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MAPS PRESERVING SOME MULTIPLICATIVE STRUCTURES ON STANDARD JORDAN OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a unital real standard Jordan operator algebra acting on a Hilbert space H of dimension at least 2. We show that every bijection ϕ on $\mathcal{A}$ satisfying $\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B)$ is of the form $\phi = \varepsilon\psi$ where ψ is an automorphism on $\mathcal{A}$ and $\varepsilon \in \{-1, 1\}$. As a consequence if $\mathcal{A}$ is the real algebra of all self-adjoint operators on a Hilbert space H, then there exists a unitary or conjugate unitary operator U on H such that $\phi(A) = \varepsilon UAU^*$ for all $A \in \mathcal{A}$.

1. Introduction

The study of linear preserver problems on matrix algebras and operator algebras is a long lasting but still a very active research area in matrix algebras and operator algebras, for a review of this subject see [15]. In a purely algebraic point of view Martindale in [11] started to study multiplicative bijections on rings and proved that every multiplicative bijection from a prime ring containing a nontrivial idempotent onto an arbitrary ring is necessarily additive. This result shows that the multiplicative structure of a ring can determine its ring structure.

When we are dealing with an algebra it is also interesting to consider its Jordan structure. Following Martindale's achievement [11] a natural question arises. When a Jordan multiplicative map on an algebra is additive? We recall that if $\mathcal{A}$ is an associative algebra, then the Jordan product on $\mathcal{A}$ is defined by $A \circ B = \frac{1}{2}(AB + BA)$ for $A, B \in \mathcal{A}$. Although this product is not associative, it satisfies

$$A \circ B = B \circ A \quad \text{and} \quad A^2 \circ (A \circ B) = A \circ (A^2 \circ B)$$

for all $A, B \in \mathcal{A}$. This means that $\mathcal{A}$ with this product is a Jordan algebra, see [7] for more about these objects. Every linear subspace $\mathcal{B}$ of an associative algebra $\mathcal{A}$ which is closed under the Jordan product, is called a Jordan

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subalgebra of $\mathcal{A}$. Any Jordan algebra isomorphic to a Jordan subalgebra of an associative algebra is said to be a special Jordan algebra, see [7, Section 2.3]. The Jordan triple product of elements A, B in a Jordan algebra $\mathcal{A}$ is defined by

$$\{ABA\} = 2A \circ (A \circ B) - A^2 \circ B.$$

One may easily check that for any two elements A, B in a Jordan subalgebra of an associative algebra (i.e., in a special Jordan algebra) we have $\{ABA\} = ABA$.

From what we have already said, it follows that an associative algebra $\mathcal{A}$ carries three important Jordan multiplicative structures

$$(A, B) \mapsto \frac{AB + BA}{2}\ , \tag{1.1}$$

$$(A, B) \mapsto AB + BA\ , \tag{1.2}$$

and

$$(A, B) \mapsto ABA = 2A \circ (A \circ B) - A^2 \circ B, \tag{1.3}$$

from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A mapping $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan multiplicative map if for each $A, B \in \mathcal{A}$ it satisfies one of the following equations

$$\phi\left(\frac{AB + BA}{2}\right) = \frac{\phi(A)\phi(B) + \phi(B)\phi(A)}{2}; \tag{1.4}$$

$$\phi\,(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A); \tag{1.5}$$

$$\phi\,(ABA) = \phi(A)\phi(B)\phi(A). \tag{1.6}$$

It is easy to see that, if ϕ is additive, then (1.4) and (1.5) are equivalent and imply (1.6). Also if $\mathcal{A}, \mathcal{B}$ are unital and ϕ is additive and unital, then these three forms of Jordan multiplicativity are equivalent. But what happens if we drop the assumption of additivity? Molnár in [13] showed that each bijection ϕ between standard operator algebras $\mathcal{A}$ and $\mathcal{B}$ satisfying (1.6) is additive and also gave the general characterization of such mappings. Later Molnár in [14] and Lu in [10] studied those bijections between standard operator algebras which satisfy conditions (1.4) and (1.5), respectively. They proved that such mappings are necessarily additive. The additivity of Jordan †-skew multiplicative maps was proved in [1]. In [2], An and Hou considered bijective Jordan multiplicative maps on the algebra of all self-adjoint operators acting on a Hilbert space H, and also on the nest algebras over H. They proved that such mappings are necessarily additive and obtained their general form. Also Ji and Liu in [9] studied Jordan multiplicative maps on Jordan operator algebras and showed that every bijection, satisfying (1.4), from a standard Jordan operator algebra acting on a Hilbert space of dimension at least 2 onto an arbitrary Jordan algebra is additive. Some other results on additivity of Jordan multiplicative maps on operator algebras can be found in [5, 6, 15].

