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SURFACE BUNDLES OVER SURFACES WITH A FIXED SIGNATURE

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Abstract. The signature of a surface bundle over a surface is known to be divisible by 4. It is also known that the signature vanishes if the fiber genus ≤ 2 or the base genus ≤ 1. In this article, we construct new smooth 4-manifolds with signature 4 which are surface bundles over surfaces with small fiber and base genera. From these we derive improved upper bounds for the minimal genus of surfaces representing the second homology classes of a mapping class group.

1. Introduction

By a surface bundle over a surface we mean an oriented fiber bundle whose fibers and base are both compact, oriented 2-manifolds. When we study the topology of fiber bundles, the fundamental question is how the topological invariants of the total space, the fiber space, and the base space are related. Even though it is an elementary fact that the Euler characteristic is multiplicative for fiber bundles, for the signature, the same does not hold in general. As the first counterexamples, Atiyah [1] and, independently, Kodaira [18] provided surface bundles over surfaces with nonvanishing signature. In these classical examples, the fiber genus f or the base genus b was fairly big. For example, in Atiyah's example, $f = 6$ and $b = 129$. After that, there have been many efforts to find out the smallest possible genera of surface bundle over surface for which the signature is nonzero, see [4, 5, 6, 8, 31].

In the early constructions of surface bundles, the signature of the total space was computed by using the signature formula for ramified coverings created by Hirzebruch [15]. However, not all of the bundles can be constructed by using the branched covering method. Instead, in general, the monodromy information of a surface bundle allows us to compute its signature, with the help of Meyer's signature cocycle [26] which is a 2-cocycle of the symplectic group $Sp(2g, \mathbb{R})$. Using the signature cocycle and Birman-Hilden's relations of mapping class

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group, Meyer proved that if the fiber genus $f \leq 2$ or the base genus $b \leq 1$, then the signature vanishes. Hence, for a nonzero signature, we only need to consider the case when $f \geq 3$ and $b \geq 2$. He also proved that for every $f \geq 3$ and every $4n \in 4\mathbb{Z}$, there exists a Σ_f bundle over Σ_b with signature $4n$ for some $b \geq 0$. Based on his result, Endo [6] studied the following refined question which is very similar to Problem 2.18A in Kirby's problem list [17].

Problem 1.1. *For each $f \geq 3$ and each $n \in \mathbb{Z}$, let $b(f,n)$ be the minimal base genus b over which a surface bundle with fiber genus f and signature $4n$ exists. Determine the value $b(f,n)$.*

In [6], Endo showed that $b(f,n) \leq 111|n|$ for any $f \geq 3$. In [31], Stipsicz showed that $b(f, 2f+2) \leq 4f+20$. In [8], Endo, Kotschick, Korkmaz, Ozbagci, and Stipsicz proved that $b(f,n) \leq 8|n|+1$ for any $f \geq 3$ and any $n \neq 0$. In this paper, we improve this upper bound for $b(f,n)$.

Theorem 1.2. (a) *For every $f \geq 3$ and $n \neq 0$, $b(f,n) \leq 7|n|+1$. In particular, there exists a smooth 4-manifold with signature 4 which is a Σ_3-bundle over Σ_8.*

(b) *For every $f \geq 5$ and $n \neq 0$, $b(f,n) \leq 6|n|+1$. In particular, there exists a smooth 4-manifold with signature 4 which is a Σ_5-bundle over Σ_7.*

(c) *For every $f \geq 6$ and $n \neq 0$, $b(f,n) \leq 5|n|+1$. In particular, there exists a smooth 4-manifold with signature 4 which is a Σ_6-bundle over Σ_6.*

Our constructions of surface bundles rely on various computations in mapping class groups, which we will introduce in Section 3. From a geometric point of view, these computations correspond to monodromy factorizations of Lefschetz fibrations. From Lefschetz fibrations, by taking neighborhoods of singular fibers out and gluing them along isomorphic boundaries via fiber-preserving diffeomorphisms, we can construct surface bundles over surfaces. This method was introduced in [8] to construct a Σ_3 bundle over Σ_9 with signature 4. A key ingredient in this paper is that a clever use of different embeddings of relations in mapping class groups gives rise to more economical, in the sense of small genera, surface bundles with a fixed signature 4.

Remark 1.3 ([23]). We may think of $b(f,n)$ as the minimal genus of the surfaces representing the n times generator of $H_2(\mathrm{Mod}(\Sigma_f);\mathbb{Z})/Tor$ for fixed $f \geq 3$ and n.

On the other hand, the lower bound for $b(f,n)$ was also investigated. Kotschick [23] proved $b(f,n) \geq \frac{2|n|}{f-1}+1$, and Hamenstadt [13] proved $b(f,n) \geq \frac{3|n|}{f-1}+1$. Combining the latter with our result, we have $3 \leq b(3,1) \leq 8$, $2 \leq b(5,1) \leq 7$, and $2 \leq b(6,1) \leq 6$.

It is not hard to see that $\frac{b(f,n)}{n}$ converges. Now we define $G_f := \lim_{n\to\infty} \frac{b(f,n)}{n}$ and improve a priori the upper bound for G_f that appeared in [8].

Theorem 1.4. *For every odd $f \geq 3$, $G_f \leq \frac{14}{f-1}$.*

Remark 1.5. As far as I know, this is the best known upper bound for $f = 3$ or every odd f of the form $3k+1, 3k+2$. In fact, for some other f's, better upper bounds are known: for even $f \geq 4$, $G_f \leq \frac{6}{f-2}$ [4], and for $f = 3k \geq 6$, $G_f \leq \frac{9}{f-2}$ [5].

2. Preliminaries

2.1. Signature

Let M be a compact oriented topological manifold of dimension $4m$. Since M is oriented, it admits the fundamental class $[M] \in H_{4m}(M, \partial M)$.

Definition. The symmetric bilinear form $Q_M : H^{2m}(M, \partial M) \times H^{2m}(M, \partial M) \to \mathbb{Z}$ defined by $Q_M(a, b) := \langle a \cup b, [M] \rangle$ is called the intersection form of M.

Remark 2.1. In the smooth case, we can understand Q_M above as the algebraic intersection number of smoothly embedded oriented submanifolds in M representing the Poincaré duals of a and b.

If a or b is a torsion element, then Q_M vanishes, and hence Q_M descends to the cohomology modulo torsion.

Definition. The signature of M, denoted by $\sigma(M)$, is defined to be the signature of the symmetric bilinear form Q_M on $H^{2m}(M, \partial M)/\mathrm{Tor}$. If the dimension of M is not divisible by 4, $\sigma(M)$ is defined to be zero.

2.2. Mapping class group

Let Σ_g^r be an oriented surface of genus g with r boundary components and let Σ_g be a closed oriented surface of genus g. The mapping class group $\mathrm{Mod}(\Sigma_g^r)$ of Σ_g^r is defined to be the group of isotopy classes of orientation preserving self-homeomorphisms which are identity on each boundary component. Based on the theorem of Dehn, we have a surjective homomorphism $\pi : F(S) \to \mathrm{Mod}(\Sigma_g)$, where $F(S)$ is the free group generated by the generating set S consisting of all the Dehn twists over all isotopy classes of simple closed curves on Σ_g. Set $R := \mathrm{Ker}\pi$ and call each word w in the generators S of $\mathrm{Mod}(\Sigma_g)$ a relation of $\mathrm{Mod}(\Sigma_g)$ if $w \in R$. Now, let us review some famous relations of mapping class groups.

