

THE GROUP OF STRONG GALOIS OBJECTS ASSOCIATED TO A COCOMMUTATIVE HOPF QUASIGROUP

JOSE N. ALONSO ÁLVAREZ, RAMON GONZÁLEZ RODRÍGUEZ,
AND JOSE M. FERNÁNDEZ VILABOA

ABSTRACT. Let H be a cocommutative faithfully flat Hopf quasigroup in a strict symmetric monoidal category with equalizers. In this paper we introduce the notion of (strong) Galois H -object and we prove that the set of isomorphism classes of (strong) Galois H -objects is a (group) monoid which coincides, in the Hopf algebra setting, with the Galois group of H -Galois objects introduced by Chase and Sweedler.

Introduction

Let R be a commutative ring with unit. The notion of Galois H -object for a commutative, cocommutative Hopf R -algebra H , which is a finitely generated projective R -module, is due to Chase and Sweedler [7]. As was pointed by Beattie [4], although the discussion of Galois H -objects in [7] is limited to commutative algebras, the main properties can be easily extended to non commutative algebras. One of more relevant is the following: if H is cocommutative, the isomorphism classes of Galois H -objects form a group denoted by $Gal(R, H)$. The product in $Gal(R, H)$ is defined by the kernel of a suitable morphism and the class of H is the identity element. This construction can be extended to symmetric closed categories with equalizers and coequalizers working with monoids instead of algebras and some of the more important properties and exact sequences involving the group $Gal(R, H)$ were obtained in this categorical setting ([9], [13], [14]).

An interesting generalization of Hopf algebras are Hopf quasigroups introduced by Klim and Majid in [8] in order to understand the structure and relevant properties of the algebraic 7-sphere. They are not associative but the lack of this property is compensated by some axioms involving the antipode. The concept of Hopf quasigroup is a particular instance of the notion of unital coassociative H -bialgebra introduced in [11] and includes the example of an

Received February 20, 2016.

2010 *Mathematics Subject Classification.* 18D10, 17A01, 16T05, 81R50, 20N05.

Key words and phrases. monoidal category, unital magma, Hopf quasigroup, (strong) Galois H -object, Galois group, normal basis.

enveloping algebra $U(L)$ of a Malcev algebra (see [8]) as well as the notion of quasigroup algebra RL of an I.P. loop L . Then, quasigroups unify I.P. loops and Malcev algebras in the same way that Hopf algebras unified groups and Lie algebras.

In this paper we are interested to answer the following question: is it possible to extend the construction of $Gal(R, H)$ to the situation where H is a cocommutative Hopf quasigroup? in other words, can we construct in a non-associative setting a group of Galois H -objects? The main obstacle to define the group is the lack of associativity because we must work with unital magmas, i.e., objects where there exists a non-associative product with unit. As we can see in the first section of this paper, Hopf quasi groups are examples of these algebraic structures.

The paper is organized as follows. We begin introducing the notion of right H -comodule magma, where H is a Hopf quasigroup, and defining the product of right H -comodule magmas. In the second section we introduce the notions of Galois H -object and strong Galois H -objects proving that, with the product defined in the first section for comodule magmas, the set of isomorphism classes forms a monoid, in the case of Galois H -objects, and a group when we work with strong Galois H -objects. In this point it appears the main difference between our Galois H -objects and the ones associated to a Hopf algebra because in the Hopf algebra setting the inverse of the class of a Galois H -object A is the class of the opposite Galois H -object A^{op} , while in the quasigroup context this property fails. We only have the following: the product of A and A^{op} is isomorphic to H only as comodules. To obtain an isomorphism of magmas we need to work with strong Galois H -objects. Then, the strong condition appears in a natural way and we want to point out that in the classical case of Galois H -objects associated to a Hopf algebra H all of them are strong. Finally, in the last section, we study the connections between Galois H -objects and invertible comodules with geometric normal basis.

Throughout this paper \mathcal{C} denotes a strict symmetric monoidal category with equalizers where \otimes denotes the tensor product, K the unit object and c the symmetry isomorphism. We denote the class of objects of \mathcal{C} by $|\mathcal{C}|$ and for each object $M \in |\mathcal{C}|$, the identity morphism by $id_M : M \rightarrow M$. For simplicity of notation, given objects M , N and P in \mathcal{C} and a morphism $f : M \rightarrow N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$. We will say that $A \in |\mathcal{C}|$ is flat if the functor $A \otimes - : \mathcal{C} \rightarrow \mathcal{C}$ preserves equalizers. If moreover $A \otimes -$ reflects isomorphisms we say that A is faithfully flat.

By a unital magma in \mathcal{C} we understand a triple $A = (A, \eta_A, \mu_A)$ where A is an object in \mathcal{C} and $\eta_A : K \rightarrow A$ (unit), $\mu_A : A \otimes A \rightarrow A$ (product) are morphisms in \mathcal{C} such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$. If μ_A is associative, that is, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$, the unital magma will be called a monoid in \mathcal{C} . For any unital magma A with \bar{A} we will denote the opposite unital magma $(A, \eta_{\bar{A}} = \eta_A, \mu_{\bar{A}} = \mu_A \circ c_{A,A})$. Given two unital magmas (monoids) $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \rightarrow B$ is a morphism of

unital magmas (monoids) if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$. By duality, a counital comagma in \mathcal{C} is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in \mathcal{C} and $\varepsilon_D : D \rightarrow K$ (counit), $\delta_D : D \rightarrow D \otimes D$ (coproduct) are morphisms in \mathcal{C} such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$. If δ_D is coassociative, that is, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$, the counital comagma will be called a comonoid. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are counital comagmas (comonoids), $f : D \rightarrow E$ is morphism of counital magmas (comonoids) if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

Finally note that if A, B are unital magmas (monoids) in \mathcal{C} , the object $A \otimes B$ is a unital magma (monoid) in \mathcal{C} where $\eta_{A \otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$. With A^e we will denote the unital magma $\overline{A} \otimes A$. In a dual way, if D, E are counital comagmas (comonoids) in \mathcal{C} , $D \otimes E$ is a counital comagma (comonoid) in \mathcal{C} where $\varepsilon_{D \otimes E} = \varepsilon_D \otimes \varepsilon_E$ and $\delta_{D \otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

1. Comodule magmas for Hopf quasigroups

This first section is devoted to the study of the notion of H -comodule magma associated to a Hopf quasigroup H . We will show that, as in the Hopf algebra setting, it is possible to define a product using suitable equalizers which induces a monoidal structure in the category of flat H -comodule magmas.

The notion of Hopf quasigroup was introduced in [8] and the following is its monoidal version.

Definition 1.1. A Hopf quasigroup H in \mathcal{C} is a unital magma (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that the following axioms hold:

- (a1) ε_H and δ_H are morphisms of unital magmas.
- (a2) There exists $\lambda_H : H \rightarrow H$ in \mathcal{C} (called the antipode of H) such that:
 - (a2-1)
$$\begin{aligned} \mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) \\ &= \varepsilon_H \otimes H \\ &= \mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H) \circ (\delta_H \otimes H). \end{aligned}$$
 - (a2-2)
$$\begin{aligned} \mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H) \\ &= H \otimes \varepsilon_H \\ &= \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H). \end{aligned}$$

If H is a Hopf quasigroup, the antipode is unique, antimultiplicative, antimultiplicative and leaves the unit and the counit invariable:

$$(1) \quad \lambda_H \circ \mu_H = \mu_H \circ (\lambda_H \otimes \lambda_H) \circ c_{H,H}, \quad \delta_H \circ \lambda_H = c_{H,H} \circ (\lambda_H \otimes \lambda_H) \circ \delta_H,$$

$$(2) \quad \lambda_H \circ \eta_H = \eta_H, \quad \varepsilon_H \circ \lambda_H = \varepsilon_H$$

([8], Proposition 4.2 and [10], Proposition 1). Note that by (a2),

$$(3) \quad \mu_H \circ (\lambda_H \otimes id_H) \circ \delta_H = \mu_H \circ (id_H \otimes \lambda_H) \circ \delta_H = \varepsilon_H \otimes \eta_H.$$

A Hopf quasigroup H is cocommutative if $c_{H,H} \circ \delta_H = \delta_H$. In this case, as in the Hopf algebra setting, we have that $\lambda_H \circ \lambda_H = id_H$ (see Proposition 4.3 of [8]).

Let H and B be Hopf quasigroups. We say that $f : H \rightarrow B$ is a morphism of Hopf quasigroups if it is a morphism of unital magmas and comonoids. In this case $\lambda_B \circ f = f \circ \lambda_H$ (see Proposition 1.5 of [1]).

Examples 1.2. The notion of Hopf quasigroup was introduced in [8] and it can be interpreted as the linearization of the concept of quasigroup. A quasigroup is a set Q together with a product such that for any two elements $u, v \in Q$ the equations $ux = v$, $xu = v$ and $uv = x$ have unique solutions in Q . A quasigroup L which contains an element e_L such that $ue_L = u = e_Lu$ for every $u \in L$ is called a loop. A loop L is said to be a loop with the inverse property (for brevity an I.P. loop) if and only if, to every element $u \in L$, there corresponds an element $u^{-1} \in L$ such that the equations $u^{-1}(uv) = v = (vu)u^{-1}$ hold for every $v \in L$.

If L is an I.P. loop, it is easy to show (see [5]) that for all $u \in L$ the element u^{-1} is unique and $u^{-1}u = e_L = uu^{-1}$. Moreover, for all $u, v \in L$, the equality $(uv)^{-1} = v^{-1}u^{-1}$ holds.

Let R be a commutative ring and L and I.P. loop. Then, by Proposition 4.7 of [8], we know that

$$RL = \bigoplus_{u \in L} Ru$$

is a cocommutative Hopf quasigroup with product given by the linear extension of the one defined in L and

$$\delta_{RL}(u) = u \otimes u, \quad \varepsilon_{RL}(u) = 1_R, \quad \lambda_{RL}(u) = u^{-1}$$

on the basis elements.

Now we briefly describe another example of Hopf quasigroup constructed working with Malcev algebras (see [12] for details). Consider a commutative and associative ring K with $\frac{1}{2}$ and $\frac{1}{3}$ in K . A Malcev algebra $(M, [,])$ over K is a free module in $K\text{-Mod}$ with a bilinear anticommutative operation $[,]$ on M satisfying that $[J(a, b, c), a] = J(a, b, [a, c])$, where $J(a, b, c) = [[a, b], c] - [[a, c], b] - [a, [b, c]]$ is the Jacobian in a, b, c . Denote by $U(M)$ the not necessarily associative algebra defined as the quotient of $K\{M\}$, the free non-associative algebra on a basis of M , by the ideal $I(M)$ generated by the set $\{ab - ba - [a, b], (a, x, y) + (x, a, y), (x, a, y) + (x, y, a) : a, b \in M, x, y \in K\{M\}\}$, where $(x, y, z) = (xy)z - x(yz)$ is the usual additive associator.

By Proposition 4.1 of [12] and Proposition 4.8 of [8], the diagonal map $\delta_{U(M)} : U(M) \rightarrow U(M) \otimes U(M)$ defined by $\delta_{U(M)}(x) = 1 \otimes x + x \otimes 1$ for all $x \in M$, and the map $\varepsilon_{U(M)} : U(M) \rightarrow K$ defined by $\varepsilon_{U(M)}(x) = 0$ for all $x \in M$, both extended to $U(M)$ as morphisms of unital magmas; together with the map $\lambda_{U(M)} : U(M) \rightarrow U(M)$, defined by $\lambda_{U(M)}(x) = -x$ for all

$x \in M$ and extended to $U(M)$ as an antimultiplicative morphism, provide a cocommutative Hopf quasigroup structure on $U(M)$.