Throughout this paper H is a Hilbert space with $\dim(H) > 1$, $\mathfrak{B}(H)$ is the C^*-algebra of all bounded linear operators acting on H and $\mathfrak{B}_s(H)$ denotes the self-adjoint part of $\mathfrak{B}(H)$. One can easily observe that $\mathfrak{B}_s(H)$ is closed under the Jordan product which means that it is a special Jordan algebra over the field of real numbers. By $\mathfrak{F}_s(H)$ we denote the Jordan ideal of all self-adjoint finite rank operators in $\mathfrak{B}_s(H)$. We recall that each self-adjoint rank one operator on H is of the form $\alpha x \otimes x$ for some $0 \neq x \in H$ and some $0 \neq \alpha \in \mathbb{R}$ and rank one projections are exactly of the form $x \otimes x$ for some unit vector $x \in H$. Moreover, each self-adjoint finite rank operator is a real linear combination of pairwise orthogonal rank one projections.

By a real Jordan operator algebra acting on a Hilbert space H, we mean a Jordan subalgebra of $\mathfrak{B}_s(H)$ and if it is norm closed, then it is called a JC-algebra, we refer the reader to [7] for more details. A real Jordan operator algebra is said to be standard if it contains $\mathfrak{F}_s(H)$. Note that if a real standard Jordan operator algebra is unital, then the unit element is necessarily the identity operator I on H because it belongs to the commutant of $\mathfrak{F}_s(H)$ in $\mathfrak{B}(H)$.

Being interested in different Jordan multiplicative structures we are going to examine the product $A^2 \circ B$ in a real standard Jordan operator algebra. This product was used in [3] to define orthogonality in a JB-algebra, see [7] for more about JB-algebras. Let $\mathcal{A}$ be a unital real standard Jordan operator algebra acting on a Hilbert space H of dimension at least 2. We will show that if $\phi : \mathcal{A} \to \mathcal{A}$ is a bijection satisfying $\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B)$ for all $A, B \in \mathcal{A}$, then $\phi = \varepsilon\psi$ where $\varepsilon \in \{-1, 1\}$ and ψ is an automorphism; that is ψ is a linear bijection satisfying $\psi(A \circ B) = \psi(A) \circ \psi(B)$ for all $A, B \in \mathcal{A}$. As a consequence, if $\mathcal{A} = \mathfrak{B}_s(H)$, then we will show that there exists a unitary or conjugate unitary operator U on H such that $\phi(A) = \varepsilon UAU^*$ for all $A \in \mathfrak{B}_s(H)$, where $\varepsilon \in \{-1, 1\}$.

If $\dim(H) = 1$, then any real standard operator algebra acting on H is trivially identified by $\mathbb{R}$. Molnár in [14] gives an example of a bijective multiplicative function $h : \mathbb{C} \to \mathbb{C}$ which is not additive. It is easy to see that the restriction of this function to $\mathbb{R}$ is a multiplicative bijection which is not additive. Thus the condition $\dim(H) > 1$ can not be dropped from our assumption.

2. Characterizing maps preserving $A^2 \circ B$

Let $\mathcal{A}, \mathcal{B}$ be special Jordan algebras. We recall that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism, that is $\phi(A \circ B) = \phi(A) \circ \phi(B)$ for all $A, B \in \mathcal{A}$, if and only if $\phi(A^2) = \phi(A)^2$ for all $A \in \mathcal{A}$.

Theorem 2.1. *Let $\mathcal{A}$ be a unital real standard Jordan operator algebra acting on a Hilbert space H of dimension > 1 and let $\phi : \mathcal{A} \to \mathcal{A}$ be a bijection satisfying*

$$\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B) \quad (A, B \in \mathcal{A}). \tag{2.1}$$

Then $\phi = \varepsilon\psi$ *where* ψ *is an automorphism on* $\mathcal{A}$ *and* $\varepsilon \in \{-1, 1\}$.

Proof. First of all we note that ϕ^{-1} also satisfies (2.1), so all the results which are proved in the sequel may apply to ϕ^{-1}, as well. Now we proceed in some steps.