Let a and b be two simple closed curves on Σ_g. If a and b are disjoint, then the supports of the Dehn twists t_a and t_b can be chosen to be disjoint. Hence, there exist commutativity relations $t_a t_b t_a^{-1} t_b^{-1}$ for any disjoint simple closed curves a and b. If a intersects b transversely at one point, then there exists a braid relation $t_a t_b t_a t_b^{-1} t_a^{-1} t_b^{-1}$. It can be derived from more general fact that $f t_a f^{-1} = t_{f(a)}$ in $\mathrm{Mod}(\Sigma_g)$ for any simple closed curve a on Σ_g and any orientation preserving homeomorphism f of Σ_g. For braid relations, we will take the latter general form $f t_a f^{-1} t_{f(a)}^{-1}$. Consider the planar surface Σ_0^4 with boundary components a, b, c, and d. On the left hand side of Figure 1, the boundary curves a, b, c, and d are in black and the interior curves x, y, and

z are in different colors. One can easily check that $t_a t_b t_c t_d = t_z t_y t_x$ holds in $\mathrm{Mod}(\Sigma_0^4)$ by applying the Alexander method, and we call $t_d^{-1} t_c^{-1} t_b^{-1} t_a^{-1} t_z t_y t_x$ the lantern relations for all embedded subsurfaces $\Sigma_0^4 \hookrightarrow \Sigma_g$. For the k-chain relations and any other details for mapping class groups, refer to [11]. One can also deduce the star relations $t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1} (t_{\alpha_1} t_{\alpha_2} t_{\alpha_3} t_\beta)^3$ supported on any embedded subsurfaces $\Sigma_1^3 \hookrightarrow \Sigma_g$. See Figure 4 as an example.

We say that two simple closed curves a and b on Σ_g are topologically equivalent if there exists a homeomorphism of Σ_g sending a to b. Similarly, the two collections $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ of simple closed curves on Σ_g are called topologically equivalent if there exists a homeomorphism of Σ_g sending a_i to b_i simultaneously for all $1 \leq i \leq n$. To simplify the notation in the rest of this paper, we will use the notation $w_1^{w_2}$ for the conjugation $w_2^{-1} w_1 w_2$.

2.3. Lefschetz fibrations and surface bundles

Definition. Let X be a compact oriented 4-manifold, and B a compact oriented 2-manifold. A smooth surjective map $f : X \to B$ is called a Lefschetz fibration if for each critical point $p \in X$ there are local complex coordinates (z_1, z_2) on X around p and z on B around $f(p)$ compatible with the orientations and such that $f(z_1, z_2) = z_1^2 + z_2^2$.

It follows that f has only finitely many critical points, and we may assume that f is injective on the critical set $C = \{p_1, \dots, p_k\}$. A fiber of f containing a critical point is called a singular fiber, and the genus of f is defined to be the genus of the regular fiber. Notice that if $\nu(f(C))$ denotes an open tubular neighborhood of the set of critical values $f(C)$, then the restriction of f to $f^{-1}(B - \nu(f(C)))$ is a smooth surface bundle over $B - \nu(f(C))$.

For a smooth surface bundle $f : E \to B$ with a fixed identification ϕ of the fiber over the base point p of B with a standard genus g surface Σ_g, the monodromy representation of f is defined to be an antihomomorphism $\chi : \pi_1(B, p) \to \mathrm{Mod}(\Sigma_g)$ defined as follows. For each loop $l : [0,1] \to B$, $l^*(E) \to [0,1]$ is trivial and hence there exists a parametrization $\Phi : [0,1] \times \Sigma_g \to f^{-1}(l[0,1]))$ with $\Phi|_{0 \times \Sigma_g} = \phi^{-1}$. Now define $\chi([l]) := [\Phi|_{0 \times \Sigma_g}^{-1} \circ \Phi|_{1 \times \Sigma_g}]$. For the genus g Lefschetz fibration $f : X \to B$ with a fixed identification of the fiber with Σ_g, we define the monodromy representation of f to be the monodromy representation of the surface bundle $f : X - f^{-1}(f(C)) \to B - f(C)$.

A Lefschetz singular fiber can be described by its monodromy. By looking at the local model of the Lefschetz critical point, one can see that the singular fiber is obtained from the regular fiber by collapsing a simple closed curve, called the vanishing cycle. One can also observe that the monodromy along the loop going around one Lefschetz critical value is given by the right-handed Dehn twist along the vanishing cycle. Hence, from the monodromy representation χ of a Lefschetz fibration, after fixing the generating system $\{a_1, b_1, \dots, a_h, b_h, l_1, \dots, l_k\}$ of $\pi_1(B - f(C), p)$, we get the global monodromy $\prod_{i=1}^h [\chi(a_i), \chi(b_i)] \prod_{j=1}^k t_{\gamma_j}$ since we have $\chi(l_j) = t_{\gamma_j}$ for each $j = 1, \dots, k$;

and when B is closed, $\prod_{i=1}^{h}[\chi(a_i),\chi(b_i)]\prod_{j=1}^{k}t_{\gamma_j}=1$ in $\mathrm{Mod}(\Sigma_g)$, and this is called the monodromy factorization of a Lefschetz fibration. Conversely, a factorization $\prod_{i=1}^{h}[\alpha_i,\beta_i]\prod_{j=1}^{k}t_{\gamma_j}=1$ of identity in $\mathrm{Mod}(\Sigma_g)$ gives rise to a genus g Lefschetz fibration over Σ_h. For this, first observe that a product $\prod_{i=1}^{h}[\alpha_i,\beta_i]$ of h commutators in $\mathrm{Mod}(\Sigma_g)$ gives a Σ_g bundle over Σ_h^1. Also, a product $\prod_{j=1}^{k}t_j$ of right-handed Dehn twists t_j in $\mathrm{Mod}(\Sigma_g)$ gives a genus g Lefshetz fibration over D^2. By combining these two constructions, a word $w=\prod_{i=1}^{h}[\alpha_i,\beta_i]\prod_{j=1}^{k}t_j\in\mathrm{Mod}(\Sigma_g)$ gives the genus g Lefschetz fibration over Σ_h^1, and if $w=1$ in $\mathrm{Mod}(\Sigma_g)$ we can close up to a Lefschetz fibration over Σ_h.

Two Lefschetz fibrations $f_1:X_1\to B_1, f_2:X_2\to B_2$ are called isomorphic if there exist orientation preserving diffeomorphisms $H:X_1\to X_2$ and $h:B_1\to B_2$ such that $f_2\circ H=h\circ f_1$. The isomorphism class of a Lefschetz fibration is determined by an equivalence class of its monodromy representation. Oriented genus g surface bundles over surfaces of genus h are classified, up to isomorphism, by homotopy classes of the classifying map $\Sigma_h\to \mathrm{BDiff}^+\Sigma_g$ since the structure group is $\mathrm{Diff}^+\Sigma_g$. If $g\geq 2$, then according to the Earle-Eells theorem and the $K(\pi,1)$ theory, they are classified by the conjugacy classes of the induced homomorphisms $\pi_1(\Sigma_h)\to\mathrm{Mod}(\Sigma_g)$. Therefore, $\prod_{i=1}^{h}[\alpha_i,\beta_i]=1$ in $\mathrm{Mod}(\Sigma_g)$, up to global conjugations, determines the genus g surface bundle over a surface of genus h.

3. Subtraction of Lefschetz fibrations

In the study of manifold theory, a common way to construct a new manifold from a given manifold is a cut-and-paste operation. To construct a new 4-manifold which is a surface bundle over a surface, H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci and A. Stipsicz introduced an operation, called the "subtraction of Lefschetz fibrations", in [8]. Let us first explain it here in a generalized version.

