p,c}$ is

$${}_2\mathtt{S}_{p,c}(x) := \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma\big(2k + p + \frac{3}{2}\big)\Gamma\big(k + \frac{3}{2}\big)} \left(\frac{x}{2}\right)^{2k+p+1}.$$

The dominant and subordinant functions will be obtained for $c = -1$, that is, for the function ${}_2\mathtt{L}_p := {}_2\mathtt{S}_{p,-1}$.

Theorem 3.4. *If $p \ge -1/2$ and $x \ge 0$, then*

$$\tag{3.14} {}_2\mathtt{L}_p(x) \le \left(\frac{x}{2}\right)^{p+1} \left(\frac{12p^2 + 48p + 29 + 16\cosh(\frac{x}{2})}{6\sqrt{\pi}\Gamma\left(p + \frac{7}{2}\right)}\right).$$

Proof. Clearly (3.14) trivially holds for $x = 0$. Let $x > 0$ and define $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) := \left(\frac{2}{x}\right)^{p+1} {}_2\mathtt{L}_p(x) - \frac{8\cosh(\frac{x}{2})}{3\sqrt{\pi}\Gamma\left(p + \frac{7}{2}\right)}.$$

It follows from (3.1) and the hyperbolic cosine series that

$$f(x)=\sum_{k=0}^{\infty}\left(\frac{1}{\Gamma\big(2k+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)}-\frac{8}{3\sqrt{\pi}\Gamma\left(p+\frac{7}{2}\right)\Gamma(2k+1)}\right)\left(\frac{x}{2}\right)^{2k}.$$

Thus

$$f'(x)=\sum_{k=1}^{\infty}\left(\frac{1}{\Gamma\big(2k+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)}-\frac{8}{3\sqrt{\pi}\Gamma\left(p+\frac{7}{2}\right)\Gamma(2k+1)}\right)2k\left(\frac{x}{2}\right)^{2k-1}. \tag{3.15}$$

We establish $f'(x)<0$ by showing each coefficient in the above series is negative.

Let $\zeta:[1,\infty)\to\mathbb{R}$ be

$$\zeta(t):=\frac{\Gamma(2t+1)}{\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(2t+p+\frac{3}{2}\right)}. \tag{3.16}$$

A logarithmic differentiation of (3.16) yields

$$\frac{\zeta'(t)}{\zeta(t)}=2\Psi(2t+1)-\Psi\left(t+\tfrac{3}{2}\right)-2\Psi\left(2t+p+\tfrac{3}{2}\right),$$

where Ψ is the digamma function. It is known that Ψ is increasing and

$$\Psi(y)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{y+n}\right),$$

where $\gamma\approx 0.5772$ is the Euler-Mascheroni constant. For $t\geq 1$ and $p\geq -1/2$, it follows that $\Psi(2t+1)\leq\Psi\left(2t+p+3/2\right)$ and

$$\Psi\left(t+\tfrac{3}{2}\right)\geq\Psi\left(\tfrac{5}{2}\right)=\left(\frac{3}{5}-\gamma\right)+\sum_{n=1}^{\infty}\frac{3}{(n+1)(2n+5)}>0.$$

Hence $\zeta'(t)\leq 0$, and $\zeta(t)\leq\zeta(1)$. In particular,

$$\frac{1}{\Gamma\big(2k+p+\frac{3}{2}\big)\Gamma\big(k+\frac{3}{2}\big)}\leq\frac{8}{3\sqrt{\pi}\Gamma\left(p+\frac{7}{2}\right)\Gamma(2k+1)},$$

which from (3.15) gives $f'(x)<0$ for all $x>0$. The result is now deduced from

$$f(x)\leq f(0)=\frac{1}{\Gamma\big(p+\frac{3}{2}\big)\Gamma\big(\frac{3}{2}\big)}-\frac{8}{3\sqrt{\pi}\Gamma\left(p+\frac{7}{2}\right)}=\frac{12p^2+48p+29}{6\sqrt{\pi}\Gamma\left(p+\frac{7}{2}\right)}.$$

□

Yet another dominant for ${}_2\mathrm{L}_p$ is given in the following result.