Definition 1.3. Let H be a Hopf quasigroup and let A be a unital magma (monoid) with a right coaction $\rho_A : A \rightarrow A \otimes H$. We will say that $\mathbb{A} = (A, \rho_A)$ is a right H -comodule magma (monoid) if (A, ρ_A) is a right H -comodule (i.e., $(\rho_A \otimes H) \circ \rho_A = (A \otimes \delta_H) \circ \rho_A$, $(A \otimes \varepsilon_H) \circ \rho_A = id_A$), and the following identities

- (b1) $\rho_A \circ \eta_A = \eta_A \otimes \eta_H$,
- (b2) $\rho_A \circ \mu_A = \mu_{A \otimes H} \circ (\rho_A \otimes \rho_A)$,

hold.

Obviously, if H is a Hopf quasigroup, the pair $\mathbb{H} = (H, \delta_H)$ is an example of right H -comodule magma.

Let \mathbb{A}, \mathbb{B} be right H -comodule magmas (monoids). A morphism of right H -comodule magmas (monoids) $f : \mathbb{A} \rightarrow \mathbb{B}$ is a morphism $f : A \rightarrow B$ in \mathcal{C} of unital magmas (monoids) and right H -comodules, that is $(f \otimes H) \circ \rho_A = \rho_B \circ f$.

Remark 1.4. Note that, if H is cocommutative, every endomorphism $\alpha : \mathbb{H} \rightarrow \mathbb{H}$ of right H -comodule magmas is an isomorphism. Indeed: First note that by the comodule condition and the cocommutativity of H we have $\alpha = ((\varepsilon_H \circ \alpha) \otimes H) \circ \delta_H = (H \otimes (\varepsilon_H \circ \alpha)) \circ \delta_H$ and then $\alpha' = (H \otimes (\varepsilon_H \circ \alpha \circ \lambda_H)) \circ \delta_H$ is the inverse of α because by the properties of H :

$$\begin{aligned} \alpha' \circ \alpha &= \alpha \circ \alpha' = (H \otimes (((\varepsilon_H \circ \alpha) \otimes (\varepsilon_H \circ \alpha \circ \lambda_H)) \circ \delta_H)) \circ \delta_H \\ &= (H \otimes (\varepsilon_H \circ \alpha \circ \mu_H \circ (H \otimes \lambda_H) \circ \delta_H)) \circ \delta_H \\ &= id_H. \end{aligned}$$

Proposition 1.5. Let H be a Hopf quasigroup and \mathbb{A}, \mathbb{B} right H -comodule magmas. The pairs $\mathbb{A} \otimes_1 \mathbb{B} = (A \otimes B, \rho_{A \otimes B}^1 = (A \otimes c_{H,B}) \circ (\rho_A \otimes B))$, $\mathbb{A} \otimes_2 \mathbb{B} = (A \otimes B, \rho_{A \otimes B}^2 = A \otimes \rho_B)$ are right H -comodule magmas. Moreover $\mathbb{A} \otimes_1 \mathbb{B}$ and $\mathbb{B} \otimes_2 \mathbb{A}$ are isomorphic right H -comodule magmas.

Proof. We give the proof only for $\mathbb{A} \otimes_1 \mathbb{B}$. The calculus for $\mathbb{A} \otimes_2 \mathbb{B}$ are analogous and we left to the reader. First note that the object $A \otimes B$ is a unital magma in \mathcal{C} . On the other hand, the pair $(A \otimes B, \rho_{A \otimes B}^1)$ is a right H -comodule because trivially $(A \otimes B \otimes \varepsilon_H) \circ \rho_{A \otimes B}^1 = id_{A \otimes B}$ and using the naturality of c we obtain that $(\rho_{A \otimes B}^1 \otimes H) \circ \rho_{A \otimes B}^1 = (A \otimes \delta_H) \circ \rho_{A \otimes B}^1$. Moreover, $\rho_{A \otimes B}^1 \circ \eta_{A \otimes B} = \eta_{A \otimes B} \otimes \eta_H$ and also by the naturality of c we have $\rho_{A \otimes B}^1 \circ \mu_{A \otimes B} = (\mu_{A \otimes B} \otimes \mu_H) \circ (A \otimes B \otimes c_{H,A \otimes B} \otimes H) \circ (\rho_{A \otimes B}^1 \otimes \rho_{A \otimes B}^1)$. Finally, $c_{A,B}$ is an isomorphism of right H -comodule magmas between $\mathbb{A} \otimes_1 \mathbb{B}$ and $\mathbb{B} \otimes_2 \mathbb{A}$ because by the naturally of c we obtain that $c_{A,B} \circ \eta_{A \otimes B} = \eta_{B \otimes A}$, $\mu_{B \otimes A} \circ (c_{A,B} \otimes c_{A,B}) = c_{A,B} \circ \mu_{A \otimes B}$ and $\rho_{B \otimes A}^2 \circ c_{A,B} = (c_{A,B} \otimes H) \circ \rho_{A \otimes B}^1$. \square

Proposition 1.6. Let H be a cocommutative Hopf quasigroup and \mathbb{A} a right H -comodule magma. Then $\overline{\mathbb{A}} = (\overline{A}, \rho_{\overline{A}} = (A \otimes \lambda_H) \circ \rho_A)$ is a right H -comodule magma.

Proof. Trivially $(A \otimes \varepsilon_H) \circ \rho_{\overline{A}} = id_A$. Using that H is cocommutative and (1) we obtain $(\rho_{\overline{A}} \otimes H) \circ \rho_{\overline{A}} = (A \otimes \delta_H) \circ \rho_{\overline{A}}$. Moreover by (b1) of Definition 1.3 and (2), the identity $\rho_{\overline{A}} \circ \eta_A = \eta_A \otimes \eta_H$ holds. Finally, by the naturality of c , (b2) of Definition 1.3 and (1) the equality $\rho_{\overline{A}} \circ \mu_{\overline{A}} = \mu_{\overline{A} \otimes H} \circ (\rho_{\overline{A}} \otimes \rho_{\overline{A}})$ follows easily. \square

Proposition 1.7. *Let H be a Hopf quasigroup and \mathbb{A}, \mathbb{B} right H -comodule magmas. The object $A \bullet B$ defined by the equalizer diagram*

$$A \bullet B \xrightarrow{i_{A \bullet B}} A \otimes B \begin{array}{c} \xrightarrow{\rho_{A \otimes B}^1} \\ \xrightarrow{\rho_{A \otimes B}^2} \end{array} A \otimes B \otimes H,$$

where $\rho_{A \otimes B}^1$ and $\rho_{A \otimes B}^2$ are the morphisms defined in Proposition 1.5, is a unital magma where $\eta_{A \bullet B}$ and $\mu_{A \bullet B}$ are the factorizations through $i_{A \bullet B}$ of the morphisms $\eta_{A \otimes B}$ and $\mu_{A \otimes B} \circ (i_{A \bullet B} \otimes i_{A \bullet B})$ respectively. Moreover, if H is flat and the coaction $\rho_{A \bullet B} : A \bullet B \rightarrow A \bullet B \otimes H$ is the factorization of $\rho_{A \otimes B}^2 \circ i_{A \bullet B}$ through $i_{A \bullet B} \otimes H$, the pair $\mathbb{A} \bullet \mathbb{B} = (A \bullet B, \rho_{A \bullet B})$ is a right H -comodule magma.

Proof. Trivially $\rho_{A \otimes B}^1 \circ \eta_{A \otimes B} = \eta_A \otimes \eta_B \otimes \eta_H = \rho_{A \otimes B}^2 \circ \eta_{A \otimes B}$. Therefore, there exists a unique morphism $\eta_{A \bullet B} : K \rightarrow A \bullet B$ such that $i_{A \bullet B} \circ \eta_{A \bullet B} = \eta_{A \otimes B}$. On the other hand, using the properties of ρ_A and ρ_B and the naturality of c we have

$$\begin{aligned} & \rho_{A \otimes B}^1 \circ \mu_{A \otimes B} \circ (i_{A \bullet B} \otimes i_{A \bullet B}) \\ &= (\mu_{A \otimes B} \otimes \mu_H) \circ (A \otimes B \otimes c_{H, A \otimes B} \otimes H) \circ ((\rho_{A \otimes B}^1 \circ i_{A \bullet B}) \otimes (\rho_{A \otimes B}^1 \circ i_{A \bullet B})) \\ &= (\mu_{A \otimes B} \otimes \mu_H) \circ (A \otimes B \otimes c_{H, A \otimes B} \otimes H) \circ ((\rho_{A \otimes B}^2 \circ i_{A \bullet B}) \otimes (\rho_{A \otimes B}^2 \circ i_{A \bullet B})) \\ &= \rho_{A \otimes B}^2 \circ \mu_{A \otimes B} \circ (i_{A \bullet B} \otimes i_{A \bullet B}). \end{aligned}$$

Then, there exists a unique morphism $\mu_{A \bullet B} : A \bullet B \otimes A \bullet B \rightarrow A \bullet B$ such that $i_{A \bullet B} \circ \mu_{A \bullet B} = \mu_{A \otimes B} \circ (i_{A \bullet B} \otimes i_{A \bullet B})$. Moreover, $\mu_{A \bullet B} \circ (\eta_{A \bullet B} \otimes A \bullet B) = id_{A \bullet B} = \mu_{A \bullet B} \circ (A \bullet B \otimes \eta_{A \bullet B})$ because $i_{A \bullet B} \circ \mu_{A \bullet B} \circ (\eta_{A \bullet B} \otimes A \bullet B) = i_{A \bullet B} = i_{A \bullet B} \circ \mu_{A \bullet B} \circ (A \bullet B \otimes \eta_{A \bullet B})$. Therefore, $A \bullet B$ is a unital magma.