Let $f:X\to B_1$ be a Lefschetz fibration with m critical values $q_1^{(1)},\dots,q_m^{(1)}$ and let $g:Y\to B_2$ be another Lefschetz fibration with $k\leq m$ critical values $q_1^{(2)},\dots,q_k^{(2)}$. Suppose that $f:f^{-1}(D_1)\to D_1$ and $g:g^{-1}(D_2)\to D_2$ are isomorphic where $D_1\subset B_1$ is a disk containing some critical values $q_1^{(1)},\dots,q_k^{(1)}$ and $D_2\subset B_2$ is a disk containing $q_1^{(2)},\dots,q_k^{(2)}$. Then, the manifolds $X\backslash f^{-1}(D_1)$ and $Y\backslash g^{-1}(D_2)$ have a diffeomorphic boundary, and after reversing the orientation of one of them, this diffeomorphism can be chosen to be fiber-preserving and orientation-reversing. Fix such a diffeomorphism ϕ and then glue $\overline{Y\backslash g^{-1}(D_2)}$, the manifold $Y\backslash g^{-1}(D_2)$ with the reversed orientation, to $X\backslash f^{-1}(D_1)$ using this diffeomorphism ϕ. Note that the resulting manifold, denoted by $X-Y$, inherits a natural orientation and admits a smooth fibration $f\cup g: X\backslash f^{-1}(D_1)\cup\overline{Y\backslash g^{-1}(D_2)}\to B_1\#B_2$. This is a Lefschetz fibration with $m-k$ singular fibers. In particular, for $k=m$, we get a surface bundle over a surface. In general, after repeatedly subtracting Lefschetz fibrations, we get $X - Y_1 - Y_2 - \cdots - Y_n$, a surface bundle over a surface, under the following assumptions. Let $f : X \to B_0$ be a Lefschetz fibration with m critical values $\{q^{(0)}_{1,1}, \ldots, q^{(0)}_{1,k_1}, q^{(0)}_{2,1}, \ldots, q^{(0)}_{2,k_2}, \ldots, q^{(0)}_{n,1}, \ldots, q^{(0)}_{n,k_n}\}$ and $g_1 : Y_1 \to B_1$, …, $g_n : Y_n \to B_n$ be Lefschetz fibrations with critical values $\{q^{(1)}_1, \ldots, q^{(1)}_{k_1}\}$, $\cdots$, $\{q^{(n)}_1, \ldots, q^{(n)}_{k_n}\}$, respectively. We assume that $k_1 + \cdots + k_n = m$ and that $f : f^{-1}(D_{0,i}) \to D_{0,i}$ is isomorphic to $g_i : g_i^{-1}(D_i) \to D_i$ for each $1 \le i \le n$, where each $D_{0,i} \subset B_0$ is a disk containing $q^{(0)}_1, \ldots q^{(0)}_{k_i}$ and $D_i \subset B_i$ is a disk containing $q^{(i)}_1, \ldots, q^{(i)}_{k_i}$.

In order to use the subtraction method explained above, we need to construct the building blocks X and Y_i's. First, we describe various gluing pieces Y_i.

Proposition 3.1 ([8])**.** *Let $f \ge 3$ and let a be a simple closed curve on Σ_f. In the mapping class group* $\mathrm{Mod}(\Sigma_f)$*,*

(a) *t_a^2 can be written as a product of two commutators,*

(b) *if a is nonseparating, then t_a^4 can be written as a product of three commutators.*

Remark 3.2. This proposition gives us two genus $f \ge 3$ Lefschetz fibrations $Y_1 \to \Sigma_2$ and $Y_2 \to \Sigma_3$ whose monodromy factorizations are given by $[f_1, g_1][f_2, g_2]t_a^2 = 1$ and $[f_3, g_3][f_4, g_4][f_5, g_5]t_a^4 = 1$ for some mapping classes $f_i, g_i \in \mathrm{Mod}(\Sigma_f)$ for $1 \le i \le 5$. Generally, for every n, we can obtain a Lefschetz fibration which has n singular fibers and the monodromy t_a^n using a daisy relation.

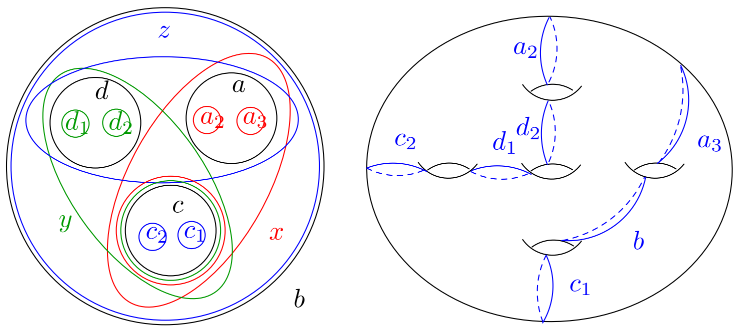

FIGURE 1. Supports of four lantern relations and an embedding of Σ_0^7 into a genus 5 surface

The following two propositions allow us to glue building blocks along more complicated monodromies in the sense that they are products of Dehn twists along distinct simple closed curves.

Proposition 3.3. *Let $f \ge 5$ and let b,c be disjoint simple closed curves on Σ_f such that $\Sigma_f - b - c$ is connected. In* $\mathrm{Mod}(\Sigma_f)$*, $t_b^2 t_c^2$ can be written as a product of three commutators.*

Proof. We may assume b and c are embedded, as shown in Figure 1, because any pair of simple closed curves whose complement in Σ_f is connected is topologically equivalent. On the planar surface Σ_0^7 in Figure 1, the following four lantern relations hold. $L_1 := t_a^{-1}t_b^{-1}t_c^{-1}t_d^{-1}t_yt_xt_z$, $L_2 := t_dt_{D_2}t_{D_1}t_{d_1}^{-1}t_{d_2}^{-1}t_c^{-1}t_y^{-1}$, $L_3 := t_x^{-1}t_{a_2}^{-1}t_{a_3}^{-1}t_c^{-1}t_at_{A_3}t_{A_2}$, $L_4 := t_z^{-1}t_{c_1}^{-1}t_{c_2}^{-1}t_b^{-1}t_ct_{C_2}t_{C_1}$. Here, D_1 is an interior curve surrounding two boundary curves except d_1, and all other curves denoted by capital letters are defined similarly. After embedding Σ_0^7 into Σ_f with $f \geq 5$, as shown in Figure 1, we have $1 = L_1 \cdot L_2^{t_yt_xt_z} \cdot L_3^{t_z} \cdot L_4 = t_b^{-1}t_c^{-1}t_{D_2}t_{d_2}^{-1}t_{D_1}t_{d_1}^{-1}t_c^{-1}t_{A_3}t_{a_3}^{-1}t_{A_2}t_{a_2}^{-1}\ t_b^{-1}t_{C_2}t_{c_2}^{-1}t_{C_1}t_{c_1}^{-1}$ in $\mathrm{Mod}(\Sigma_f)$. Since both $\Sigma_f - D_2 - d_2$ and $\Sigma_f - D_1 - d_1$ are connected, $\{d_2, D_2\}$ and $\{D_1, d_1\}$ are topologically equivalent and then $t_{D_2}t_{d_2}^{-1}t_{D_1}t_{d_1}^{-1} = [t_{D_2}t_{d_2}^{-1}, \phi_1]$ for some $\phi_1 \in \mathrm{Mod}(\Sigma_f)$. Similarly, $t_{A_3}t_{a_3}^{-1}t_{A_2}t_{a_2}^{-1} = [t_{A_3}t_{a_3}^{-1}, \phi_2]$ and $t_{C_2}t_{c_2}^{-1}t_{C_1}t_{c_1}^{-1} = [t_{C_2}t_{c_2}^{-1}, \phi_3]$ for some $\phi_2, \phi_3 \in \mathrm{Mod}(\Sigma_f)$.

Therefore, $t_b^2t_c^2 = [t_{D_2}t_{d_2}^{-1}, \phi_1]^{(t_bt_c)^{-1}}[t_{A_3}t_{a_3}^{-1}, \phi_2]^{t_b^{-1}}[t_{C_2}t_{c_2}^{-1}, \phi_3]$. □

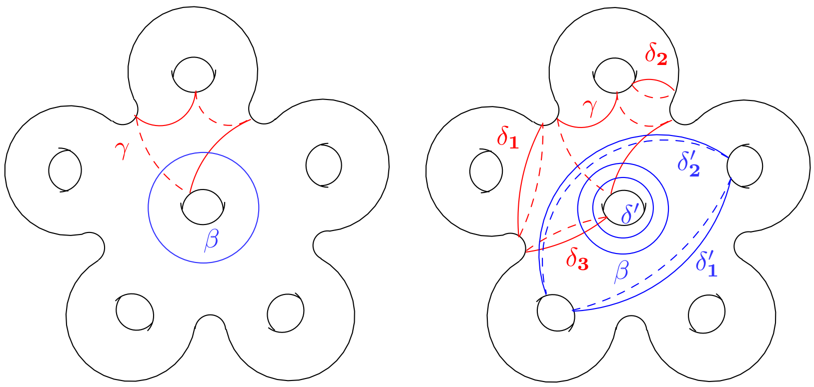

FIGURE 2. Supports of two lantern relations embedded in a genus 6 surface

Proposition 3.4. *Let $f \geq 6$ and let β,γ be simple closed curves on Σ_f embedded, as shown in Figure 2. In $\mathrm{Mod}(\Sigma_f)$, $t_\beta t_\gamma$ can be written as a product of three commutators.*