Theorem 3.5. *If $p\geq -1/2$ and $x\geq 0$, then*

$${}_2\mathrm{L}_p(x)\leq\frac{\sinh(x)}{\sqrt{\pi}\Gamma\left(p+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{p}.$$

Proof. The Legendre duplication formula (2.13) yields

$$\Gamma(2k+2) = \frac{2^{2k+1}}{\sqrt{\pi}}\Gamma(k+1)\Gamma\left(k+\tfrac{3}{2}\right).$$

Thus the hyperbolic sine series can be expressed in the form

$$\sinh(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(2k+2)} x^{2k+1} = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(k+1)\Gamma\left(k+\frac{3}{2}\right)} \left(\tfrac{x}{2}\right)^{2k+1}. \tag{3.17}$$

The proof follows along similar lines as in Theorem 3.4. Here

$$\zeta(t) := \frac{\Gamma(t+1)}{\Gamma\left(2t+p+\frac{3}{2}\right)}, \quad t \geq 1,$$

and

$$\frac{\zeta'(t)}{\zeta(t)} < -\Psi(t+1) \leq -\Psi(2) < 0.$$

We omit the remaining details. □

Neither dominant in Theorem 3.4 and the preceding theorem is smaller than the other; they depend on the choice of the parameter p as well as the range x.

To establish the final result, we use the Chebyshev integral inequality [18, p. 40], which states the following: suppose f and g are two integrable functions and monotonic in the same sense (either both decreasing or both increasing). Let $p : (a, b) \to \mathbb{R}$ be a positive integrable function. Then

$$\left(\int_a^b p(t)f(t)dt\right)\left(\int_a^b p(t)g(t)dt\right) \leq \left(\int_a^b p(t)dt\right)\left(\int_a^b p(t)f(t)g(t)dt\right). \tag{3.18}$$

The inequality in (3.18) is reversed if f and g are monotonic but in the opposite sense.

Theorem 3.6. *Let $x \in (0, 2\pi)$ and $p > 0$. Then*

$$\begin{aligned} {}_2\mathtt{H}_p(x) &\geq \frac{\pi x \left(\Gamma\left(\frac{p}{2}\right)\right)^2}{4\sqrt{2}\,\Gamma(\frac{2p-1}{4})\Gamma(\frac{2p+1}{4})}\ \mathtt{H}^2_{\frac{p-1}{2}}\left(\tfrac{x}{2}\right), \\ {}_2\mathtt{L}_p(x) &\geq \frac{\pi x \left(\Gamma\left(\frac{p}{2}\right)\right)^2}{4\sqrt{2}\,\Gamma(\frac{2p-1}{4})\Gamma(\frac{2p+1}{4})}\ \mathtt{L}^2_{\frac{p-1}{2}}\left(\tfrac{x}{2}\right). \end{aligned} \tag{3.19}$$

Proof. From (2.14) with $a = 2$, the integral form for ${}_2\mathtt{H}_p := {}_2\mathtt{S}_{p,1}$ is

$$\begin{aligned} {}_2\mathtt{H}_p(x) = \frac{x^p}{2^{2p-\frac{3}{2}}\Gamma\left(\frac{2p-1}{4}\right)\Gamma\left(\frac{2p+1}{4}\right)} &\left(\int_0^1 (1-t^2)^{\frac{2p-5}{4}} \sin\left(\tfrac{xt}{2}\right) dt\right) \\ \times &\left(\int_0^1 (1-t^2)^{\frac{2p-3}{4}} \sin\left(\tfrac{xt}{2}\right) dt\right). \end{aligned} \tag{3.20}$$

The integral form for ${}_2\mathtt{S}_{p,-1} = {}_2\mathtt{L}_p$ is obtained from (3.20) by replacing the sine function with hyperbolic sine.