Moreover,

$$A \bullet B \otimes H \xrightarrow{i_{A \bullet B} \otimes H} A \otimes B \otimes H \begin{array}{c} \xrightarrow{\rho_{A \otimes B}^1 \otimes H} \\ \xrightarrow{\rho_{A \otimes B}^2 \otimes H} \end{array} A \otimes B \otimes H \otimes H$$

is an equalizer diagram, because $- \otimes H$ preserves equalizers, and by the properties of ρ_A and ρ_B and the naturality of c we obtain $(\rho_{A \otimes B}^1 \otimes H) \circ \rho_{A \otimes B}^2 \circ i_{A \bullet B} = (\rho_{A \otimes B}^2 \otimes H) \circ \rho_{A \otimes B}^2 \circ i_{A \bullet B}$. As a consequence, there exists a unique morphism $\rho_{A \bullet B} : A \bullet B \rightarrow A \bullet B \otimes H$ such that $(i_{A \bullet B} \otimes H) \circ \rho_{A \bullet B} = \rho_{A \otimes B}^2 \circ i_{A \bullet B}$. Then, the pair $(A \bullet B, \rho_{A \bullet B})$ is a right H -comodule because $(i_{A \bullet B} \otimes \varepsilon_H) \circ \rho_{A \bullet B} = i_{A \bullet B}$ and also $((i_{A \bullet B} \otimes H) \circ \rho_{A \bullet B}) \otimes H = (i_{A \bullet B} \otimes \delta_H) \circ \rho_{A \bullet B}$. Finally, (b1) and (b2) of Definition 1.3 follow, by a similar reasoning, from $(i_{A \bullet B} \otimes H) \circ$

$$\rho_{A \bullet B} \circ \eta_{A \bullet B} = (i_{A \bullet B} \otimes H) \circ (\eta_{A \bullet B} \otimes \eta_H) \text{ and } (i_{A \bullet B} \otimes H) \circ \rho_{A \bullet B} \circ \mu_{A \bullet B} = (i_{A \bullet B} \otimes H) \circ \mu_{A \bullet B \otimes H} \circ (\rho_{A \bullet B} \otimes \rho_{A \bullet B}). \quad \square$$

Proposition 1.8. *Let H be a flat Hopf quasigroup and $f : \mathbb{A} \rightarrow \mathbb{B}$, $g : \mathbb{T} \rightarrow \mathbb{D}$ morphisms of right H -comodule magmas. Then the morphism $f \bullet g : A \bullet T \rightarrow B \bullet D$, obtained as the factorization of $(f \otimes g) \circ i_{A \bullet T} : A \bullet T \rightarrow B \otimes D$ through the equalizer $i_{B \bullet D}$, is a morphism of right H -comodule magmas between $\mathbb{A} \bullet \mathbb{T}$ and $\mathbb{B} \bullet \mathbb{D}$. Moreover, if f and g are isomorphisms, so is $f \bullet g$.*

Proof. Using that f and g are comodule morphisms we obtain $\rho_{B \otimes D}^1 \circ (f \otimes g) \circ i_{A \bullet T} = \rho_{B \otimes D}^2 \circ (f \otimes g) \circ i_{A \bullet T}$ and as a consequence there exist a unique morphism $(f \bullet g) : A \bullet T \rightarrow B \bullet D$ such that $i_{B \bullet D} \circ (f \bullet g) = (f \otimes g) \circ i_{A \bullet T}$. The morphism $f \bullet g$ is a morphism of unital magmas because $i_{B \bullet D} \circ \eta_{B \bullet D} = i_{B \bullet D} \circ (f \bullet g) \circ \eta_{A \bullet T}$ and for the product the equality $i_{B \bullet D} \circ \mu_{B \bullet D} \circ ((f \bullet g) \otimes (f \bullet g)) = i_{B \bullet D} \circ (f \bullet g) \circ \mu_{A \bullet T}$ holds. Also, it is a comodule morphism because $(i_{B \bullet D} \otimes H) \circ \rho_{B \bullet D} \circ (f \bullet g) = (i_{B \bullet D} \otimes H) \circ ((f \bullet g) \otimes H) \circ \rho_{A \bullet T}$.

Finally, it is easy to show that, if f and g are isomorphisms, $f \bullet g$ is an isomorphism with inverse $f^{-1} \bullet g^{-1}$. \square

Proposition 1.9. *Let H be a flat Hopf quasigroup and \mathbb{A}, \mathbb{B} right H -comodule magmas. Then $\mathbb{A} \bullet \mathbb{B}$ and $\mathbb{B} \bullet \mathbb{A}$ are isomorphic as right H -comodule magmas.*

Proof. First note that by the naturality of c and the properties of the equaliser morphism $i_{A \bullet B}$ we have that $\rho_{B \otimes A}^1 \circ c_{A,B} \circ i_{A \bullet B} = \rho_{B \otimes A}^2 \circ c_{A,B} \circ i_{A \bullet B}$ and then there exists a morphism $\tau_{A,B} : A \bullet B \rightarrow B \bullet A$ such that $i_{B \bullet A} \circ \tau_{A,B} = c_{A,B} \circ i_{A \bullet B}$. Also there exists a unique morphism $\tau_{B,A} : B \bullet A \rightarrow A \bullet B$ such that $i_{A \bullet B} \circ \tau_{B,A} = c_{B,A} \circ i_{B \bullet A}$. Then $i_{A \bullet B} \circ \tau_{B,A} \circ \tau_{A,B} = c_{B,A} \circ c_{A,B} \circ i_{A \bullet B} = i_{A \bullet B}$ and similarly $i_{B \bullet A} \circ \tau_{A,B} \circ \tau_{B,A} = i_{B \bullet A}$. Thus $\tau_{A,B}$ is an isomorphism with inverse $\tau_{B,A}$. Moreover, $i_{B \bullet A} \circ \tau_{A,B} \circ \eta_{A \bullet B} = c_{A,B} \circ i_{A \bullet B} \circ \eta_{A \bullet B} = \eta_{B \otimes A} = i_{B \bullet A} \circ \eta_{B \bullet A}$ and

$$\begin{aligned} i_{B \bullet A} \circ \tau_{A,B} \circ \mu_{A \bullet B} &= \mu_{B \otimes A} \circ ((c_{A,B} \circ i_{A \bullet B}) \otimes (c_{A,B} \circ i_{A \bullet B})) \\ &= \mu_{B \otimes A} \circ ((i_{B \bullet A} \circ \tau_{A,B}) \otimes (i_{B \bullet A} \circ \tau_{A,B})) \\ &= i_{B \bullet A} \circ \mu_{B \bullet A} \circ (\tau_{A,B} \otimes \tau_{A,B}). \end{aligned}$$

Therefore, $\tau_{A,B}$ is a morphism of unital magmas and finally it is a morphism of right H -comodules because $((i_{B \bullet A} \circ \tau_{A,B}) \otimes H) \circ \rho_{A \bullet B} = (i_{B \bullet A} \otimes H) \circ \rho_{B \bullet A} \circ \tau_{A,B}$. \square

Proposition 1.10. *Let H be a flat Hopf quasigroup and $\mathbb{A}, \mathbb{B}, \mathbb{D}$ right H -comodule magmas such that A and D are flat. Then $\mathbb{A} \bullet (\mathbb{B} \bullet \mathbb{D})$ and $(\mathbb{A} \bullet \mathbb{B}) \bullet \mathbb{D}$ are isomorphic as right H -comodule magmas.*

Proof. First, note that

$$A \otimes B \bullet D \xrightarrow{A \otimes i_{B \bullet D}} A \otimes B \otimes D \begin{array}{c} \xrightarrow{A \otimes \rho_{B \otimes D}^1} \\ \xrightarrow{A \otimes \rho_{B \otimes D}^2} \end{array} A \otimes B \otimes D \otimes H$$

and

$$A \bullet B \otimes D \xrightarrow{i_{A \bullet B \otimes D}} A \otimes B \otimes D \xrightarrow[(A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^2 \otimes D)]{(A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^1 \otimes D)} A \otimes B \otimes D \otimes H$$

are equalizer diagrams because A and D are flat and $A \otimes B \otimes c_{H,D}$ an isomorphism. On the other hand, it is easy to show that

$$(A \otimes i_{B \bullet D} \otimes H) \circ \rho_{A \otimes B \bullet D}^1 = (A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^1 \otimes D) \circ (A \otimes i_{B \bullet D})$$

and

$$(A \otimes i_{B \bullet D} \otimes H) \circ \rho_{A \otimes B \bullet D}^2 = (A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^2 \otimes D) \circ (A \otimes i_{B \bullet D}).$$

Therefore

$$\begin{aligned} & (A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^1 \otimes D) \circ (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \\ &= (A \otimes B \otimes c_{H,D}) \circ (\rho_{A \otimes B}^2 \otimes D) \circ (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \end{aligned}$$

and as a consequence there exists a unique morphism $h : A \bullet (B \bullet D) \rightarrow (A \bullet B) \otimes D$ such that

$$(4) \quad (i_{A \bullet B} \otimes D) \circ h = (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)}.$$

The diagram

$$A \bullet (B \bullet D) \xrightarrow{h} (A \bullet B) \otimes D \xrightarrow[\rho_{A \bullet B \otimes D}^2]{\rho_{A \bullet B \otimes D}^1} A \bullet B \otimes D \otimes H$$

is an equalizer diagram. Indeed, it is easy to see that $\rho_{A \bullet B \otimes D}^1 \circ h = \rho_{A \bullet B \otimes D}^1 \circ h$ and, if $f : C \rightarrow A \bullet B \otimes D$ is a morphism such that $\rho_{(A \bullet B) \otimes D}^1 \circ f = \rho_{(A \bullet B) \otimes D}^2 \circ f$, we have that

$$(A \otimes \rho_{B \otimes D}^1) \circ (i_{A \bullet B} \otimes D) \circ f = (A \otimes \rho_{B \otimes D}^2) \circ (i_{A \bullet B} \otimes D) \circ f$$

because

$$(A \otimes \rho_{B \otimes D}^1) \circ (i_{A \bullet B} \otimes D) = (i_{A \bullet B} \otimes D \otimes H) \circ \rho_{A \bullet B \otimes D}^1$$

and

$$(A \otimes \rho_{B \otimes D}^2) \circ (i_{A \bullet B} \otimes D) = (i_{A \bullet B} \otimes D \otimes H) \circ \rho_{A \bullet B \otimes D}^2.$$

Then, there exists a unique morphism $t : C \rightarrow A \otimes B \bullet D$ such that $(A \otimes i_{B \bullet D}) \circ t = (i_{A \bullet B} \otimes D) \circ f$. The morphism t factorizes through the equalizer $i_{A \bullet (B \bullet D)}$ because

$$(A \otimes i_{B \bullet D} \otimes H) \circ \rho_{A \otimes B \bullet D}^1 \circ t = (A \otimes i_{B \bullet D} \otimes H) \circ \rho_{A \otimes B \bullet D}^2 \circ t$$

and then

$$\rho_{A \otimes B \bullet D}^1 \circ t = \rho_{A \otimes B \bullet D}^2 \circ t$$

holds. Thus, there exists a unique morphism $g : C \rightarrow A \bullet (B \bullet D)$ satisfying the equality $i_{A \bullet (B \bullet D)} \circ g = t$. As a consequence

$$(i_{A \bullet B} \otimes D) \circ h \circ g = (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \circ g$$

$$= (A \otimes i_{B \bullet D}) \circ t = (i_{A \bullet B} \otimes D) \circ f$$

and then $h \circ g = f$. Moreover, g is the unique morphism such that $h \circ g = t$, because if $d : C \rightarrow A \bullet (B \bullet D)$ satisfies $h \circ d = f$, we obtain that $i_{A \bullet (B \bullet D)} \circ d = t$ and therefore $d = g$.