Proof. Choose two lantern relations with their supports on Σ_f, as shown in Figure 2: $L_1 := t_\gamma^{-1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_yt_xt_z$ and $L_2 := t_{x'}t_{z'}t_{y'}t_{\delta'}^{-1}t_{\delta_1'}^{-1}t_{\delta_2'}^{-1}t_\beta^{-1}$. For interior curves, see Figure 3. It follows that $1 = L_1 \cdot L_2 = t_\gamma^{-1}t_{\delta_2}^{-1}t_yt_{\delta_3}^{-1}t_xt_zt_{\delta_1}^{-1} \cdot t_{x'}t_{\delta_2'}^{-1}t_{z'}t_{\delta'}^{-1}t_{y'}t_{\delta_1'}^{-1}t_\beta^{-1}$. In Figure 2 and Figure 3, we can see that δ_1 and x' are separating curves on Σ_f and that both $\Sigma_f - z - \delta_1$ and $\Sigma_f - \delta_2' - x'$ are homeomorphic to $\Sigma_1^1 \cup \Sigma_{f-2}^3$. Hence, we have $t_zt_{\delta_1}^{-1}t_{x'}t_{\delta_2'}^{-1} = [t_zt_{\delta_1}^{-1}, \phi_2]$ for some

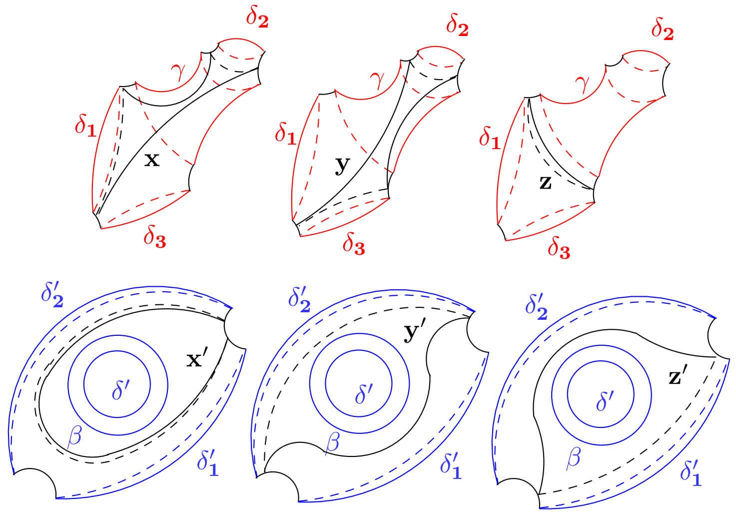

FIGURE 3. Interior curves for two lantern relations

ϕ_2. Similarly, we have $t_{\delta_2}^{-1} t_y t_{\delta_3}^{-1} t_x = [t_{\delta_2}^{-1} t_y, \phi_1]$ and $t_{z'} t_{\delta'}^{-1} t_{y'} t_{\delta'_1}^{-1} = [t_{z'} t_{\delta'}^{-1}, \phi_3]$ for some ϕ_1 and ϕ_3.

Therefore, $t_\beta t_\gamma = [t_{\delta_2}^{-1} t_y, \phi_1]^{t_\beta^{-1}} [t_z t_{\delta_1}^{-1}, \phi_2]^{t_\beta^{-1}} [t_{z'} t_{\delta'}^{-1}, \phi_3]^{t_\beta^{-1}}$. □

In Proposition 11 of [8], they constructed a genus $f \geq 3$ Lefschetz fibration over a torus with 10 singular fibers using a two-holed torus relation which is also called a 3-chain relation. In the following three Propositions, we generalize this construction of a Lefschetz fibration.

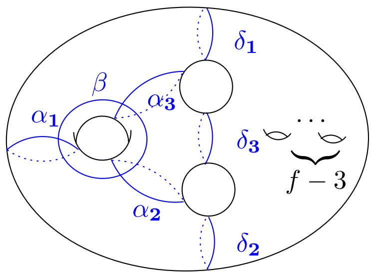

FIGURE 4. Support of a star relation

Proposition 3.5. *Let $f \geq 3$ and let $\{\alpha_1, \alpha_2\}$ be any pair of nonseparating simple closed curves on Σ_f such that $\Sigma_f - \alpha_1 - \alpha_2$ is connected. Then there exists a genus f Lefschetz fibration X over Σ_3 which has six singular fibers, four of which have monodromy t_{α_1} and two of which have monodromy t_{α_2}.*

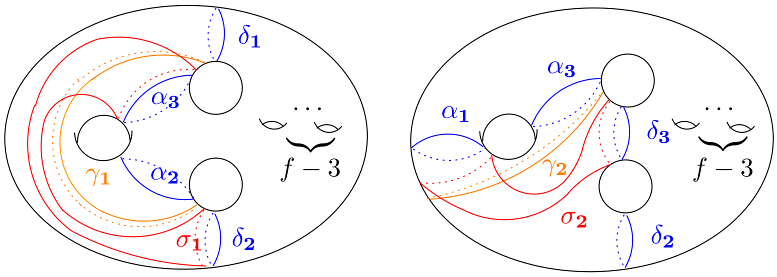

FIGURE 5. Supports of two lantern relations

Proof. We use the star relation $E := t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}(t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta)^3$ supported on $\Sigma_1^3 \hookrightarrow \Sigma_f$ (Figure 4). Also, consider the following lantern relations whose supports are given in Figure 5 : $L_1 := t_{\alpha_3}^{-1}t_{\alpha_2}^{-1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}$, $L_2 := t_{\alpha_3}^{-1}t_{\alpha_1}^{-1}t_{\delta_2}^{-1}t_{\delta_3}^{-1} t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}$. Let $W_0 := t_\beta(t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta)^2$, $W_1 := t_\beta t_{\alpha_1}t_{\alpha_2}t_{\alpha_3}t_\beta$, and $W_2 := t_\beta$. Then, by using commutativity relations and braid relations,

$$\begin{aligned}
1 &= E\cdot(W_0^{-1}L_1W_0)\cdot(W_1^{-1}L_1W_1)\cdot(W_2^{-1}L_2W_2)\\
&= t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{\alpha_1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}t_\beta t_{\alpha_1}t_{\delta_1}^{-1}t_{\delta_2}^{-1}t_{\sigma_1}t_{\alpha_1}t_{\gamma_1}t_\beta t_{\alpha_2}t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta\\
&= t_{\alpha_1}t_{\delta_1}^{-1}t_{\sigma_1}t_{\delta_2}^{-1}t_{\alpha_1}t_{\gamma_1}t_{\delta_1}^{-1}t_\beta t_{\alpha_1}t_{\delta_1}^{-1}t_{\sigma_1}t_{\delta_2}^{-1}t_{\alpha_1}t_{\gamma_1}t_{\delta_2}^{-1}t_\beta t_{\alpha_2}t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_3}^{-1}t_\beta\\
&= t_{\alpha_1}^2\{t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1}t_{\gamma_1}t_{\delta_1}^{-1}t_\beta\}t_{\alpha_1}^2\{t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1}t_{\gamma_1}t_{\delta_2}^{-1}t_\beta\}\\
&\quad t_{\alpha_2}^2\{t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_3}^{-1}t_\beta\}\\
&= t_{\alpha_1}^2[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_1]t_{\alpha_1}^2[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_2]t_{\alpha_2}^2[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3].
\end{aligned}$$