Step 1. $\phi(0) = 0$.
Since ϕ is surjective, there exists $A \in \mathcal{A}$ such that $\phi(A) = 0$. Thus we have $\phi(0) = \phi\big(\frac{1}{2}(A^2 0 + 0A^2)\big) = \frac{1}{2}\big(\phi(A)^2\phi(0) + \phi(0)\phi(A)^2\big) = 0$.

Step 2. $\phi(I) = \varepsilon I$ where $\varepsilon \in \{-1, 1\}$.
There exists $B \in \mathcal{A}$ such that $\phi(B) = I$. We have

$$\phi(B) = \phi\left(\frac{I^2B + BI^2}{2}\right) = \frac{\phi(I)^2\phi(B) + \phi(B)\phi(I)^2}{2} = \phi(I)^2.$$

Thus $\phi(I)^2 = I$. Also, for each $A \in \mathcal{A}$ we have

$$\phi(A^2) = \phi(A^2 \circ I) = \frac{\phi(A)^2\phi(I) + \phi(I)\phi(A)^2}{2}.$$

Therefore

$$\phi(A^2)\phi(I) = \phi(I)\phi(A^2) \quad (A \in \mathcal{A}). \tag{2.2}$$

Now for each $A \in \mathcal{A}$

$$\begin{aligned}
\phi(A^4) &= \frac{1}{2}\left(\phi(A)^2\phi(A^2) + \phi(A^2)\phi(A)^2\right)\\
&= \frac{1}{2}\left(\phi(A)^2\Big(\frac{\phi(A)^2\phi(I) + \phi(I)\phi(A)^2}{2}\Big) + \Big(\frac{\phi(A)^2\phi(I) + \phi(I)\phi(A)^2}{2}\Big)\phi(A)^2\right)\\
&= \frac{1}{4}\phi(A)^4\phi(I) + \frac{1}{2}\phi(A)^2\phi(I)\phi(A)^2 + \frac{1}{4}\phi(I)\phi(A)^4.
\end{aligned}$$

Since $\phi(I)^2 = I$ we get

$$\phi(I)\phi(A^4) = \frac{1}{4}\phi(I)\phi(A)^4\phi(I) + \frac{1}{2}\phi(I)\phi(A)^2\phi(I)\phi(A)^2 + \frac{1}{4}\phi(A)^4,$$

and

$$\phi(A^4)\phi(I) = \frac{1}{4}\phi(A)^4 + \frac{1}{2}\phi(A)^2\phi(I)\phi(A)^2\phi(I) + \frac{1}{4}\phi(I)\phi(A)^4\phi(I).$$

It follows from (2.2) that

$$\phi(A)^2\phi(I)\phi(A)^2\phi(I) = \phi(I)\phi(A)^2\phi(I)\phi(A)^2 \quad (A \in \mathcal{A}).$$

In particular, since ϕ is surjective, for every projection $P \in \mathcal{A}$ we have $P\phi(I)P\phi(I) = \phi(I)P\phi(I)P$. Let x be an arbitrary unit vector in H. Then for each $y \in H$

$$(x \otimes x)\phi(I)(x \otimes x)\phi(I)(y) = \phi(I)(x \otimes x)\phi(I)(x \otimes x)(y)$$

and hence

$$\langle \phi(I)y, x\rangle\langle \phi(I)x, x\rangle x = \langle y, x\rangle\langle \phi(I)x, x\rangle\phi(I)(x) \quad (y \in H).$$

By taking $y = x$ we arrive at

$$\langle \phi(I)x, x\rangle^2 x = \langle \phi(I)x, x\rangle \phi(I)(x) \quad (x \in H). \tag{2.3}$$

Take $\lambda_x = \langle \phi(I)x, x\rangle$ then by (2.3), $\lambda_x^2 x = \lambda_x \phi(I)(x)$ and since $\phi(I)^2 = I$, $\lambda_x^2 \phi(I)x = \lambda_x x$. Therefore

$$\lambda_x = \lambda_x \langle x, x\rangle = \langle \lambda_x x, x\rangle = \langle \lambda_x^2 \phi(I)x, x\rangle = \lambda_x^2 \langle \phi(I)x, x\rangle = \lambda_x^3.$$

Hence $\lambda_x \in \{-1, 0, 1\}$. Let $W(\cdot)$ denote the numerical range. Since the unit vector $x \in H$ was chosen arbitrarily

$$W(\phi(I)) = \{\langle \phi(I)x, x\rangle; \| x \| = 1\} = \{\lambda_x; \| x \| = 1\} \subseteq \{-1, 0, 1\}.$$

We know that the numerical range of an operator is a convex set so, $W(\phi(I)) = \{1\}$ or $\{-1\}$ or $\{0\}$. But if $W(\phi(I)) = \{0\}$, then $\phi(I) = 0$, a contradiction. Thus $\phi(I) = I$ or $\phi(I) = -I$.