As a consequence, there exists an isomorphism $n_{A,B,C} : A \bullet (B \bullet D) \rightarrow (A \bullet B) \bullet D$ such that

$$(5) \quad i_{(A \bullet B) \bullet D} \circ n_{A,B,D} = h.$$

The isomorphism $n_{A,B,C}$ is a morphism of unital magmas because by (4), (5) and the naturality of c we have

$$\begin{aligned} & (i_{A \bullet B} \otimes D) \circ i_{(A \bullet B) \bullet D} \circ n_{A,B,D} \circ \eta_{A \bullet (B \bullet D)} \\ &= (i_{A \bullet B} \otimes D) \circ h \circ \eta_{A \bullet (B \bullet D)} \\ &= (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \circ \eta_{A \bullet (B \bullet D)} \\ &= \eta_A \otimes \eta_B \otimes \eta_D \\ &= (i_{A \bullet B} \otimes D) \circ i_{(A \bullet B) \bullet D} \circ \eta_{(A \bullet B) \bullet D} \end{aligned}$$

and

$$\begin{aligned} & (i_{A \bullet B} \otimes D) \circ i_{(A \bullet B) \bullet D} \circ n_{A,B,D} \circ \mu_{A \bullet (B \bullet D)} \\ &= (i_{A \bullet B} \otimes D) \circ h \circ \mu_{A \bullet (B \bullet D)} \\ &= (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \circ \mu_{A \bullet (B \bullet D)} \\ &= (A \otimes i_{B \bullet D}) \circ \mu_{A \otimes (B \bullet D)} \circ (i_{A \bullet (B \bullet D)} \otimes i_{A \bullet (B \bullet D)}) \\ &= \mu_{A \otimes B \otimes D} \circ (((A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)}) \otimes ((A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)})) \\ &= \mu_{A \otimes B \otimes D} \circ (((i_{A \bullet B} \otimes D) \circ h) \otimes ((i_{A \bullet B} \otimes D) \circ h)) \\ &= (i_{A \bullet B} \otimes D) \circ \mu_{A \bullet B \otimes D} \circ (h \otimes h) \\ &= (i_{A \bullet B} \otimes D) \circ \mu_{A \bullet B \otimes D} \circ ((i_{(A \bullet B) \bullet D} \circ n_{A,B,C}) \otimes (i_{(A \bullet B) \bullet D} \circ n_{A,B,C})) \\ &= (i_{A \bullet B} \otimes D) \circ i_{(A \bullet B) \bullet D} \circ \mu_{(A \bullet B) \bullet D} \circ (n_{A,B,D} \otimes n_{A,B,D}). \end{aligned}$$

Finally, using a similar reasoning, we obtain that $n_{A,B,C}$ is a morphism of right H -comodules because

$$\begin{aligned} & (i_{A \bullet B} \otimes D \otimes H) \circ (i_{(A \bullet B) \bullet D} \otimes H) \circ \rho_{(A \bullet B) \bullet D} \circ n_{A,B,D} \\ &= (A \otimes B \otimes \rho_D) \circ (i_{A \bullet B} \otimes D) \circ i_{(A \bullet B) \bullet D} \circ n_{A,B,C} \\ &= (A \otimes B \otimes \rho_D) \circ (i_{A \bullet B} \otimes D) \circ h \\ &= (A \otimes B \otimes \rho_D) \circ (A \otimes i_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \\ &= (A \otimes i_{B \bullet D} \otimes H) \circ (A \otimes \rho_{B \bullet D}) \circ i_{A \bullet (B \bullet D)} \\ &= (A \otimes i_{B \bullet D} \otimes H) \circ (i_{A \bullet (B \bullet D)} \otimes H) \circ \rho_{A \bullet (B \bullet D)} \\ &= (i_{A \bullet B} \otimes D \otimes H) \circ (h \otimes H) \circ \rho_{A \bullet (B \bullet D)} \\ &= (i_{A \bullet B} \otimes D \otimes H) \circ (i_{(A \bullet B) \bullet D} \otimes H) \circ (n_{A,B,D} \otimes H) \circ \rho_{A \bullet (B \bullet D)}. \quad \square \end{aligned}$$

Proposition 1.11. *Let H be a cocommutative Hopf quasigroup and \mathbb{A} a right H -comodule magma. Then*

$$(6) \quad A \xrightarrow{\rho_A} A \otimes H \begin{array}{c} \xrightarrow{\rho_{A \otimes H}^1} \\ \xrightarrow{\rho_{A \otimes H}^2} \end{array} A \otimes H \otimes H,$$

is an equalizer diagram. If H is flat $\mathbb{A} \bullet \mathbb{H}$ and \mathbb{A} are isomorphic as right H -comodule magmas.

Proof. We will begin by showing that (6) is an equalizer diagram. Indeed, if H is cocommutative we have that $\rho_{A \otimes H}^1 \circ \rho_A = (A \otimes (c_{H,H} \circ \delta_H)) \circ \rho_A = \rho_{A \otimes H}^2 \circ \rho_A$. Moreover, if there exists a morphism $f : D \rightarrow A \otimes H$ such that $\rho_{A \otimes H}^1 \circ f = \rho_{A \otimes H}^2 \circ f$, we have that $\rho_A \circ (A \otimes \varepsilon_H) \circ f = f$ and if $g : D \rightarrow A$ is a morphism such that $\rho_A \circ g = f$ this gives $g = (A \otimes \varepsilon_H) \circ f$. Therefore there is a unique isomorphism $r_A : A \bullet H \rightarrow A$ satisfying $\rho_A \circ r_A = i_{A \bullet H}$.

In the second step we show that r_A is a morphism of right H -comodule magmas. Trivially, $r_A \circ \eta_{A \bullet H} = \eta_A$ because $\rho_A \circ r_A \circ \eta_{A \bullet H} = i_{A \bullet H} \circ \eta_{A \bullet H} = \eta_A \otimes \eta_H = \rho_A \circ \eta_A$. Also $\mu_A \circ (r_A \otimes r_A) = r_A \circ \mu_{A \bullet H}$ because $\rho_A \circ \mu_A \circ (r_A \otimes r_A) = \rho_A \circ r_A \circ \mu_{A \bullet H}$ and as a consequence r_A is a morphism of unital magmas. Finally, the H -comodule condition follows from $((\rho_A \circ r_A) \otimes H) \circ \rho_{A \bullet H} = (\rho_A \otimes H) \circ \rho_A \circ r_A$. \square

Remark 1.12. Note that, under the conditions of the previous proposition, the coaction for $\mathbb{A} \bullet \mathbb{H}$ is $i_{A \bullet B}$. On the other hand, Proposition 1.11 gives that

$$H \xrightarrow{\delta_H} H \otimes H \begin{array}{c} \xrightarrow{\rho_{H \otimes H}^1} \\ \xrightarrow{\rho_{H \otimes H}^2} \end{array} A \otimes H \otimes H,$$

is an equalizer diagram.

Proposition 1.13. *Let H be a cocommutative Hopf quasigroup and $\text{Mag}_f(\mathcal{C}, H)$ be the category whose objects are flat H -comodule magmas and whose arrows are the morphism of H -comodule magmas. Then $\text{Mag}_f(\mathcal{C}, H)$ is a symmetric monoidal category.*

Proof. The category $\text{Mag}_f(\mathcal{C}, H)$ is a monoidal category with the tensor product defined by the product “ \bullet ” introduced in Proposition 1.7, with unit \mathbb{H} , with associative constraints $\mathbf{a}_{\mathbb{A}, \mathbb{B}, \mathbb{D}} = n_{\mathbb{A}, \mathbb{B}, \mathbb{D}}^{-1}$, where $n_{\mathbb{A}, \mathbb{B}, \mathbb{D}}$ is the isomorphism defined in Proposition 1.10, and right unit constraints and left unit constraints $\mathbf{r}_A = r_A$, $\mathbf{l}_A = r_A \circ \tau_{H, A}$ respectively, where r_A is the isomorphism defined in Proposition 1.11 and $\tau_{H, A}$ the one defined in Proposition 1.9. It is easy but tedious, and we leave the details to the reader, to show that associative constraints and right and left unit constraints are natural and satisfy the Pentagon Axiom and the Triangle Axiom. Finally the tensor product of two morphisms

is defined by Proposition 1.8 and, of course, the symmetry isomorphism is the transformation τ defined in Proposition 1.9. \square

2. The group of strong Galois objects

The aim of this section is to introduce the notion of strong Galois H -object for a cocommutative Hopf quasigroup H . We will prove that the set of isomorphism classes of strong H -Galois objects is a group that becomes the classical Galois group when H is a cocommutative Hopf algebra.

Definition 2.1. Let H be a Hopf quasigroup and \mathbb{A} a right H -comodule magma. We will say that \mathbb{A} is a Galois H -object if

- (c1) A is faithfully flat.
- (c2) The canonical morphism $\gamma_A = (\mu_A \otimes H) \circ (A \otimes \rho_A) : A \otimes A \rightarrow A \otimes H$ is an isomorphism.

If moreover, $f_A = \gamma_A^{-1} \circ (\eta_A \otimes H) : H \rightarrow A^e$ is a morphism of unital magmas, we will say that \mathbb{A} is a strong Galois H -object.

A morphism between to (strong) Galois H -objects is a morphism of right H -comodule magmas.

Note that if \mathbb{A} is a strong Galois H -object and \mathbb{B} is a Galois H -object isomorphic to \mathbb{A} as Galois H -objects, then \mathbb{B} is also a strong Galois H -object because if $g : A \rightarrow B$ is the isomorphism, we have $\gamma_B \circ (g \otimes g) = (g \otimes H) \circ \gamma_A$ and it follows that $f_B = (g \otimes g) \circ f_A$. Then, f_B is a morphism of unital magmas and \mathbb{B} is strong.

Example 2.2. If H is a faithfully flat Hopf quasigroup, \mathbb{H} is a strong Galois H -object because $\gamma_H = (\mu_H \otimes H) \circ (H \otimes \delta_H)$ is an isomorphism with inverse $\gamma_H^{-1} = ((\mu_H \circ (H \otimes \lambda_H)) \otimes H) \circ (H \otimes \delta_H)$ and $f_H = (\lambda_H \otimes H) \circ \delta_H : H \rightarrow H^e$ is a morphism of unital magmas.

Remark 2.3. If H is a Hopf algebra and \mathbb{A} is a right H -comodule monoid, we say that \mathbb{A} is a Galois H -object when A is faithfully flat and the canonical morphism γ_A is an isomorphism. In this setting every Galois H -object is a strong Galois H -object because

$$\begin{aligned} & \gamma_A \circ \mu_{A^e} \circ ((\gamma_A^{-1} \circ (\eta_A \otimes H)) \otimes (\gamma_A^{-1} \circ (\eta_A \otimes H))) \\ &= (\mu_A \otimes H) \circ (A \otimes \mu_{A \otimes H}) \circ (A \otimes (\gamma_A \circ \gamma_A^{-1} \circ (\eta_A \otimes H))) \otimes \rho_A \\ & \quad \circ (c_{H,A} \otimes A) \circ (H \otimes (\gamma_A^{-1} \circ (\eta_A \otimes H))) \\ &= (A \otimes \mu_H) \circ (c_{H,A} \otimes H) \circ (H \otimes (\gamma_A \circ \gamma_A^{-1} \circ (\eta_A \otimes H))) \\ &= \eta_A \otimes \mu_H \\ &= \gamma_A \circ \gamma_A^{-1} \circ (\eta_A \otimes \mu_H), \end{aligned}$$

where the equalities follow by (b2) of Definition 1.3, the naturality of c and the associativity of μ_A .

Proposition 2.4. *Let H be a Hopf quasigroup and \mathbb{A} a Galois H -object. Then*

$$(7) \quad \begin{array}{ccccc} K & \xrightarrow{\eta_A} & A & \begin{array}{c} \xrightarrow{\rho_A} \\ \xrightarrow{A \otimes \eta_H} \end{array} & A \otimes H \end{array}$$

is an equalizer diagram.