For the last equality, we need to verify that there exists a self-homeomorphism ϕ_1 of Σ_f sending δ_1, $t_{\alpha_1}^{-1}(\sigma_1)$, and δ_2 to β, δ_1, and γ_1, respectively. First, it is easy to check that $\sigma_1 = t_\beta^{-1}t_{\alpha_2}^{-1}t_{\alpha_1}t_{\alpha_3}^{-1}(\beta)$. Hence, the self-homeomorphism $t_{\alpha_3}t_{\alpha_1}^{-1}t_{\alpha_2}t_\beta t_{\alpha_1}$ sends δ_1, $t_{\alpha_1}^{-1}(\sigma_1)$, and δ_2 to δ_1, β, and δ_2, respectively. Also, there exists a homeomorphism sending δ_1, β, and δ_2 to β, δ_1, and γ_1, respectively, because both $\Sigma_f - \delta_1 - \beta - \delta_2$ and $\Sigma_f - \beta - \delta_1 - \gamma_1$ are homeomorphic to Σ_{f-3}^6. The composition of these two homeomorphisms is the required ϕ_1. The existence of ϕ_2 and ϕ_3 can be proven in a similar way because $\sigma_2 = t_\beta^{-1}t_{\alpha_1}^{-1}t_{\alpha_3}^{-1}t_{\alpha_2}(\beta)$. Finally, we get the required Lefschetz fibration over Σ_3 with fiber Σ_f whose monodromy factorization is given by $[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_1]^{t_{\alpha_1}^{-2}}[t_{\delta_1}^{-1}t_{t_{\alpha_1}^{-1}(\sigma_1)}t_{\delta_2}^{-1},\phi_2]^{t_{\alpha_1}^{-4}}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1},\phi_3]^{t_{\alpha_2}^{-2}t_{\alpha_1}^{-4}}t_{\alpha_1}^4t_{\alpha_2}^2 = 1.$ □

Proposition 3.6. *Let $f \geq 4$ and let $\{\alpha_2,\alpha_3\}$ be any pair of nonseparating simple closed curves on Σ_f such that $\Sigma_f - \alpha_2 - \alpha_3$ is connected. Then there is a genus f Lefschetz fibration Z over Σ_4 which has four singular fibers, two of which have monodromy t_{α_2} and another two of which have monodromy t_{α_3}.*

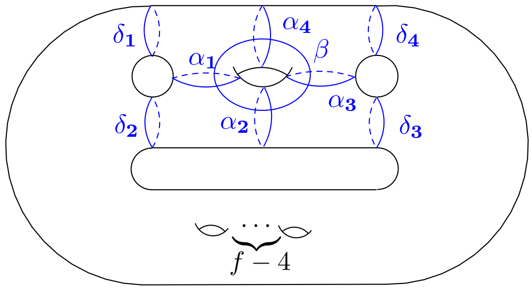

FIGURE 6. Support of a four-holed torus relation embedded in a genus 4 surface

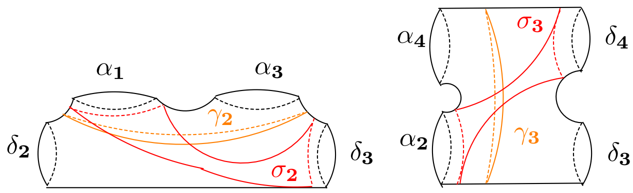

FIGURE 7. Supports of two lantern relations

Proof. We use the 4-holed torus relation [21] and lantern relations. Let $E_2 := t_{\delta_4}^{-1}t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta$. We embed the support of this relation into Σ_f, as shown in Figure 6. Let $L_5 := t_{\alpha_3}^{-1}t_{\alpha_1}^{-1}t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}$ and $L_6 := t_{\alpha_4}^{-1}t_{\alpha_2}^{-1}t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}$. For the supports of lanterns, see Figure 7. Let $w_1 := t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta$, $w_2 := t_\beta t_{\alpha_1}t_{\alpha_3}t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta$, and $w_3 := t_\beta t_{\alpha_2}t_{\alpha_4}t_\beta$. Then, from commutativity relations and braid relations,

$$
\begin{aligned}
1 &= E_2 \cdot L_5^{w_1} \cdot L_6^{w_2} \cdot L_5^{w_3} \cdot L_6^{t_\beta} \\
&= t_{\delta_4}^{-1}t_{\delta_3}^{-1}t_{\delta_2}^{-1}t_{\delta_1}^{-1}(t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta)(t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}t_\beta) \\
&\quad (t_{\delta_2}^{-1}t_{\delta_3}^{-1}t_{\sigma_2}t_{\alpha_2}t_{\gamma_2}t_\beta)(t_{\delta_3}^{-1}t_{\delta_4}^{-1}t_{\sigma_3}t_{\alpha_3}t_{\gamma_3}t_\beta) \\
&= (t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_1}^{-1}t_\beta)(t_{\delta_3}^{-1}t_{\sigma_3}t_{\delta_4}^{-1}t_{\alpha_3}t_{\gamma_3}t_{\delta_4}^{-1}t_\beta) \\
&\quad (t_{\delta_2}^{-1}t_{\sigma_2}t_{\delta_3}^{-1}t_{\alpha_2}t_{\gamma_2}t_{\delta_2}^{-1}t_\beta)(t_{\delta_3}^{-1}t_{\sigma_3}t_{\delta_4}^{-1}t_{\alpha_3}t_{\gamma_3}t_{\delta_3}^{-1}t_\beta) \\
&= t_{\alpha_2}(t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_1}^{-1}t_\beta)t_{\alpha_3}(t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}t_{\gamma_3}t_{\delta_4}^{-1}t_\beta) \\
&\quad t_{\alpha_2}(t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}t_{\gamma_2}t_{\delta_2}^{-1}t_\beta)t_{\alpha_3}(t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}t_{\gamma_3}t_{\delta_3}^{-1}t_\beta) \\
&= t_{\alpha_2}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}, \phi_1]t_{\alpha_3}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}, \phi_2] \\
&\quad t_{\alpha_2}[t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}, \phi_3]t_{\alpha_3}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}, \phi_4] \\
&= [t_{\delta_2}^{-1}t_{t_{\alpha_2}^{-1}(\sigma_2)}t_{\delta_3}^{-1}, \phi_1]^{t_{\alpha_2}^{-1}}[t_{\delta_3}^{-1}t_{t_{\alpha_3}^{-1}(\sigma_3)}t_{\delta_4}^{-1}, \phi_2]^{t_{\alpha_3}^{-1}t_{\alpha_2}^{-1}}
\end{aligned}
$$

$$[t_{\delta_2}^{-1} t_{t_{\alpha_2}^{-1}(\sigma_2)} t_{\delta_3}^{-1}, \phi_3]^{t_{\alpha_3}^{-1} t_{\alpha_2}^{-2}} [t_{\delta_3}^{-1} t_{t_{\alpha_3}^{-1}(\sigma_3)} t_{\delta_4}^{-1}, \phi_4]^{t_{\alpha_3}^{-2} t_{\alpha_2}^{-2}} t_{\alpha_2}^2 t_{\alpha_3}^2.$$

For the fifth equality, we need to find certain ϕ_1, ϕ_2, ϕ_3 and ϕ_4. For ϕ_1, it is sufficient to verify that $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\beta, \delta_1, \gamma_2\}$. This is because $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\delta_2, \beta, \delta_3\}$, and then $\{\delta_2, \beta, \delta_3\}$ to $\{\beta, \delta_1, \gamma_2\}$. The arguments for ϕ_2, ϕ_3, and ϕ_4 are similar. For these, we can check that $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3), \delta_4\}$ is topologically equivalent to $\{\beta, \delta_4, \gamma_3\}$, $\{\delta_2, t_{\alpha_2}^{-1}(\sigma_2), \delta_3\}$ is topologically equivalent to $\{\beta, \delta_2, \gamma_2\}$, and $\{\delta_3, t_{\alpha_3}^{-1}(\sigma_3), \delta_4\}$ is topologically equivalent to $\{\beta, \delta_3, \gamma_3\}$. □

Proposition 3.7. *Let $f \geq 6$ and let β, γ be simple closed curves on Σ_f embedded, as shown in Figure 2. Then there is a genus f Lefschetz fibration W over Σ_3 which has two singular fibers, one of which has monodromy t_β and another has monodromy t_γ.*