Step 3. *We may assume that $\phi(I) = I$. In this case $\phi(-I) = -I$.*
If $\phi(I) = -I$, then $\psi = -\phi$ also satisfies (2.1) and $\psi(I) = I$. So without loss of generality, we assume that $\phi(I) = I$.

Now suppose that $B \in \mathcal{A}$ satisfies $\phi(B) = -I$. Then

$$\phi(B) = \phi\big(B \circ (-I)^2\big) = \phi(B) \circ \phi(-I)^2 = -\phi(-I)^2.$$

Thus $\phi(-I)^2 = I$. For each $A \in \mathcal{A}$ we have

$$\phi(-A^2) = \frac{\phi(-I)\phi(A)^2 + \phi(A)^2\phi(-I)}{2} \ . \tag{2.4}$$

Multiplying (2.4) by $\phi(-I)$ from right and left, respectively, we get

$$\phi(-A^2)\phi(-I) = \phi(-I)\phi(-A^2) \quad (A \in \mathcal{A}) \ . \tag{2.5}$$

Now a similar argument as in Step 2 shows that

$$\phi(-A^4) = \frac{\phi(-I)\phi(A)^4 + 2\phi(A)^2\phi(-I)\phi(A)^2 + \phi(A)^4\phi(-I)}{4} \ .$$

Again by multiplying the last equation by $\phi(-I)$ from right and left, respectively, and using (2.5) we arrive at

$$\phi(A)^2\phi(-I)\phi(A)^2\phi(-I) = \phi(-I)\phi(A)^2\phi(-I)\phi(A)^2 \quad (A \in \mathcal{A}) \ .$$

The same reasoning as in Step 2 and using the fact that ϕ is injective implies that $\phi(-I) = -I$.

Step 4. *For each $A \in \mathcal{A}$, $\phi(A^2) = \phi(A)^2$ and therefore ϕ preserves projections and positive elements in both directions.*
It is clear from Step 3.

Step 5. *For every positive finite rank $A \in \mathcal{A}$ we have*

$$\phi(A \circ B) = \phi(A) \circ \phi(B) \quad (B \in \mathcal{A}).$$

In particular, $\phi(-A) = -\phi(A)$.

Let A be a positive finite rank operator in $\mathcal{A}$. Therefore, the positive square root C of A is also of finite rank and hence belongs to $\mathcal{A}$. By Step 4

$$\begin{aligned}\phi(A \circ B) &= \phi(C^2 \circ B)\\ &= \phi(C)^2 \circ \phi(B)\\ &= \phi(C^2) \circ \phi(B)\\ &= \phi(A) \circ \phi(B).\end{aligned}$$

The last assertion is now directly obtained by taking $B = -I$.

Step 6. *ϕ preserves orthogonality of projections in both directions.* Suppose P, Q are projections in $\mathcal{A}$ satisfying $PQ = QP = 0$. We have

$$0 = \phi(PQ) = \phi(P \circ Q) = \frac{\phi(P)\phi(Q) + \phi(Q)\phi(P)}{2}.$$

By Step 4, $\phi(P)$ and $\phi(Q)$ are projections. Now, multiplying the above equality by $\phi(Q)$ form left and right, respectively, implies that $\phi(Q)\phi(P) = \phi(P)\phi(Q) = 0$.

Step 7. *ϕ preserves the order of projections in both directions.* If $P, Q \in \mathcal{A}$ are projections and $P \leqslant Q$, then $PQ = QP = P$ and by Step 4

$$\phi(P) = \phi\big(\frac{1}{2}(PQ + QP)\big) = \frac{1}{2}\big(\phi(P)\phi(Q) + \phi(Q)\phi(P)\big).$$

Multiplying this equality by $\phi(Q)$ from right and left, respectively, shows that

$$\phi(P)\phi(Q) = \phi(Q)\phi(P).$$

Therefore, $\phi(P)\phi(Q) = \phi(Q)\phi(P) = \phi(P)$ and the result follows.