Proof. First note that

$$A \xrightarrow{A \otimes \eta_A} A \otimes A \begin{array}{c} \xrightarrow{A \otimes A \otimes \eta_A} \\ \xrightarrow{A \otimes \eta_A \otimes A} \end{array} A \otimes A \otimes A$$

is an equalizer diagram. Then, using that A is faithfully flat, so is

$$K \xrightarrow{\eta_A} A \begin{array}{c} \xrightarrow{A \otimes \eta_A} \\ \xrightarrow{\eta_A \otimes A} \end{array} A \otimes A.$$

On the other hand, $\gamma_A \circ (A \otimes \eta_A) = A \otimes \eta_H$, $\gamma_A \circ (\eta_A \otimes A) = \rho_A$. Therefore, if γ_A is an isomorphism, (7) is an equalizer diagram. \square

Lemma 2.5. *Let H be a Hopf quasigroup and let \mathbb{A} a Galois H -object. The following equalities hold:*

- (i) $\rho_{A \otimes A}^2 \circ \gamma_A^{-1} = (\gamma_A^{-1} \otimes H) \circ (A \otimes \delta_H)$.
- (ii) $\rho_{A \otimes A}^1 \circ \gamma_A^{-1} = (\gamma_A^{-1} \otimes H) \circ (A \otimes c_{H,H}) \circ (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H))$.

Proof. The proof for (i) follows from the identity $(\gamma_A \otimes H) \circ \rho_{A \otimes A}^2 = (A \otimes \delta_H) \circ \gamma_A$. To obtain (ii), first we prove that

$$(8) \quad (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H)) \circ \gamma_A = (A \otimes c_{H,H}) \circ (\gamma_A \otimes H) \circ \rho_{A \otimes A}^1.$$

Indeed,

$$\begin{aligned} & (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H)) \circ \gamma_A \\ &= (A \otimes (\mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes \delta_H)) \otimes H) \circ (\mu_A \otimes H \otimes \delta_H) \\ & \quad \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\ &= (\mu_A \otimes H \otimes ((\varepsilon_H \otimes H) \circ \delta_H)) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\ &= (\mu_A \otimes H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\ &= (\mu_A \otimes c_{H,H}) \circ (\rho_{A \otimes A}^2 \otimes H) \circ \rho_{A \otimes A}^1 \\ &= (A \otimes c_{H,H}) \circ (\gamma_A \otimes H) \circ \rho_{A \otimes A}^1, \end{aligned}$$

where the first equality follows by the comodule condition for A , the second one by (a2-2) of Definition 1.1, the third one by the counit condition, the fourth and the last ones by the symmetry of c and the naturality of the braiding.

Then, by (8) we obtain

$$\begin{aligned} & \rho_{A \otimes A}^1 \circ \gamma_A^{-1} \\ &= (\gamma_A^{-1} \otimes H) \circ (A \otimes (c_{H,H} \circ c_{H,H})) \circ (\gamma_A \otimes H) \circ \rho_{A \otimes A}^1 \circ \gamma_A^{-1} \\ &= (\gamma_A^{-1} \otimes H) \circ (A \otimes c_{H,H}) \circ (A \otimes \mu_H \otimes H) \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H)) \end{aligned}$$

and (ii) holds. \square

Proposition 2.6. *Let H be a cocommutative faithfully flat Hopf quasigroup. The following assertions hold:*

- (i) *If \mathbb{A} and \mathbb{B} are Galois H -objects so is $\mathbb{A} \bullet \mathbb{B}$.*
- (ii) *If \mathbb{A} and \mathbb{B} are strong Galois H -objects so is $\mathbb{A} \bullet \mathbb{B}$.*

Proof. First we prove (i). Let \mathbb{A} and \mathbb{B} be Galois H -objects. By Proposition 1.7 we know that $A \bullet B$ is a unital magma where $\eta_{A \bullet B}$ and $\mu_{A \bullet B}$ are the factorizations through $i_{A \bullet B}$ of the morphisms $\eta_{A \otimes B}$ and $\mu_{A \otimes B} \circ (i_{A \bullet B} \otimes i_{A \bullet B})$ respectively. Moreover, using that H is flat we have that $A \bullet B$ is a right H -comodule magma where the coaction $\rho_{A \bullet B} : A \bullet B \rightarrow A \bullet B \otimes H$ is the factorization of $\rho_{A \otimes B}^2 \circ i_{A \bullet B}$ (or $\rho_{A \otimes B}^1 \circ i_{A \bullet B}$) through $i_{A \bullet B} \otimes H$.

The objects A and B are faithfully flat and then so is $A \otimes B$. Therefore

$$A \otimes B \otimes A \bullet B \xrightarrow{A \otimes B \otimes i_{A \bullet B}} A \otimes B \otimes A \otimes B \begin{array}{c} \xrightarrow{A \otimes B \otimes \rho_{A \otimes B}^1} \\ \xrightarrow{A \otimes B \otimes \rho_{A \otimes B}^2} \end{array} A \otimes B \otimes A \otimes B \otimes H$$

is an equalizer diagram. On the other hand, if H is cocommutative

$$A \otimes B \otimes H \xrightarrow{A \otimes B \otimes \delta_H} A \otimes B \otimes H \otimes H \begin{array}{c} \xrightarrow{A \otimes B \otimes \rho_{H \otimes H}^1} \\ \xrightarrow{A \otimes B \otimes \rho_{H \otimes H}^2} \end{array} A \otimes B \otimes H \otimes H \otimes H$$

is an equalizer diagram (see Remark 1.12).

Let $\Gamma_{A \otimes B} : A \otimes B \otimes A \otimes B \rightarrow A \otimes B \otimes H \otimes H$ be the morphism defined by

$$\Gamma_{A \otimes B} = (A \otimes c_{H,B} \otimes H) \circ (\gamma_A \otimes \gamma_B) \circ (A \otimes c_{B,A} \otimes B).$$

Trivially $\Gamma_{A \otimes B}$ is an isomorphism with inverse

$$\Gamma_{A \otimes B}^{-1} = (A \otimes c_{A,B} \otimes B) \circ (\gamma_A^{-1} \otimes \gamma_B^{-1}) \circ (A \otimes c_{B,H} \otimes H)$$

and satisfies

$$\begin{aligned} & (A \otimes B \otimes \rho_{H \otimes H}^1) \circ \Gamma_{A \otimes B} \circ (A \otimes B \otimes i_{A \bullet B}) \\ &= (\mu_{A \otimes B} \otimes \rho_{H \otimes H}^1) \circ (A \otimes B \otimes (((\rho_{A \otimes B}^1 \circ i_{A \bullet B}) \otimes H) \circ \rho_{A \bullet B})) \\ &= (\mu_{A \otimes B} \otimes \rho_{H \otimes H}^1) \circ (A \otimes B \otimes ((i_{A \bullet B} \otimes H \otimes H) \circ (\rho_{A \bullet B} \otimes H) \circ \rho_{A \bullet B})) \\ &= (\mu_{A \otimes B} \otimes H \otimes (c_{H,H} \circ \delta_H)) \circ (A \otimes B \otimes ((i_{A \bullet B} \otimes \delta_H) \circ \rho_{A \bullet B})) \\ &= (\mu_{A \otimes B} \otimes H \otimes \delta_H) \circ (A \otimes B \otimes ((i_{A \bullet B} \otimes H \otimes H) \circ (\rho_{A \bullet B} \otimes H) \circ \rho_{A \bullet B})) \\ &= (A \otimes B \otimes \rho_{H \otimes H}^2) \circ \Gamma_{A \otimes B} \circ (A \otimes B \otimes i_{A \bullet B}), \end{aligned}$$

where the first equality follows by the naturality of c and the properties of $\rho_{A \bullet B}$, the second and the third ones by the comodule structure of $A \bullet B$, the fourth one by the cocommutativity of H and the last one was obtained repeating the same calculus with $\rho_{H \otimes H}^2$.

As a consequence, there exists a unique morphism $h : A \otimes B \otimes A \bullet B \rightarrow A \otimes B \otimes H$ such that

$$(9) \quad (A \otimes B \otimes \delta_H) \circ h = \Gamma_{A \otimes B} \circ (A \otimes B \otimes i_{A \bullet B}).$$

On the other hand, in an analogous way the morphism $\Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H) : A \otimes B \otimes H \rightarrow A \otimes B \otimes A \bullet B$ factorizes through through the equalizer $A \otimes B \otimes i_{A \bullet B}$ because by (i) of Lemma 2.5, the naturality and symmetry of c and the cocommutativity of H we have

$$\begin{aligned} & (A \otimes B \otimes \rho_{A \otimes B}^1) \circ \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H) \\ &= (A \otimes c_{A,B} \otimes c_{H,B}) \circ (A \otimes A \otimes c_{H,B} \otimes B) \circ (((A \otimes \rho_A) \circ \gamma_A^{-1}) \otimes \gamma_B^{-1}) \\ & \quad \circ (A \otimes c_{B,H} \otimes H) \circ (A \otimes B \otimes \delta_H) \\ &= (A \otimes c_{A,B} \otimes c_{H,B}) \circ (A \otimes A \otimes c_{H,B} \otimes B) \circ (((\gamma_A^{-1} \otimes H) \circ (A \otimes \delta_H)) \otimes \gamma_B^{-1}) \\ & \quad \circ (A \otimes c_{B,H} \otimes H) \circ (A \otimes B \otimes \delta_H) \\ &= (A \otimes c_{A,B} \otimes B \otimes H) \circ (\gamma_A^{-1} \otimes ((\gamma_B^{-1} \otimes H) \circ (B \otimes \delta_H))) \circ (A \otimes c_{B,H} \otimes H) \\ & \quad \circ (A \otimes B \otimes \delta_H) \\ &= (A \otimes B \otimes \rho_{A \otimes B}^2) \circ \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H). \end{aligned}$$

Thus, let g be the unique morphism such that

$$(10) \quad (A \otimes B \otimes i_{A \bullet B}) \circ g = \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H).$$

By (9) and (10)

$$\begin{aligned} (A \otimes B \otimes \delta_H) \circ h \circ g &= \Gamma_{A \otimes B} \circ (A \otimes B \otimes i_{A \bullet B}) \circ g \\ &= \Gamma_{A \otimes B} \circ \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H) \\ &= A \otimes B \otimes \delta_H, \\ (A \otimes B \otimes i_{A \bullet B}) \circ g \circ h &= \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H) \circ h \\ &= \Gamma_{A \otimes B}^{-1} \circ \Gamma_{A \otimes B} \circ (A \otimes B \otimes i_{A \bullet B}) \\ &= A \otimes B \otimes i_{A \bullet B} \end{aligned}$$

and then we obtain that h is an isomorphism with inverse g . As a consequence $A \bullet B$ is faithfully flat because A , B and H are faithfully flat.

The morphism $\Gamma_{A \otimes B}^{-1} \circ (i_{A \bullet B} \otimes \delta_H) : A \bullet B \otimes H \rightarrow A \otimes B \otimes A \bullet B$ admits a factorization $\alpha_{A,B} : A \bullet B \otimes H \rightarrow A \otimes B \otimes A \bullet B$ through the equalizer $A \otimes B \otimes i_{A \bullet B}$ because as we saw in the previous lines $\Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes \delta_H)$ admits a factorization through $A \otimes B \otimes i_{A \bullet B}$.