Proof. There is a 9-holed torus relation $E_7 := t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_8}^{-1} t_{\gamma_9}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_6} t_{\alpha_{10}} t_{\beta_5} t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\beta_2} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3}$ (see its support in orange in Figure 8 and see Figure 9 for its interior curves), where we use the identification $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9) \to (\alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ to go from Figure 9 in [21] to Figure 9 in this article. Here, each $\beta_i = t_{\alpha_i}(\beta)$ as in [21]. If we combine this relation E_7 and one more lantern relation $L_8 := t_{\delta_9}^{-1} t_{\delta_{10}}^{-1} t_{\gamma_9} t_{\sigma_9} t_{\alpha_9} t_{\alpha_8}^{-1} t_{\alpha_{10}}^{-1}$ (see its support in blue in Figure 8), then we get the following 10-holed torus relation $E_8 := t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\alpha_8}^{-1} t_{\alpha_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_6} t_{\alpha_{10}} t_{\beta_5} t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\beta_2} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9}$. Let $\beta_5' = (t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9})^{-1}(\beta_5)$ and $\beta_2' = (t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9})^{-1}(\beta_2)$. Then, by using commutativity relations and braid relations,

$$\begin{aligned}
1 &= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_6} t_{\alpha_{10}} t_{\sigma_4} t_{\sigma_7} t_{\alpha_6} t_{\sigma_5} t_{\sigma_8} t_{\alpha_3} t_{\sigma_9} t_{\alpha_9} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} t_{\beta_8} t_{\sigma_3} t_{\sigma_4} t_{\sigma_5} t_{\sigma_6} t_{\sigma_7} t_{\sigma_8} t_{\alpha_{10}} t_{\alpha_6} t_{\alpha_3} t_{\alpha_9} t_{t_{\alpha_9}^{-1}(\sigma_9)} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= t_{\delta_1}^{-1} t_{\delta_2}^{-1} \cdots t_{\delta_{10}}^{-1} (t_{\sigma_3} t_{\sigma_4} t_{\sigma_5} t_{\sigma_6} t_{\sigma_7} t_{\sigma_8} t_{\alpha_{10}} t_{\alpha_6} t_{\alpha_3} t_{\alpha_9})^{t_{\beta_8}^{-1}} \\
&\quad t_{\beta_8} t_{t_{\alpha_9}^{-1}(\sigma_9)} t_{\beta_5'} t_{\alpha_8}^{-1} t_{t_{\alpha_8}(\beta_2')} t_{\alpha_{10}}^{-1} \\
&= \{t_{\delta_1}^{-1} \cdot t_{t_{\beta_8}(\sigma_3)} \cdot t_{\delta_3}^{-1} \cdot t_{t_{\beta_8}(\sigma_4)} \cdot t_{\delta_{10}}^{-1} \cdot t_{t_{\beta_8}(\sigma_5)} \cdot t_{\delta_2}^{-1} \cdot t_{t_{\beta_8}(\sigma_6)} \cdot t_{\delta_7}^{-1} \cdot t_{t_{\beta_8}(\sigma_7)} \\
&\quad \cdot t_{\delta_9}^{-1} \cdot t_{t_{\beta_8}(\sigma_8)}\} \{t_{\delta_5}^{-1} \cdot t_{t_{\beta_8}(\alpha_3)} \cdot t_{\delta_8}^{-1} \cdot t_{t_{\beta_8}(\alpha_{10})} \cdot t_{\delta_6}^{-1} \cdot t_{t_{\beta_8}(\alpha_6)} \cdot t_{\delta_4}^{-1} \cdot t_{t_{\beta_8}(\alpha_9)}\} \\
&\quad \{t_{\beta_8} \cdot t_{t_{\alpha_9}^{-1}(\sigma_9)}\} \{t_{\beta_5'} \cdot t_{\alpha_8}^{-1} \cdot t_{t_{\alpha_8}(\beta_2')} \cdot t_{\alpha_{10}}^{-1}\} \\
&= [t_{\delta_1}^{-1} \cdot t_{t_{\beta_8}(\sigma_3)} \cdot t_{\delta_3}^{-1} \cdot t_{t_{\beta_8}(\sigma_4)} \cdot t_{\delta_{10}}^{-1} \cdot t_{t_{\beta_8}(\sigma_5)}, \phi_1] \\
&\quad [t_{\delta_5}^{-1} \cdot t_{t_{\beta_8}(\alpha_3)} \cdot t_{\delta_8}^{-1} \cdot t_{t_{\beta_8}(\alpha_{10})}, \phi_2] \cdot t_{\beta_8} \cdot t_{t_{\alpha_9}^{-1}(\sigma_9)} \cdot [t_{\beta_5'} t_{\alpha_8}^{-1}, \phi_3].
\end{aligned}$$

For the last equality, we need to verify that $\{\delta_1, t_{\beta_8}(\sigma_3), \delta_3, t_{\beta_8}(\sigma_4), \delta_{10}, t_{\beta_8}(\sigma_5)\}$ is topologically equivalent to $\{t_{\beta_8}(\sigma_8), \delta_9, t_{\beta_8}(\sigma_7), \delta_7, t_{\beta_8}(\sigma_6), \delta_2\}$. This follows from the fact that both $\Sigma_f - \delta_1 - \delta_3 - \delta_{10} - \sigma_3 - \sigma_4 - \sigma_5$ and $\Sigma_f - \delta_2 - \delta_7 - \delta_9 - \sigma_6 - \sigma_7 - \sigma_8$ are connected. For ϕ_2 and ϕ_3, it is easy to check that $\Sigma_f - \delta_5 - \alpha_3 - \delta_8 - \alpha_{10} \approx \Sigma_{f-4}^8 \approx \Sigma_f - \alpha_9 - \delta_4 - \alpha_6 - \delta_6$ and that $\{\beta_5', \alpha_8\}$ is topologically equivalent to $\{\beta, \alpha_8\}$ and $\{\alpha_{10}, t_{\alpha_8}(\beta_2')\}$ is topologically equivalent to $\{\alpha_{10}, \beta\}$. Finally, observe that $\{\beta_8, t_{\alpha_9}^{-1}(\sigma_9)\}$ is topologically equivalent to $\{\beta, t_{\alpha_9}^{-1}(\sigma_9)\}$ and $t_{\alpha_9}^{-1}(\sigma_9) = \gamma$. □

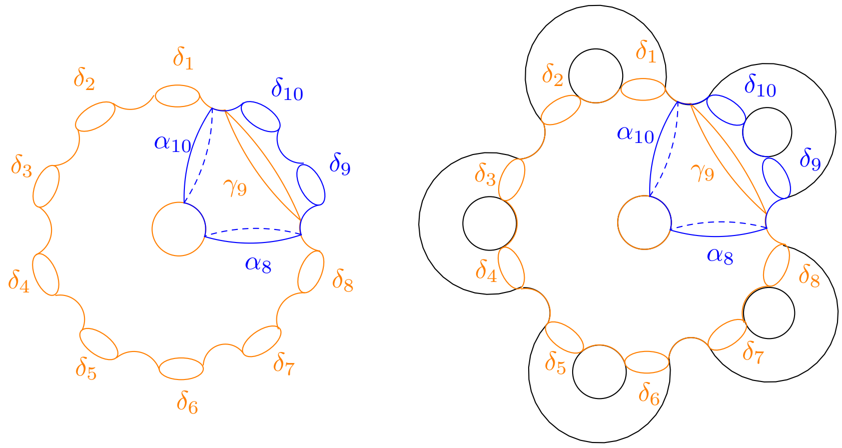

FIGURE 8. Supports for a 9-holed torus relation and a lantern relation and their embeddings into a genus 6 surface

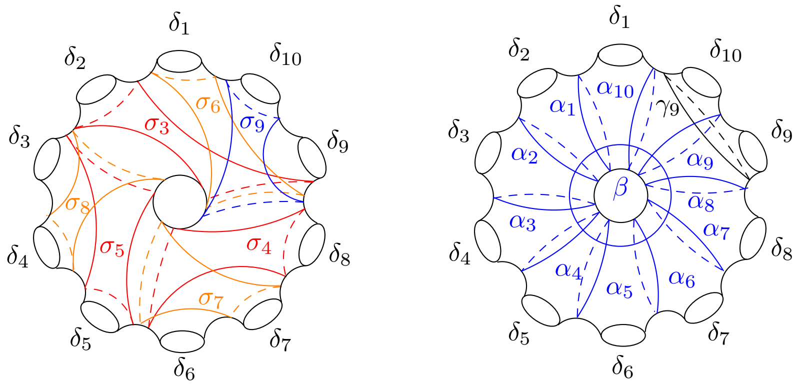

FIGURE 9. Interior curves for a 10-holed torus relation

4. Signature computation

In order to compute the signature of the total space of surface bundles, we first review the definition of Meyer's signature cocycle.