To establish the subordinant for ${}_2\mathtt{H}_p$ in (3.19), let

$$p(t) = (1-t^2)^{\frac{2p-3}{4}} \sin\left(\tfrac{xt}{2}\right), \quad \text{and} \quad f(t) = g(t) := (1-t^2)^{-\frac{1}{4}}; \quad 0 < t < 1.$$

Then

$$\int_0^1 p(t)f(t)dt = \int_0^1 p(t)g(t)dt = \int_0^1 (1-t^2)^{\frac{2p-4}{4}} \sin\left(\tfrac{xt}{2}\right) dt.$$

It is known that for Re $p > -1/2$, the classical Struve function $\mathtt{H}_p$ has the integral representation

$$\mathtt{H}_p(y) = \frac{2^{1-p}y^p}{\sqrt{\pi}\,\Gamma\left(p+\frac{1}{2}\right)} \int_0^1 \left(1-t^2\right)^{p-\frac{1}{2}} \sin(yt)\, dt.$$

Replacing y by $x/2$ and p by $(p-1)/2$, it follows that

$$\left(\int_0^1 p(t)f(t)dt\right)\left(\int_0^1 p(t)g(t)dt\right) = 2^{2p-4}\pi x^{1-p} \left(\Gamma\left(\tfrac{p}{2}\right)\right)^2 \mathtt{H}^2_{\frac{p-1}{2}}\left(\tfrac{x}{2}\right).$$

Since f and g both are increasing on $(0,1)$, it is evident from (3.18) that

$$\begin{aligned} {}_2\mathtt{H}_p(x) &= \frac{x^p}{2^{2p-\frac{3}{2}}\Gamma\left(\frac{2p-1}{4}\right)\Gamma\left(\frac{2p+1}{4}\right)} \\ &\quad \times \left(\int_0^1 (1-t^2)^{\frac{2p-5}{4}} \sin\left(\tfrac{xt}{2}\right) dt\right)\left(\int_0^1 (1-t^2)^{\frac{2p-3}{4}} \sin\left(\tfrac{xt}{2}\right) dt\right) \\ &= \frac{x^p}{2^{2p-\frac{3}{2}}\Gamma\left(\frac{2p-1}{4}\right)\Gamma\left(\frac{2p+1}{4}\right)} \left(\int_0^1 p(t)f(t)g(t)dt\right)\left(\int_0^1 p(t)dt\right) \\ &\geq \frac{\pi x \left(\Gamma\left(\frac{p}{2}\right)\right)^2}{4\sqrt{2}\Gamma\left(\frac{2p-1}{4}\right)\Gamma\left(\frac{2p+1}{4}\right)} \mathtt{H}^2_{\frac{p-1}{2}}\left(\tfrac{x}{2}\right). \end{aligned}$$

The subordinant for ${}_2\mathtt{L}_p$ in (3.19) is similarly established by choosing

$$p(t) = (1-t^2)^{\frac{2p-3}{4}} \sinh\left(\tfrac{xt}{2}\right), \quad \text{and} \quad f(t) = g(t) := (1-t^2)^{-\frac{1}{4}}; \ 0 < t < 1. \ \square$$

Remark 3.3. Inequalities (3.19) can also be established using the generalized Schwarz inequality (see [17]):

$$\left(\int_a^b g(t)(f(t))^m dt\right)\left(\int_a^b g(t)(f(t))^n dt\right) \geq \left(\int_a^b g(t)(f(t))^{\frac{m+n}{2}} dt\right)^2, \tag{3.21}$$

where f and g are two nonnegative functions, and m and n are real numbers such that the integrals in (3.21) exist.

The subordinant (3.19) for ${}_2\mathtt{H}_p$ follows by choosing

$$g(t) = \sin\left(\tfrac{xt}{2}\right), \ f(t) = 1-t^2, \ n = \frac{2p-5}{4} \quad \text{and} \quad m = \frac{2p-3}{4}, \quad 0 < t < 1,$$

while for ${}_2\mathrm{L}_p$ is obtained by letting

$$g(t) = \sinh\left(\tfrac{xt}{2}\right),\ f(t) = 1 - t^2,\ n = \frac{2p-5}{4} \quad \text{and} \quad m = \frac{2p-3}{4}, \quad 0 < t < 1.$$

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Rosihan M. Ali
School of Mathematical Sciences
Universiti Sains Malaysia
11800 USM Penang, Malaysia
E-mail address: `rosihan@usm.my`

Saiful R. Mondal
Department of Mathematics and Statistics
Collage of Science
King Faisal University
Al-Hasa 31982, Saudi Arabia
E-mail address: `smondal@kfu.edu.sa`

Kottakkaran S. Nisar
Department of Mathematics
College of Arts and Science
Prince Sattam bin Abdulaziz University
Riyadh Region, Wadi Al Dawser, Saudi Arabia
E-mail address: `ksnisar1@gmail.com`