Now consider the equalizer diagram:

$$A \bullet B \otimes A \bullet B \xrightarrow{i_{A \bullet B} \otimes A \bullet B} A \otimes B \otimes A \bullet B \begin{array}{c} \xrightarrow{\rho_{A \otimes B}^1 \otimes A \bullet B} \\ \xrightarrow{\rho_{A \otimes B}^2 \otimes A \bullet B} \end{array} A \otimes B \otimes H \otimes A \bullet B$$

We have that

$$\begin{aligned} & (\rho_{A \otimes B}^1 \otimes i_{A \bullet B}) \circ \alpha_{A,B} \\ &= (\rho_{A \otimes B}^1 \otimes A \otimes B) \circ (A \otimes c_{A,B} \otimes B) \circ (\gamma_A^{-1} \otimes \gamma_B^{-1}) \circ (A \otimes c_{B,H} \otimes H) \\ & \quad \circ (i_{A \bullet B} \otimes \delta_H) \\ &= (A \otimes ((B \otimes c_{A,H}) \circ (c_{A,B} \otimes H) \circ (A \otimes c_{H,B})) \otimes B) \\ & \quad \circ (((\rho_{A \otimes H}^1 \circ \gamma_A^{-1}) \otimes \gamma_B^{-1}) \circ (A \otimes c_{B,H} \otimes H) \circ (i_{A \bullet B} \otimes \delta_H)) \\ &= (A \otimes ((B \otimes c_{A,H}) \circ (c_{A,B} \otimes H) \circ (A \otimes c_{H,B})) \otimes B) \\ & \quad \circ (((\gamma_A^{-1} \otimes H) \circ (A \otimes c_{H,H}) \circ (A \otimes \mu_H \otimes H)) \\ & \quad \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H))) \otimes \gamma_B^{-1} \circ (A \otimes c_{B,H} \otimes H) \circ (i_{A \bullet B} \otimes \delta_H) \\ &= (A \otimes ((B \otimes c_{A,H}) \circ (c_{A,B} \otimes H) \circ (A \otimes c_{H,B})) \otimes B) \circ (\gamma_A^{-1} \otimes H \otimes \gamma_B^{-1}) \\ & \quad \circ (A \otimes ((c_{H,H} \otimes B) \circ (\mu_H \otimes c_{B,H}) \circ (H \otimes c_{B,H} \otimes H) \\ & \quad \circ (c_{B,H} \otimes ((\lambda_H \otimes H) \circ \delta_H))) \otimes H) \circ ((\rho_{A \otimes B}^1 \circ i_{A \bullet B}) \otimes \delta_H) \\ &= (A \otimes ((B \otimes c_{A,H}) \circ (c_{A,B} \otimes H) \circ (A \otimes c_{H,B})) \otimes B) \circ (\gamma_A^{-1} \otimes H \otimes B \otimes B) \\ & \quad \circ (A \otimes ((c_{H,H} \otimes \gamma_B^{-1}) \circ (H \otimes c_{B,H} \otimes H) \circ (c_{B,H} \otimes H \otimes H) \\ & \quad \circ (B \otimes ((\mu_H \circ (H \otimes \lambda_H)) \otimes H) \otimes H))) \\ & \quad \circ ((\rho_{A \otimes B}^2 \circ i_{A \bullet B}) \otimes ((\delta_H \otimes H) \circ \delta_H)) \\ &= (A \otimes ((c_{H,B} \otimes A) \circ (H \otimes c_{A,B}) \circ (c_{A,H} \otimes B)) \otimes B) \circ (\gamma_A^{-1} \otimes H \otimes \gamma_B^{-1}) \\ & \quad \circ (A \otimes ((H \otimes c_{B,H} \otimes H) \circ (c_{B,H} \otimes (c_{H,H} \circ c_{H,H})) \\ & \quad \circ (B \otimes c_{H,H} \otimes H))) \circ (A \otimes B \otimes \mu_H \otimes (c_{H,H} \circ \delta_H)) \\ & \quad \circ ((\rho_{A \otimes B}^2 \circ i_{A \bullet B}) \otimes ((\lambda_H \otimes H) \circ \delta_H)) \\ &= (A \otimes ((B \otimes c_{A,H} \otimes B) \circ (c_{A,B} \otimes c_{B,H})) \circ (\gamma_A^{-1} \otimes \gamma_B^{-1} \otimes H) \\ & \quad \circ (A \otimes c_{B,H} \otimes c_{H,H}) \circ (A \otimes B \otimes (c_{H,H} \circ (\mu_H \otimes H)) \otimes H) \\ & \quad \circ (((A \otimes \rho_B) \circ i_{A \bullet B}) \otimes ((\lambda_H \otimes (c_{H,H} \circ \delta_H)) \circ \delta_H)) \\ &= (A \otimes ((B \otimes c_{A,H} \otimes B) \circ (c_{A,B} \otimes c_{B,H}))) \circ (\gamma_A^{-1} \otimes ((\gamma_B^{-1} \otimes H) \\ & \quad \circ (B \otimes c_{H,H}) \circ (B \otimes \mu_H \otimes H) \circ (\rho_B \otimes ((\lambda_H \otimes H) \circ \delta_H)))) \\ & \quad \circ (A \otimes c_{B,H} \otimes H) \circ (i_{A \bullet B} \otimes (c_{H,H} \circ \delta_H)) \\ &= (A \otimes ((B \otimes c_{A,H}) \circ (c_{A,B} \otimes H) \circ (A \otimes \rho_B)) \otimes B) \circ (\gamma_A^{-1} \otimes \gamma_B^{-1}) \\ & \quad \circ (A \otimes c_{B,H} \otimes H) \circ (i_{A \bullet B} \otimes \delta_H) \\ &= (\rho_{A \otimes B}^2 \otimes i_{A \bullet B}) \circ \alpha_{A,B}, \end{aligned}$$

where the first equality follows by the definition, the second, the fourth and the fifth ones by the naturality and symmetry of c , the third and the ninth ones by (ii) of Lemma 2.5, the sixth one by the cocommutativity of H and, finally, the eighth and the tenth ones by the naturality of c .

Then, there exists a unique morphism $\beta_{A,B} : A \bullet B \otimes H \rightarrow A \bullet B \otimes A \bullet B$ such that

$$(11) \quad (i_{A \bullet B} \otimes A \bullet B) \circ \beta_{A,B} = \alpha_{A,B}$$

and then

$$(12) \quad (i_{A \bullet B} \otimes i_{A \bullet B}) \circ \beta_{A,B} = \Gamma_{A \otimes B}^{-1} \circ (i_{A \bullet B} \otimes \delta_H).$$

The morphism $\beta_{A,B}$ satisfies

$$\begin{aligned} & (i_{A \bullet B} \otimes H) \circ \gamma_{A \bullet B} \circ \beta_{A,B} \\ &= (\mu_{A \otimes B} \otimes H) \circ (i_{A \bullet B} \otimes ((i_{A \bullet B} \otimes H) \circ \rho_{A \bullet B})) \circ \beta_{A,B} \\ &= (\mu_{A \otimes B} \otimes H) \circ (i_{A \bullet B} \otimes ((A \otimes \rho_B) \circ i_{A \bullet B})) \circ \beta_{A,B} \\ &= (\mu_A \otimes \gamma_B) \circ (A \otimes c_{B,A} \otimes B) \circ \Gamma_{A \otimes B}^{-1} \circ (i_{A \bullet B} \otimes \delta_H) \\ &= (((A \otimes \varepsilon_H) \circ \gamma_A \circ \gamma_A^{-1}) \otimes B \otimes H) \circ (A \otimes c_{B,H} \otimes H) \circ (i_{A \bullet B} \otimes \delta_H) \\ &= i_{A \bullet B} \otimes H \end{aligned}$$

and by the cocommutativity of H we have

$$\begin{aligned} & (i_{A \bullet B} \otimes i_{A \bullet B}) \circ \beta_{A,B} \circ \gamma_{A \bullet B} \\ &= \Gamma_{A \otimes B}^{-1} \circ (i_{A \bullet B} \otimes \delta_H) \circ (\mu_{A \bullet B} \otimes H) \circ (A \bullet B \otimes \rho_{A \bullet B}) \\ &= \Gamma_{A \otimes B}^{-1} \circ (\mu_{A \otimes B} \otimes \delta_H) \circ (i_{A \bullet B} \otimes ((A \otimes \rho_B) \circ i_{A \bullet B})) \\ &= \Gamma_{A \otimes B}^{-1} \circ (A \otimes B \otimes c_{H,H}) \circ (\mu_A \otimes \gamma_B \otimes H) \circ (A \otimes c_{B,A} \otimes \rho_B) \\ & \quad \circ (i_{A \bullet B} \otimes i_{A \bullet B}) \\ &= (A \otimes c_{A,B} \otimes B) \circ (\gamma_A^{-1} \otimes B \otimes B) \circ (A \otimes c_{B,H} \otimes B) \\ & \quad \circ (\mu_A \otimes B \otimes (c_{B,H} \circ \rho_B)) \circ (A \otimes c_{B,A} \otimes B) \circ (i_{A \bullet B} \otimes i_{A \bullet B}) \\ &= (A \otimes c_{A,B} \otimes B) \circ (\gamma_A^{-1} \otimes B \otimes B) \circ (\mu_A \otimes c_{B,H} \otimes B) \\ & \quad \circ (A \otimes c_{B,A} \otimes H \otimes B) \circ (i_{A \bullet B} \otimes ((\rho_A \otimes B) \circ i_{A \bullet B})) \\ &= (((A \otimes c_{A,B}) \circ ((\gamma_A^{-1} \circ \gamma_A) \otimes B)) \otimes B) \circ (A \otimes c_{B,A} \otimes B) \circ (i_{A \bullet B} \otimes i_{A \bullet B}) \\ &= i_{A \bullet B} \otimes i_{A \bullet B}. \end{aligned}$$

Taking into account that H is flat and that $A \bullet B$ is faithfully flat we obtain that $\beta_{A,B}$ is the inverse of the canonical morphism $\gamma_{A \bullet B}$.

Now we assume that \mathbb{A} and \mathbb{B} are strong Galois H -objects. To prove that $\mathbb{A} \bullet \mathbb{B}$ is a strong Galois H -object we only need to show that $f_{A \bullet B} : H \rightarrow (A \bullet B)^e$ is a morphism of unital magmas. If f_A and f_B are morphisms of unital magmas, by the properties of $i_{A \bullet B}$ and the naturality of c we have

$$(i_{A \bullet B} \otimes i_{A \bullet B}) \circ f_{A \bullet B} \circ \eta_H = (A \otimes c_{A,B} \otimes B) \circ ((f_A \circ \eta_H) \otimes (f_B \circ \eta_H))$$

$$= \eta_A \otimes \eta_B \otimes \eta_A \otimes \eta_B = (i_{A \bullet B} \otimes i_{A \bullet B}) \circ \eta_{(A \bullet B)^e}$$

and

$$\begin{aligned} & (i_{A \bullet B} \otimes i_{A \bullet B}) \circ \mu_{(A \bullet B)^e} \circ (f_{A \bullet B} \otimes f_{A \bullet B}) \\ &= (A \otimes c_{A,B} \otimes B) \circ ((\mu_{A^e} \circ (f_A \otimes f_A)) \otimes (\mu_{B^e} \circ (f_B \otimes f_B))) \circ \delta_{H \otimes H} \\ &= (A \otimes c_{A,B} \otimes B) \circ (f_A \otimes f_B) \circ \delta_H \circ \mu_H = (i_{A \bullet B} \otimes i_{A \bullet B}) \circ f_{A \bullet B} \circ \mu_H. \end{aligned}$$

Therefore, $f_{A \bullet B} \circ \eta_H = \eta_{(A \bullet B)^e}$ and $\mu_{(A \bullet B)^e} \circ (f_{A \bullet B} \otimes f_{A \bullet B}) = f_{A \bullet B} \circ \mu_H$. \square

Proposition 2.7. *Let H be a cocommutative Hopf quasigroup and \mathbb{A} a Galois H -object. Then the right H -comodule magma $\overline{\mathbb{A}}$ defined in Proposition 1.6 is a Galois H -object. Moreover, if \mathbb{A} is strong so is $\overline{\mathbb{A}}$.*

Proof. To prove that $\overline{\mathbb{A}}$ is a Galois H -object we only need to show that $\gamma_{\overline{\mathbb{A}}}$ is an isomorphism. We begin by proving the following identity:

$$(13) \quad (A \otimes (\mu_H \circ c_{H,H} \circ (\lambda_H \otimes H))) \circ (\rho_A \otimes H) \circ \gamma_A = \gamma_{\overline{\mathbb{A}}} \circ c_{A,A}.$$

Indeed:

$$\begin{aligned} & (A \otimes (\mu_H \circ c_{H,H} \circ (\lambda_H \otimes H))) \circ (\rho_A \otimes H) \circ \gamma_A \\ &= (\mu_A \otimes (\mu_H \circ (H \otimes \mu_H) \circ (c_{H,H} \otimes H) \circ (H \otimes c_{H,H}) \circ (c_{H,H} \otimes H) \\ & \quad \circ (\lambda_H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H))) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\ &= (\mu_A \otimes (\mu_H \circ (\lambda_H \otimes \mu_H) \circ (\delta_H \otimes H) \circ c_{H,H} \circ (\lambda_H \otimes H))) \\ & \quad \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\ &= (\mu_A \otimes H) \circ (A \otimes c_{H,A}) \circ (\rho_{\overline{\mathbb{A}}} \otimes A) \\ &= \gamma_{\overline{\mathbb{A}}} \circ c_{A,A}, \end{aligned}$$

where the first equality follows by (b2) of Definition 1.3, (1) and the naturality of c , the second one by the cocommutativity of H and the naturality of c , the third one by (a2-1) of Definition 1.1 and the last one by the symmetry and naturality of c .