Definition. For any given $A, B \in Sp(2g, \mathbb{R})$, consider the subspace

$$V_{A,B} := \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} | (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}$$

of the real vector space $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ and the bilinear form $\langle , \rangle_{A,B} : (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \times (\mathbb{R}^{2g} \times \mathbb{R}^{2g}) \to \mathbb{R}$ defined by $\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := (x_1 + y_1) \cdot J(I_{2g} - B)y_2$, where $\cdot$ is the inner product of $\mathbb{R}^{2g}$ and J is the matrix representing the multiplication by $-\sqrt{-1}$ on $\mathbb{R}^{2g} = \mathbb{C}^g$. Since the restriction of $\langle , \rangle_{A,B}$ on $V_{A,B}$ is symmetric, we can define $\tau_g(A, B) := sign(\langle , \rangle_{A,B}, V_{A,B})$.

We denote by $\psi : \mathrm{Mod}(\Sigma_g) \to Sp(2g; \mathbb{R})$ the symplectic representation of the mapping class group.

Theorem 4.1 ([25])**.** *Let $E_{A,B} \to P$ be an oriented Σ_g bundle over a pair of pants P whose monodromy representation χ composed with the symplectic representation ψ is given by $\psi \circ \chi : \pi_1(P, *) \to Sp(2g : \mathbb{R})$ sending one generator to A and the other to B. Then $\sigma(E_{A,B}) = -\tau_g(A, B)$.*

We can easily check that τ_g is a 2-cocycle on the symplectic group $Sp(2g, \mathbb{R})$ using Novikov's additivity. We call this τ_g Meyer's signature cocycle. The pants decomposition of any base surface gives the following signature formula.

Theorem 4.2 ([26])**.** *Let $f : E \to \Sigma_h^r$ be an oriented surface bundle with fiber Σ_g and monodromy representation $\chi : \pi_1(\Sigma_h^r) \to \mathrm{Mod}(\Sigma_g)$. Fix a standard presentation of $\pi_1(\Sigma_h^r)$ as follows*:

$$\pi_1(\Sigma_h^r) = \langle a_1, b_1, \dots, a_h, b_h, c_1, \dots, c_r | \prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1 \rangle$$

and let τ_g be Meyer's signature cocycle. Then the signature of E is given by the formula

$$\sigma(E) = \sum_{i=1}^{h} \tau_g(\kappa_i, \beta_i) - \sum_{i=2}^{h} \tau_g(\kappa_1 \cdots \kappa_{i-1}, \kappa_i) - \sum_{j=1}^{r-1} \tau_g(\kappa_1 \cdots \kappa_h \gamma_1 \cdots \gamma_{j-1}, \gamma_j),$$

where $\alpha_i = \psi \circ \chi(a_i), \beta_i = \psi \circ \chi(b_i), \gamma_i = \psi \circ \chi(c_i)$ and $\kappa_i = [\alpha_i, \beta_i]$.

By applying this formula, we can compute the signatures of surface bundles obtained by taking out some neighborhoods of singular fibers from the Lefschetz fibrations constructed in Section 3. We used Mathematica for computing each term in the above formula.

Meyer also provided another interpretation of the above signature formula. For this, we start with the following diagram.

$$\begin{array}{ccccccccc} 1 & \to & \widetilde{R} & \to & \widetilde{F} & \xrightarrow{\widetilde{\pi}} & \pi_1(\Sigma_h) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow \chi & & \\ 1 & \to & R & \to & F & \xrightarrow{\pi} & \mathrm{Mod}(\Sigma_g) & \to & 1 \end{array}$$

Here, $\pi_1(\Sigma_h) = \langle a_1, \dots, a_h, b_1, \dots, b_h | \prod_{i=1}^{h} [a_i, b_i] = 1 \rangle$, $\widetilde{F} = \langle \widetilde{a_1}, \dots, \widetilde{a_h}, \widetilde{b_1}, \dots, \widetilde{b_h} \rangle$, $\widetilde{R}$ is the normal closure of $\widetilde{r} = \prod_{i=1}^{h} [\widetilde{a_i}, \widetilde{b_i}]$, and $\widetilde{\pi} : \widetilde{a_i} \mapsto a_i, \widetilde{b_i} \mapsto$

b_i. The second row corresponds to the finite presentation of $\mathrm{Mod}(\Sigma_g)$ due to Wajnryb. $F = F(S)$, where $S = \{y_1, y_2, u_1, \ldots, u_g, z_1, \ldots, z_{g-1}\}$ and R is the normal closure of $A^k_{i,j}$'s, B^k_i's, C^1, D^1, E^1 (cf. [6, §3]). If we have a monodromy representation $\chi : \pi_1(\Sigma_h) \to \mathrm{Mod}(\Sigma_g)$, then there exists a homomorphism $\widetilde{\chi} : \widetilde{F} \to F$ such that $\chi \circ \widetilde{\pi} = \pi \circ \widetilde{\chi}$ since π is surjective and $\widetilde{F}$ is free. Hence we have $\widetilde{\chi}(\widetilde{r}) \in R \cap [F, F]$. Now define the 1-cochain $c : F \to \mathbb{Z}$ cobounding the 2-cocycle $-\pi^*\psi^*(\tau_g)$ as follows.

$$c(x) := \sum_{j=1}^{m} \tau_g(\psi(\pi(\widetilde{x_{j-1}})), \psi(\pi(x_j))),$$

$$(x = \prod_{j=1}^{m} x_j, \quad x_j \in S \cup S^{-1}, \quad \widetilde{x_j} = \prod_{i=1}^{j} x_i).$$

Since $\pi^*\psi^*(\tau_g) \mid_{R\times R}= 0$, the restriction $c \mid_R: R \to \mathbb{Z}$ is a homomorphism. The values of c for the relations of Wajnryb's presentation were calculated in [6].

Theorem 4.3 ([26]). *Let $p : E \to \Sigma_h$ be a Σ_g-bundle over Σ_h and $\chi : \pi_1(\Sigma_h) \to \mathrm{Mod}(\Sigma_g)$ be its monodromy homomorphism. Then the signature of the total space E is given as follows*:

$$\sigma(E) = -c \mid_R (\widetilde{\chi}(\widetilde{r})) \quad (= -\langle \psi^*[\tau_g], \widetilde{\chi}(\widetilde{r})[R, F]\rangle),$$

where $\langle , \rangle$ is a pairing on the second cohomology and homology of $\mathrm{Mod}(\Sigma_g)$.

Now, we are ready to prove our main theorem.