Step 8. *ϕ preserves the rank of finite rank projections in both directions and it is orthogonally additive on finite rank projections. Moreover, $\phi(\mathfrak{F}_s(H)) \subseteq \mathfrak{F}_s(H)$.* For the first two assertions, the same argument as in Step 4 and Step 5 of Theorem 2.1 in [2] gives the result. If $A \in \mathfrak{F}_s(H)$, then there exists a finite rank projection P such that $PA = AP = A$. By Step 5 we have

$$\phi(A) = \phi\big(\frac{1}{2}(AP + PA)\big) = \frac{1}{2}\big(\phi(A)\phi(P) + \phi(P)\phi(A)\big),$$

which implies that $\phi(A)$ is also of finite rank.

Step 9. *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and let $P_1, \ldots, P_n$ be pairwise orthogonal finite rank projections. Then*

$$\phi(\sum_{i=1}^{n} \lambda_i P_i) = \sum_{i=1}^{n} \phi(\lambda_i P_i).$$

This equality follows by the same argument as in Step 6 in [2, Theorem 2.2] as well as a part of the proof of Theorem 1 in [14].

Step 10. *For each rank one projection P and each $\lambda \in \mathbb{R}$ there exists $\mu \in \mathbb{R}$ such that $\lambda\phi(P) = \phi(\mu P)$.*

Suppose that P is a rank one projection, $\lambda \in \mathbb{R}$, and $A \in \mathcal{A}$ satisfies $\phi(A) = \lambda\phi(P)$. By Step 8, $\mathrm{rank}(\phi(A)) = 1$. Thus there exists a nonzero vector $z \in H$ and $0 \neq \alpha \in \mathbb{R}$ such that $A = \alpha z \otimes z$. We have $\phi(P)\phi(A) = \phi(A)\phi(P) = \lambda\phi(P)$ and so $\phi(A \circ P) = \lambda\phi(P) = \phi(A)$. Since ϕ is injective $A \circ P = A$ and it follows that

$$PA = AP = A \circ P = A. \tag{2.6}$$

Let x be the unit vector in H such that $P = x \otimes x$. By (2.6) $(x \otimes x)(z \otimes z) = z \otimes z$, hence for each $y \in H$

$$\langle y, z\rangle\langle z, x\rangle x = \langle y, z\rangle z$$

which implies that x and z are linearly dependent. So there exists $\mu \in \mathbb{R}$ such that $A = \mu P$.

Step 11. *$\phi(PTP) = \phi(P)\phi(T)\phi(P)$ for all positive T in $\mathfrak{F}_s(H)$ and all projections P in $\mathfrak{F}_s(H)$.*

Let $T \in \mathfrak{F}_s(H)$ be a positive operator and let $P \in \mathfrak{F}_s(H)$ be a projection. Choose a finite rank projection $Q \in \mathfrak{F}_s(H)$ such that $QP = PQ = P$ and $TQ = QT = T$. We have

$$P \circ \big((2P - Q) \circ T\big) = PTP.$$

Thus by Step 5 we get

$$\phi(P) \circ \big(\phi(2P - Q) \circ \phi(T)\big) = \phi(PTP).$$

We show that $\phi(2P-Q) = 2\phi(P)-\phi(Q)$. Since $Q-P$ is a projection orthogonal to P and $P \leq Q$, by Step 9, Step 5 and Step 8

$$\begin{aligned}
\phi(2P - Q) &= \phi\big(P - (Q - P)\big)\\
&= \phi(P) + \phi\big(-(Q - P)\big)\\
&= \phi(P) - \phi(Q - P)\\
&= \phi(P) - \big(\phi(Q) - \phi(P)\big)\\
&= 2\phi(P) - \phi(Q).
\end{aligned}$$

So, we have $\phi(PTP) = \phi(P)\phi(T)\phi(P)$.