Define the morphism $\gamma'_{\overline{\mathbb{A}}}: A \otimes H \rightarrow A \otimes A$ by

$$(14) \quad \gamma'_{\overline{\mathbb{A}}} = c_{A,A} \circ \gamma_{\overline{\mathbb{A}}}^{-1} \circ (A \otimes (\mu_H \circ c_{H,H})) \circ (\rho_A \otimes H).$$

Then, by (13), the naturality of c , the cocommutativity of H and (a2-2) of Definition 1.1, we have the following:

$$\begin{aligned} & \gamma_{\overline{\mathbb{A}}} \circ \gamma'_{\overline{\mathbb{A}}} \\ &= \gamma_{\overline{\mathbb{A}}} \circ c_{A,A} \circ \gamma_{\overline{\mathbb{A}}}^{-1} \circ (A \otimes (\mu_H \circ c_{H,H})) \circ (\rho_A \otimes H) \\ &= (A \otimes (\mu_H \circ c_{H,H} \circ (\lambda_H \otimes H))) \circ (\rho_A \otimes (\mu_H \circ c_{H,H})) \circ (\rho_A \otimes H) \\ &= (A \otimes (\mu_H \circ (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (c_{H,H} \otimes H) \circ (\lambda_H \otimes c_{H,H}) \\ & \quad \circ (\delta_H \otimes H))) \circ (\rho_A \otimes H) \\ &= (A \otimes ((\mu_H \circ (\mu_H \otimes H) \circ (H \otimes \lambda_H \otimes H) \circ (H \otimes \delta_H)) \circ c_{H,H})) \circ (\rho_A \otimes H) \\ &= id_{A \otimes H}. \end{aligned}$$

Moreover, by a similar reasoning but using (a2-1) of Definition 1.1 instead of (a2-2) we obtain

$$\begin{aligned}
& \gamma_{\overline{A}} \circ \gamma_{\overline{A}} \\
&= c_{A,A} \circ \gamma_{\overline{A}}^{-1} \circ ((\mu_A \circ c_{A,A}) \otimes (\mu_H \circ (H \otimes \mu_H) \circ (H \otimes \lambda_H \otimes H)) \\
&\quad \circ (\delta_H \otimes H) \circ c_{H,H}) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \\
&= c_{A,A} \circ \gamma_{\overline{A}}^{-1} \circ \gamma_{\overline{A}} \circ c_{A,A} \\
&= id_{A \otimes A}.
\end{aligned}$$

Therefore, $\gamma_{\overline{A}}$ is an isomorphism and \overline{A} a Galois H -object.

Finally, it is easy to show that $f_{\overline{A}} = c_{A,A} \circ f_A$. Then, if f_A is a morphism of unital magmas, so is $f_{\overline{A}}$. Thus if \mathbb{A} is strong, $\overline{\mathbb{A}}$ is strong. \square

Proposition 2.8. *Let H be a cocommutative flat Hopf quasigroup and \mathbb{A} a Galois H -object. Then $\mathbb{A} \bullet \overline{\mathbb{A}}$ is isomorphic to \mathbb{H} as right H -comodules. Moreover, if \mathbb{A} is strong, the previous isomorphism is a morphism of right H -comodule magmas.*

Proof. First note that, by Proposition 2.4, we know that (7) is an equalizer diagram and then so is

$$H \xrightarrow{\eta_A \otimes H} A \otimes H \xrightleftharpoons[A \otimes \eta_H \otimes H]{\rho_A \otimes H} A \otimes H \otimes H$$

because H is flat. For the morphism $\gamma_{\overline{A}} \circ i_{A \bullet \overline{A}} : A \bullet \overline{A} \rightarrow A \otimes H$ we have the following:

$$\begin{aligned}
& (\rho_A \otimes H) \circ \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}} \\
&= (\mu_{\overline{A} \otimes \overline{H}} \otimes \lambda_H) \circ (\rho_A \otimes ((A \otimes \delta_H) \circ \rho_A)) \circ i_{A \bullet \overline{A}} \\
&= (\mu_{\overline{A} \otimes \overline{H}} \otimes \lambda_H) \circ (A \otimes H \otimes ((A \otimes \delta_H) \circ \rho_A)) \\
&\quad \circ (A \otimes (c_{A,H} \circ (A \otimes \lambda_H) \circ \rho_A)) \circ i_{A \bullet \overline{A}} \\
&= (\mu_{\overline{A}} \otimes \mu_{\overline{H}} \otimes \lambda_H) \circ (A \otimes ((A \otimes H \otimes \delta_H) \circ (A \otimes c_{H,H})) \\
&\quad \circ (A \otimes ((H \otimes \lambda_H) \circ \delta_H))) \circ (A \otimes \rho_A) \circ i_{A \bullet \overline{A}} \\
&= (\mu_{\overline{A}} \otimes ((\mu_H \circ ((\lambda_H \otimes H) \circ \delta_H) \otimes \lambda_H) \circ \delta_H)) \circ (A \otimes \rho_A) \circ i_{A \bullet \overline{A}} \\
&= (A \otimes \eta_H \otimes H) \circ \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}},
\end{aligned}$$

where the first equality follows by the naturality of c and (b2) of Definition 1.3, the second one because $\rho_{A \otimes \overline{A}}^1 \circ i_{A \bullet \overline{A}} = \rho_{A \otimes \overline{A}}^2 \circ i_{A \bullet \overline{A}}$, the third one relies on the symmetry and the naturality of c , the fourth one follows by (1) and the last one by (3).

Therefore, there exists a unique morphism $h_A : A \bullet \overline{A} \rightarrow H$ such that

$$(15) \quad (\eta_A \otimes H) \circ h_A = \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}}.$$

The morphism h_A is a right comodule morphism because by the cocommutativity of H , (1) and the comodule properties of A , we have

$$\begin{aligned}
& \eta_A \otimes ((h_A \otimes H) \circ \rho_{A \bullet \overline{A}}) \\
&= ((\gamma_{\overline{A}} \circ i_{A \bullet \overline{A}}) \otimes H) \circ \rho_{A \bullet \overline{A}} \\
&= ((\mu_A \circ c_{A,A}) \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ (A \otimes \rho_A) \circ i_{A \bullet \overline{A}} \\
&= ((\mu_A \circ c_{A,A}) \otimes ((\lambda_H \otimes \lambda_H) \circ c_{H,H} \circ \delta_H)) \circ (A \otimes \rho_A) \circ i_{A \bullet \overline{A}} \\
&= ((\mu_A \circ c_{A,A}) \otimes (\delta_H \circ \lambda_H)) \circ (A \otimes \rho_A) \circ i_{A \bullet \overline{A}} \\
&= (A \otimes \delta_H) \circ \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}} \\
&= \eta_A \otimes (\delta_H \circ h_A)
\end{aligned}$$

and using that $\eta_A \otimes H \otimes H$ is an equalizer morphism we obtain $(h \otimes H) \circ \rho_{A \bullet \overline{A}} = \delta_H \circ h_A$.

On the other hand, for $f_{\overline{A}}: H \rightarrow A \otimes A$ we have the following

$$\begin{aligned}
& (\gamma_{\overline{A}} \otimes H) \circ \rho_{A \otimes \overline{A}}^1 \circ f_{\overline{A}} \\
&= (\mu_A \otimes \lambda_H \otimes H) \circ (c_{A,A} \otimes H \otimes H) \circ (A \otimes \rho_A \otimes H) \circ (A \otimes c_{H,A}) \\
&\quad \circ (\rho_A \otimes A) \circ c_{A,A} \circ f_A \\
&= (\mu_A \otimes \lambda_H \otimes H) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \circ f_A \\
&= (\mu_A \otimes ((\lambda_H \otimes H) \circ \gamma_H^{-1} \circ \gamma_H)) \circ (A \otimes c_{H,A} \otimes H) \circ (\rho_A \otimes \rho_A) \circ f_A \\
&= (A \otimes (\lambda_H \circ \mu_H) \otimes H) \circ (\rho_A \otimes ((\lambda_H \otimes H) \circ \delta_H)) \circ \gamma_A \circ f_A \\
&= \eta_A \otimes (((\lambda_H \circ \mu_H) \otimes H) \circ \delta_H) \\
&= \eta_A \otimes \delta_H,
\end{aligned}$$

where the first equality follows because $f_{\overline{A}} = c_{A,A} \circ f_A$, the second one by the symmetry and the naturality of c . In the third one we used that \mathbb{H} is a Galois H -object and the fourth and the sixth ones are a consequence of (b1) of Definition 1.3. Finally, in the fifth one we applied that \mathbb{A} is a Galois H -object, and the last one relies on the cocommutativity of H . Also

$$\begin{aligned}
& (\gamma_{\overline{A}} \otimes H) \circ \rho_{A \otimes \overline{A}}^2 \circ f_{\overline{A}} \\
&= ((\mu_A \circ c_{A,A}) \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ (A \otimes \rho_A) \circ c_{A,A} \circ f_A \\
&= (\mu_A \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ (A \otimes c_{H,A}) \circ (\rho_A \otimes A) \circ f_A \\
&= (\mu_A \otimes ((\varepsilon_H \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ c_{H,H} \circ ((\mu_H \circ (\mu_H \otimes \lambda_H)) \\
&\quad \circ (H \otimes \delta_H)) \otimes H)) \circ (A \otimes c_{H,A} \otimes \delta_H) \circ (\rho_A \otimes \rho_A) \circ f_A \\
&= (A \otimes ((\lambda_H \otimes \lambda_H) \circ \delta_H)) \circ (A \otimes \mu_H) \circ ((\rho_A \circ \eta_A) \otimes \lambda_H) \\
&= \eta_A \otimes \delta_H,
\end{aligned}$$

where the first equality follows by (b1) of Definition 1.3 and the comodule properties of A , the second one by the naturality of c , the third one by (a2-2) of Definition 1.1 and the counit properties, the fourth one by (b2) of Definition

1.3 and in the last one we used that \mathbb{A} is a Galois H -object, (b1) of Definition 1.3 and the cocommutativity of H .