Proof of Theorem 1.2. (a) We apply the subtraction operation to the Lefschetz fibrations $X \to \Sigma_3$, $Y_1 \to \Sigma_2$, and $Y_2 \to \Sigma_3$ constructed in Propositions 3.5 and Proposition 3.1. Let $N_1 \subset X$ be the neighborhood of four singular fibers with coinciding vanishing cycles and $N_2 \subset X$ be the neighborhood of two singular fibers with coinciding vanishing cycles. Then the complement $X \setminus N_1 \setminus N_2$ is the Σ_f bundle over Σ^2_3, and its signature can be computed by applying Theorem 4.2 to this bundle. More precisely to its monodromy representation $\chi : \pi_1(\Sigma^2_3) \to \mathrm{Mod}(\Sigma_f)$ given by $\chi(a_1) = (t^{-1}_{\delta_1} \cdot t_{t^{-1}_{\alpha_1}(\sigma_1)} \cdot t^{-1}_{\delta_2})^{t^{-2}_{\alpha_1}}$, $\chi(b_1) = (\phi_1)^{t^{-2}_{\alpha_1}}$, $\chi(a_2) = (t^{-1}_{\delta_1} \cdot t_{t^{-1}_{\alpha_1}(\sigma_1)} \cdot t^{-1}_{\delta_2})^{t^{-4}_{\alpha_1}}$, $\chi(b_2) = (\phi_2)^{t^{-4}_{\alpha_1}}$, $\chi(a_3) = (t^{-1}_{\delta_2} \cdot t_{t^{-1}_{\alpha_2}(\sigma_2)} \cdot t^{-1}_{\delta_3})^{t^{-4}_{\alpha_1} t^{-2}_{\alpha_2}}$, $\chi(b_3) = (\phi_3)^{t^{-4}_{\alpha_1} t^{-2}_{\alpha_2}}$, $\chi(c_1) = t^4_{\alpha_1}$, and $\chi(c_2) = t^2_{\alpha_2}$. Now, by computations using Mathematica we have $\tau(\kappa_1, \beta_1) = \tau(\kappa_2, \beta_2) = \tau(\kappa_3, \beta_3) = 2$, $-\tau(\kappa_1, \kappa_2) = -\tau(\kappa_1\kappa_2, \kappa_3) = -2$, and $-\tau(\kappa_1\kappa_2\kappa_3, \gamma_1) = 0$. Hence, $\sigma(X \setminus N_1 \setminus N_2) = 3 \cdot 2 + 2 \cdot (-2) + 0 = 2$. By taking out the neighborhood M_i of all singular fibers from Y_i (for $i = 1, 2$), we get $Y_i \setminus M_i$, the Σ_f bundles (one over Σ^1_2 and another over Σ^1_3), both with signature -1. For signature computation, we can directly apply Theorem 4.2 to these two bundles as above. Alternatively, we can first compute the signature of Lefschetz fibrations: $\sigma(Y_1) = -2$ and $\sigma(Y_2) = -4$ (cf. Proposition 15 and Proposition 16 in [8]). In order to compute the signature of taken out parts, apply Theorem 4.1 several times and use the fact that $\sigma(N(\text{a nonseparating singular fiber})) = 0$ (cf. [28]). From these, we have $\sigma(Y_1 \setminus M_1) = (-2)-(-1) = -1$ and $\sigma(Y_2 \setminus M_2) = (-4)-(-3) = -1$. Therefore, $X - Y_1 - Y_2$ is the $\Sigma_{f\geq 3}$ bundle over Σ_8, and $\sigma(X - Y_1 - Y_2) = \sigma(X \setminus N_1 \setminus N_2) + \sigma(\overline{Y_1 \setminus M_1}) + \sigma(\overline{Y_2 \setminus M_2}) = 2 + 1 + 1 = 4$ by Novikov additivity. Moreover, if we pullback this bundle (or, with opposite orientation) to unramified coverings of Σ_8 of degree $|n|$, then we get $b(f \geq 3, n) \leq 7|n| + 1$.

(b) Apply the subtraction operation to the Lefschetz fibrations $Z \to \Sigma_4$ and $Y_3 \to \Sigma_3$, constructed in Proposition 3.6 and Proposition 3.3, respectively. Then, $Z - Y_3$ is the required $\Sigma_{f\geq 5}$ bundle over Σ_7. Let N be the neighborhood of all singular fibers in Z and let M be the neighborhood of all singular fibers in Y_3. By applying Theorem 4.2 to two surface bundles $Z \setminus N$ and $Y_3 \setminus M$, we get $\sigma(Z - Y_3) = \sigma(Z \setminus N) + \sigma(\overline{Y_3 \setminus M}) = 2 + 2 = 4$. Let me give you another proof for verifying $\sigma(Z - Y_3) = 4$ using Theorem 4.3. From Proposition 3.6 and Proposition 3.3, we have $\widetilde{\chi}(\widetilde{r}) \equiv (E_2 \cdot L_5^{w_1} \cdot L_6^{w_2} \cdot L_5^{w_3} \cdot L_6^{t_\beta})(L_1 \cdot L_2^{t_y t_x t_z} \cdot L_3^{t_z} \cdot L_4)^g$ modulo commutativity and braid relations, where g is a self-homeomorphism of $\Sigma_{f\geq 5}$ such that $g(\alpha_3) = b$ and $g(\alpha_2) = c$. Moreover, from [21], $E_2 \equiv L_{10} \cdot (L_9 \cdot ((C^1)^{-1})^{z_0})^{z_1}$ for some mapping classes z_0, z_1, modulo commutativity and braid relations. Observe that for each L_i, four boundary curves are nonseparating and $\Sigma_f \setminus supp(L_i)$ is connected. Since the same holds for the relation $(D^1)^{-1}$, there exists a self-homeomorphism f_i of Σ_f sending the $\text{supp}((D^1)^{-1})$ to the $\text{supp}(L_i)$ for each i. Therefore, $\widetilde{\chi}(\widetilde{r}) \equiv ((D^1)^{-1})^{f_{10}}((D^1)^{-1})^{f_9\circ z_1} \cdot ((C^1)^{-1})^{z_0\circ z_1} \cdot ((D^1)^{-1})^{f_5\circ w_1} \cdot ((D^1)^{-1})^{f_6\circ w_2} \cdot ((D^1)^{-1})^{f_5\circ w_3} \cdot ((D^1)^{-1})^{f_6\circ t_\beta} \cdot ((D^1)^{-1})^{f_1\circ g} \cdot ((D^1)^{-1})^{f_2\circ(t_y t_x t_z)\circ g} \cdot ((D^1)^{-1})^{f_3\circ t_z\circ g} \cdot ((D^1)^{-1})^{f_4\circ g}$ modulo commutativity and braid relations and hence $\sigma(Z - Y_3) = -c(\widetilde{\chi}(\widetilde{r})) = c(C^1) + 10 \cdot c(D^1) = (-6) + 10 = 4$. For the upper bound for the genus function $b(f \geq 5, n)$, use the same argument as before.

(c) Apply the subtraction operation to the Lefschetz fibrations $W \to \Sigma_3$ and $Y_4 \to \Sigma_3$, constructed in Proposition 3.7 and Proposition 3.4, respectively. Then $W - Y_4$ is the required $\Sigma_{f\geq 6}$ bundle over Σ_6 with signature 4. From Proposition 3.4 and Proposition 3.7, $\widetilde{\chi}(\widetilde{r}) \equiv E_8 \cdot (L_1 \cdot L_2)^h$ modulo braid and commutativity relations, where h is a self-homeomorphism of Σ_f such that $h\{\beta_8, t_{\alpha_9}^{-1}(\sigma_9)\} = \{\beta, \gamma\}$. Moreover, $E_8 \equiv (\prod_{j=1}^{8}((D^1)^{-1})^{z_j}) \cdot ((C^1)^{-1})^{z_0}$ for some $z_0, \ldots, z_8$ (cf. [21] and Proposition 3.7). Therefore, $\sigma(W - Y_4) = -c(\widetilde{\chi}(\widetilde{r})) = c(C^1) + 10 \cdot c(D^1) = (-6) + 10 = 4$. For the upper bound for the genus function $b(f \geq 6, n)$, use the same argument as before. □

Proof of Theorem 1.4. Every odd genus surface is a covering of genus three surface. By Morita [27], after replacing a given surface bundle by a pullback to some covering of the base, the resulting surface bundle admits a fiberwise covering of any given degree. After applying this to the genus 3 surface bundle over $\Sigma_{b_3(1)}$ with signature 4 and the degree of the covering $\Sigma_f \to \Sigma_3$, we obtain $b_f(\frac{f-1}{2}n) \leq n(b_3(1) - 1) + 1$. Hence, $G_f := \lim_{n\to\infty} \frac{b_f(n)}{n} \leq \lim_{n\to\infty} \frac{2n(b_3(1)-1)+2}{(f-1)n} \leq \lim_{n\to\infty} \frac{14n+2}{(f-1)n} = \frac{14}{f-1}$. □

Remark 4.4. In [14, 22, 30], it was proven that $H_2(\mathrm{Mod}(\Sigma_g);\mathbb{Z}) \cong \mathbb{Z}$ for every $g \geq 4$ and $H_2(\mathrm{Mod}(\Sigma_g);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ for $g = 3$. Meyer [26] proved that each generator of $H_2(\mathrm{Mod}(\Sigma_g))/\mathrm{Tor}$ gives us signature 4 relying on the Theorem 4.3. In order to prove this, Meyer used Birman-Hilden's presentation of $\mathrm{Mod}(\Sigma_g)$, and Endo [6] reproved this using a simpler presentation due to Wajnryb [32]. By taking $\widetilde{\chi}(\widetilde{r})$ as different representatives for a generator of $H_2(\mathrm{Mod}(\Sigma_g))/\mathrm{Tor}$, we can construct various surface bundles with a fixed signature 4 as we have seen in the proof of Theorem 1.2. Therefore, the problem to determine $b(f,n)$ is to find the most effective representative $\tilde{\chi}(\tilde{r})$, in the sense of commutator length, for n times generator of $H_2(\mathrm{Mod}(\Sigma_f))/\mathrm{Tor}$.

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