Step 12. *$\phi(\lambda I) = \lambda\phi(I)$ for all $\lambda \in \mathbb{R}$.*

Fix an arbitrary real number λ, and let $B_\lambda \in \mathcal{A}$ satisfy $B_\lambda = \phi(\lambda I)$. We show that $B_\lambda = \lambda I$. Let x be a unit vector in H and $P = x \otimes x$. Since ϕ^{-1} satisfies (2.1) it also satisfies all the steps proved to this stage. By Step 5 and Step 10 we have $\phi^{-1}(B_\lambda \circ P) = \phi^{-1}(B_\lambda) \circ \phi^{-1}(P) = \lambda\phi^{-1}(P) = \phi^{-1}(\mu_{x,\lambda} P)$ for some $\mu_{x,\lambda} \in \mathbb{R}$. Since ϕ^{-1} is injective, $P \circ B_\lambda = \mu_{x,\lambda} P$. It is easily seen that $B_\lambda P = PB_\lambda = P \circ B_\lambda = \mu_{x,\lambda} P$, and so

$$B_\lambda(x \otimes x(y)) = \mu_{x,\lambda}\, x \otimes x\,(y) \quad (y \in H).$$

Take $y = x$, we get $B_\lambda x = \mu_{x,\lambda} x$. We have shown that, for every $\lambda \in \mathbb{R}$ and every unit vector $x \in H$, there exists $\mu_{x,\lambda} \in \mathbb{R}$ such that $\phi(\lambda I)(x) = \mu_{x,\lambda} x$.

First of all, we show that $\mu_{x,\lambda}$ does not depend on x. Let x, y be unit vectors in H with $x \neq y$, $\alpha = \|x - y\|$ and $z = \alpha^{-1}(x - y)$. Then z is a unit vector and

$$B_\lambda(x) - B_\lambda(y) = \alpha B_\lambda(z) = \alpha \mu_{z,\lambda} z = \mu_{z,\lambda}(x - y),$$

hence

$$(\mu_{x,\lambda} - \mu_{z,\lambda})x = (\mu_{y,\lambda} - \mu_{z,\lambda})y. \tag{2.7}$$

Now, if x and y are linearly independent, then by (2.7), $\mu_{x,\lambda} = \mu_{z,\lambda} = \mu_{y,\lambda}$; and if $y = \beta x$ for some scalar β, then $\mu_{y,\lambda} y = B_\lambda(y) = \beta B_\lambda(x) = \beta \mu_{x,\lambda} x = \mu_{x,\lambda} y$ and hence $\mu_{x,\lambda} = \mu_{y,\lambda}$. This means that $\mu_{x,\lambda}$ does not depend on x. Since λ was arbitrarily chosen, it follows that there exists a function $h : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(\lambda I) = h(\lambda) I \quad (\lambda \in \mathbb{R}).$$

By Step 3, $h(1) = 1$ and $h(-1) = -1$. Also from Step 5 for every finite rank projection P we have $\phi(\lambda P) = \phi(\lambda I \circ P) = h(\lambda)\phi(P)$ for all $\lambda \in \mathbb{R}$. If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \geq 0$, then again by Step 5 for each finite rank projection P

$$h(\lambda_1\lambda_2)\phi(P) = \phi(\lambda_1\lambda_2 P) = \phi(\lambda_2 I) \circ \phi(\lambda_1 P) = h(\lambda_1)h(\lambda_2)\phi(P).$$

Hence

$$h(\lambda_1\lambda_2) = h(\lambda_1)h(\lambda_2) \quad (\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \geq 0). \tag{2.8}$$

Now suppose that $\lambda_1, \lambda_2 \leq 0$. Since $h(-1) = -1$ by (2.8) for each projection $P \in \mathfrak{F}_s(H)$

$$\begin{aligned} h(\lambda_1\lambda_2)\phi(P) = \phi(\lambda_1\lambda_2 P) &= \phi(|\lambda_1|\,|\lambda_2| P) \\ &= h(|\lambda_1|)h(|\lambda_2|)\phi(P) \\ &= \big(-h(|\lambda_1|)\big)\big(-h(|\lambda_2|)\big)\phi(P) \\ &= h(\lambda_1)h(\lambda_2)\phi(P). \end{aligned}$$

Therefore, $h(\lambda_1\lambda_2) = h(\lambda_1)h(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$. Now, the same reasoning as in Step 8 of [2, Theorem 2.2] shows that $h(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$ and it follows that $\phi(\lambda I) = \lambda I$ for all $\lambda \in \mathbb{R}$.

Step 13. *$\phi(\lambda A) = \lambda\phi(A)$ for every $\lambda \in \mathbb{R}$ and $A \in \mathfrak{F}_s(H)$.*
By Step 12, for all rank one projections $P \in \mathfrak{F}_s(H)$ and all $\lambda \in \mathbb{R}$, $\phi(\lambda P) = \lambda\phi(P)$ and the result follows by Step 9.

Step 14. *ϕ is a Jordan homomorphism on $\mathfrak{F}_s(H)$.*
Using Step 4 and Step 13, it is enough to show that ϕ is additive on $\mathfrak{F}_s(H)$. Let A, B be positive finite rank operators and let $P = x \otimes x$ be a rank one projection. It follows from Step 11 that

$$\begin{aligned} \phi(P)\phi(A + B)\phi(P) &= \langle (A + B)x, x\rangle \phi(P) \\ &= \langle Ax, x\rangle \phi(P) + \langle Bx, x\rangle \phi(P) \\ &= \phi(PAP) + \phi(PBP) \\ &= \phi(P)\phi(A)\phi(P) + \phi(P)\phi(B)\phi(P) \end{aligned}$$

$$= \phi(P)\big(\phi(A) + \phi(B)\big)\phi(P).$$

Therefore $\phi(P)\phi(A + B)\phi(P) = \phi(P)(\phi(A) + \phi(B))\phi(P)$ for each rank one projection P. Thus

$$\phi(A + B) = \phi(A) + \phi(B) \quad (A, B \in \mathfrak{F}_s(H),\ A, B \geq 0). \tag{2.9}$$

Next we show that $\phi(A - B) = \phi(A) - \phi(B)$, if $A, B \in \mathfrak{F}_s(H)$ are positive operators. First let $A \geqslant B \geqslant 0$. Since $A - B \geqslant 0$, by Step 11 for every rank one projection P

$$\phi(P)\phi(A - B)\phi(P) = \phi\big(P(A - B)P\big).$$

A similar argument as for $\phi(A + B)$ shows that

$$\phi(P)\phi(A - B)\phi(P) = \phi(P)\big(\phi(A) - \phi(B)\big)\phi(P)$$

for each rank one projection P. Hence

$$\phi(A - B) = \phi(A) - \phi(B) \quad \big(A,\ B \in \mathfrak{F}_s(H),\ A \geq B \geq 0\big). \tag{2.10}$$

Now, let A and B be arbitrary positive operators in $\mathfrak{F}_s(H)$. There exist orthogonal rank one projections $P_1, \ldots, P_n$ in $\mathfrak{F}_s(H)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $A - B = \sum_{i=1}^n \lambda_i P_i$. Let $E \subseteq \{1, \ldots, n\}$ be the set of all indices i for which $\lambda_i \geq 0$ and $S = \{1, \ldots, n\} \setminus E$. Then

$$A - B = \sum_{i\in E} \lambda_i P_i + \sum_{i\in S} \lambda_i P_i.$$

We have

$$-\sum_{i\in S} \lambda_i P_i \geq 0, \quad A - \sum_{i\in S} \lambda_i P_i \geq B \geq 0.$$

Thus by (2.10) and (2.9)

$$\phi\big(\sum_{i\in E} \lambda_i P_i\big) = \phi\big(A - \sum_{i\in S} \lambda_i P_i)\big) - \phi(B) = \phi(A) + \phi\big(-\sum_{i\in S} \lambda_i P_i\big) - \phi(B). \tag{2.11}$$

On the other hand by Step 9 and Step 5 and (2.11)

$$\begin{aligned}
\phi(A - B) &= \sum_{i=1}^n \phi(\lambda_i P_i) \\
&= \sum_{i\in E} \phi(\lambda_i P_i) + \sum_{i\in S} \phi(\lambda_i P_i) \\
&= \phi\big(\sum_{i\in E} \lambda_i P_i\big) - \phi\big(-\sum_{i\in S} \lambda_i P_i\big) \\
&= \phi(A) - \phi(B).
\end{aligned}$$

Thus

$$\phi(A - B) = \phi(A) - \phi(B) \quad (A, B \in \mathfrak{F}_s(H),\ A, B \geq 0). \tag{2.12}$$

Finally, let $A, B \in \mathfrak{F}_s(H)$ be arbitrary elements. Then $A = A_1 - A_2$ and $B = B_1 - B_2$ where A_1, A_2, B_1, B_2 are positive finite rank operators. Hence by (2.12) and (2.9)

$$\begin{aligned}\phi(A+B) &= \phi\big(A_1 + B_1 - (A_2 + B_2)\big)\\ &= \phi(A_1 + B_1) - \phi(A_2 + B_2)\\ &= \phi(A_1) - \phi(A_2) + \phi(B_1) - \phi(B_2)\\ &= \phi(A) + \phi(B).\end{aligned}$$

Step 15. *ϕ is a Jordan automorphism on $\mathcal{A}$.*
First we show that $\phi(\lambda A) = \lambda\phi(A)$ for all $A \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$. Let P be an arbitrary rank one projection, $A \in \mathcal{A}$ and $\lambda \in \mathbb{R}$. Then by Step 5 and Step 13 we have

$$\phi(\lambda A) \circ \phi(P) = \phi(\lambda A \circ P) = \lambda\phi(A \circ P) = \lambda\phi(A) \circ \phi(P).$$

Thus $\phi(\lambda A) = \lambda\phi(A)$. Also from Step 5 and Step 14, for all $A, B \in \mathcal{A}$ and all finite rank projections P,

$$\begin{aligned}\phi(A+B) \circ \phi(P) &= \phi((A+B) \circ P)\\ &= \phi(A \circ P + B \circ P)\\ &= \phi(A \circ P) + \phi(B \circ P)\\ &= \phi(A) \circ \phi(P) + \phi(B) \circ \phi(P)\\ &= \big(\phi(A) + \phi(B)\big) \circ \phi(P).\end{aligned}$$

Therefore $\phi(A+B) = \phi(A) + \phi(B)$, for all $A, B \in \mathcal{A}$ and so ϕ is additive on $\mathcal{A}$. It follows that ϕ is linear and since $\phi(A^2) = \phi(A)^2$ for all $A \in \mathcal{A}$, ϕ is an automorphism on the special Jordan algebra $\mathcal{A}$.

Finally, if in Step 3 we assume that $\phi(I) = -I$, then $-\phi$ is an automorphism. So in general, $\phi = \varepsilon\psi$ where ψ is an automorphism and $\varepsilon \in \{-1, 1\}$. □

Corollary 2.2. *Let H be a Hilbert space with $\dim H > 1$ and let $\mathcal{A}$ be a unital standard JC-subalgebra of $\mathfrak{B}_s(H)$ and let ϕ be a bijection on $\mathcal{A}$ satisfying*

$$\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B) \quad (A, B \in \mathcal{A}).$$

Then $\phi = \varepsilon\psi$ where ψ is an automorphism and $\varepsilon \in \{-1, 1\}$. Moreover, ϕ is an isometry.

Proof. It is a well known result that any isomorphism between JC-algebras is an isometry. □

We recall from [4] that two self-adjoint operators A, B acting on a Hilbert space H are said to be adjacent if $A - B$ is a rank one operator. A map $\phi : \mathfrak{B}_s(H) \to \mathfrak{B}_s(H)$ preserves adjacency if for each $A, B \in \mathfrak{B}_s(H)$, $\phi(A)$ is adjacent to $\phi(B)$ whenever A, B are so.

Corollary 2.3. *Let H be a Hilbert space with $\dim H > 1$ and let $\phi : \mathfrak{B}_s(H) \to \mathfrak{B}_s(H)$ be a bijection. Then the following statements are equivalent.*

(i) $\phi(A^2 \circ B) = \phi(A)^2 \circ \phi(B)$ *for all* $A, B \in \mathcal{B}_s(H)$.

(ii) *There exists a unitary or conjugate unitary operator* U *on* H *such that* $\phi(A) = \varepsilon UAU^*$ *for all* $A \in \mathfrak{B}_s(H)$, *where* $\varepsilon \in \{-1, 1\}$.

Proof. If ϕ satisfies (i), then by Theorem 2.1 and Corollary 2.2, ϕ clearly preserves adjacency in both directions and is continuous. Therefore, by Theorem 1.3 and Corollary 1.1 of [4] or Step 11 of Theorem 2.2 in [2] there exists a unitary or conjugate unitary operator $U : H \to H$, such that $\phi(A) = \varepsilon UAU^*$ for all $A \in \mathfrak{B}_s(H)$. The reverse conclusion is trivial □

Remark 2.4. There is also an alternative argument for the proof of Corollary 2.3. Consider the automorphism ψ in Corollary 2.2 then define $\Psi : \mathfrak{B}(H) \to \mathfrak{B}(H)$ by $\Psi(A + iB) = \psi(A) + i\psi(B)$ for $A, B \in \mathfrak{B}_{sa}(H)$. Then Ψ is a Jordan $*$-automorphism on $\mathfrak{B}(H)$. It is a classical well known result that Ψ must be a $*$-automorphism or a $*$-antiautomorphism (both with respect to the associative structure of $\mathfrak{B}(H)$) and now the result follows from [15, Theorem A.8].

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