Then, $\rho_{A \otimes \overline{A}}^1 \circ f_{\overline{A}} = \rho_{A \otimes \overline{A}}^2 \circ f_{\overline{A}}$ and, as a consequence, there exists a unique morphism $h'_A : H \rightarrow A \bullet \overline{A}$ such that

$$(16) \quad i_{A \bullet \overline{A}} \circ h'_A = f_{\overline{A}}.$$

Therefore, by (15) and (16) we have

$$i_{A \bullet \overline{A}} \circ h'_A \circ h_A = f_{\overline{A}} \circ h_A = \gamma_{\overline{A}}^{-1} \circ (\eta_A \otimes H) \circ h_A = \gamma_{\overline{A}}^{-1} \circ \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}} = i_{A \bullet \overline{A}}$$

and

$$(\eta_A \otimes H) \circ h_A \circ h'_A = \gamma_{\overline{A}} \circ i_{A \bullet \overline{A}} \circ h'_A = \gamma_{\overline{A}} \circ f_{\overline{A}} = \gamma_{\overline{A}} \circ \gamma_{\overline{A}}^{-1} \circ (\eta_A \otimes H) = (\eta_A \otimes H).$$

Then, $h'_A \circ h_A = id_{A \bullet \overline{A}}$ and $h_A \circ h'_A = id_H$ and h is an isomorphism.

Finally, assume that \mathbb{A} is strong. By (16) and the equality $f_{\overline{A}} = c_{A,A} \circ f_A$ we obtain that h'_A is a morphism of unital magmas. Then h_A is a morphism of unital magmas and the proof is finished. \square

Remark 2.9. Note that, in the Hopf algebra setting, for any Galois H -object \mathbb{A} , the morphism h_A obtained in the previous proposition is a morphism of monoids because this property can be deduced from the associativity of the product defined in A . In the Hopf quasigroup world this proof does not work because A is a magma.

Theorem 2.10. *Let H be a cocommutative faithfully flat Hopf quasigroup. The set of isomorphism classes of Galois H -objects is a commutative monoid. Moreover, the set of isomorphism classes of strong Galois H -objects is a commutative group.*

Proof. Let $Gal_{\mathcal{C}}(H)$ be the set of isomorphism classes of Galois H -objects. For a Galois H -object \mathbb{A} we denote its class in $Gal_{\mathcal{C}}(H)$ by $[\mathbb{A}]$. By Propositions 2.6 and 1.8, the product

$$(17) \quad [\mathbb{A}] \cdot [\mathbb{B}] = [\mathbb{A} \bullet \mathbb{B}]$$

is well-defined. By Propositions 1.10, 1.9 and 1.11 we obtain that $Gal_{\mathcal{C}}(H)$ is a commutative monoid with unit $[\mathbb{H}]$.

If we denote by $Gal_{\mathcal{C}}^s(H)$ the set of isomorphism classes of strong Galois H -objects, with the product defined in (17) for Galois H -objects, $Gal_{\mathcal{C}}^s(H)$ is a commutative group because by (ii) of Proposition 2.6 the product of strong Galois H -objects is a strong Galois H -object, by Example 2.2 we know that \mathbb{H} is a strong Galois H -object and by Propositions 2.7 and 2.8, the inverse of $[\mathbb{A}]$ in $Gal_{\mathcal{C}}^s(H)$ is $[\overline{\mathbb{A}}]$. \square

Definition 2.11. Let H be a cocommutative faithfully flat Hopf quasigroup. If \mathbb{A} is a (strong) Galois H -object, we will say that \mathbb{A} has a normal basis if (A, ρ_A) is isomorphic to (H, δ_H) as right H -comodules. We denote by n_A the H -comodule isomorphism between A and H .

Obviously, $N_{\mathcal{C}}(H)$, the set of isomorphism classes of Galois H -objects with normal basis, is a submonoid of $Gal_{\mathcal{C}}(H)$ because $\mathbb{H} = (H, \delta_H)$ is a Galois H -object with normal basis and if \mathbb{A}, \mathbb{B} are Galois H -objects with normal basis and associated isomorphisms n_A, n_B respectively, then $\mathbb{A} \bullet \mathbb{B}$ is a Galois H -object with normal basis and associated H -comodule isomorphism $n_{\mathbb{A} \bullet \mathbb{B}} = r_H \circ n_A \bullet n_B$ where $n_A \bullet n_B$ is defined as in Proposition 1.8 and r_H is the isomorphism defined in Proposition 1.11. Moreover, for a strong Galois H -object with normal basis \mathbb{A} , with associated isomorphism n_A , we have that $\overline{\mathbb{A}} = (\overline{A}, \rho_{\overline{A}})$ is also a strong Galois H -object with normal basis, where $n_{\overline{A}} = \lambda_H \circ n_A$, and then, if we denote by $N_{\mathcal{C}}^s(H)$ the set of isomorphism classes of strong Galois H -objects with normal basis, $N_{\mathcal{C}}^s(H)$ is a subgroup of $Gal_{\mathcal{C}}^s(H)$.

Note that, if H is a Hopf algebra we have that $Gal_{\mathcal{C}}^s(H) = Gal_{\mathcal{C}}(H)$ and $N_{\mathcal{C}}^s(H) = N_{\mathcal{C}}(H)$. Therefore, in the associative setting we recover the classical group of Galois H -objects.

Remark 2.12. In this remark we use some classical results of algebraic K -theory (see [3] for the details). Let $G(\mathcal{C}, H)$ and $G^s(\mathcal{C}, H)$ be the categories of Galois H -objects and strong Galois H -objects, respectively. Then, by Proposition 1.13 these categories are symmetric monoidal and then they are categories with product. The Grothendieck group of $G(\mathcal{C}, H)$ is the abelian group generated by the isomorphism classes of objects \mathbb{A} of $G(\mathcal{C}, H)$ module the relations $[\mathbb{A} \bullet \mathbb{B}] = [\mathbb{A}] \cdot [\mathbb{B}]$. This group will be denoted by $K_0G(\mathcal{C}, H)$ and, by the general theory of Grothendieck groups, we know that for \mathbb{A}, \mathbb{B} in $G(\mathcal{C}, H)$, $[\mathbb{A}] = [\mathbb{B}]$ in $K_0G(\mathcal{C}, H)$ if and only if there exists a \mathbb{D} in $G(\mathcal{C}, H)$ such that $\mathbb{A} \bullet \mathbb{D}$ is isomorphic in $G(\mathcal{C}, H)$ to $\mathbb{B} \bullet \mathbb{D}$. The unit of $K_0G(\mathcal{C}, H)$ is $[\mathbb{H}]$. In a similar way we can define $K_0G^s(\mathcal{C}, H)$, but in this case $K_0G^s(\mathcal{C}, H) = Gal_{\mathcal{C}}^s(H)$ because the set of isomorphism classes of objects of $G^s(\mathcal{C}, H)$ is a group.

The inclusion functor $i : G^s(\mathcal{C}, H) \rightarrow G(\mathcal{C}, H)$ is a product preserving functor and then we have a group morphism $K_0i : Gal_{\mathcal{C}}^s(H) \rightarrow K_0G(\mathcal{C}, H)$. If $[\mathbb{A}] \in Ker(K_0i)$ we have that $[\mathbb{A}] = [\mathbb{H}]$ in $K_0G(\mathcal{C}, H)$. Then there exists \mathbb{D} in $G(\mathcal{C}, H)$ such that $\mathbb{A} \bullet \mathbb{D} \cong \mathbb{H} \bullet \mathbb{D} \cong \mathbb{D}$ in $G(\mathcal{C}, H)$. As a consequence $\mathbb{A} \bullet \mathbb{D} \bullet \overline{\mathbb{D}} \cong \mathbb{D} \bullet \overline{\mathbb{D}}$ in $G(\mathcal{C}, H)$. Then, By Proposition 2.8, $\mathbb{A} \cong \mathbb{H}$ as right H -comodules. Therefore \mathbb{A} is a strong Galois H -object with normal basis and $Ker(K_0i)$ is a subgroup of $N_{\mathcal{C}}^s(H)$.

The full subcategory $\mathcal{H} = \{\mathbb{H}\}$ of $G^s(\mathcal{C}, H)$ is cofinal because, for all \mathbb{A} in $G^s(\mathcal{C}, H)$, $\mathbb{A} \bullet \overline{\mathbb{A}} \cong \mathbb{H}$ as right H -comodule magmas. Therefore, the Whitehead group of $G^s(\mathcal{C}, H)$ is isomorphic to the Whitehead group of \mathcal{H} . Therefore,

$$K_1G^s(\mathcal{C}, H) \cong Aut_{G^s(\mathcal{C}, H)}(\mathbb{H}).$$

The group $Aut_{G^s(\mathcal{C}, H)}(\mathbb{H})$ admits a good explanation in terms of grouplike elements of a suitable Hopf quasigroup if H is finite, that is, if there exists an object H^* in \mathcal{C} and an adjunction $H \otimes - \dashv H^* \otimes -$. For this adjunction we will denote with $a_H : id_{\mathcal{C}} \rightarrow H^* \otimes H \otimes -$ and $b_H : H \otimes H^* \otimes - \rightarrow id_{\mathcal{C}}$ the unit and the counit respectively. The object H^* will be called the dual of H .

A Hopf coquasigroup D in \mathcal{C} is a monoid (D, η_D, μ_D) and a counital comagma $(D, \varepsilon_D, \delta_D)$ such that the following axioms hold:

- (d1) ε_D and δ_D are morphisms of monoids.
- (d2) There exists $\lambda_D : D \rightarrow D$ in \mathcal{C} (called the antipode of D) such that:
 - (d2-1) $(\mu_D \otimes D) \circ (\lambda_D \otimes \delta_D) \circ \delta_D$
 $= \eta_D \otimes D$
 $= (\mu_D \otimes D) \circ (D \otimes ((\lambda_D \otimes D) \circ \delta_D)) \circ \delta_D.$
 - (d2-2) $(D \otimes \mu_D) \circ (\delta_D \otimes \lambda_D) \circ \delta_D$
 $= D \otimes \eta_D$
 $= (D \otimes \mu_D) \circ (((D \otimes \lambda_D) \circ \delta_D) \otimes D) \circ \delta_D.$

As in the case of quasigroups, the antipode is unique, antimultiplicative, anticomultiplicative, leaves the unit and the counit invariable and satisfies (3).

If D is a Hopf coquasigroup we define $G(D)$ as the set of morphisms $h : K \rightarrow D$ such that $\delta_D \circ h = h \otimes h$ and $\varepsilon_D \circ h = id_K$. If D is commutative, $G(D)$ with the convolution $h * g = \mu_D \circ (h \otimes g)$ is a commutative group, called the group of grouplike morphisms of D . Note that the unit element of $G(D)$ is η_D and the inverse of $h \in G(D)$ is $h^{-1} = \lambda_D \circ h$.

It is easy to show that, if H is a finite cocommutative Hopf quasigroup, its dual H^* is a commutative finite Hopf coquasigroup where:

$$\begin{aligned} \eta_{H^*} &= (H^* \otimes \varepsilon_H) \circ a_H, \\ \mu_{H^*} &= (H^* \otimes b_H) \circ (H^* \otimes H \otimes b_H \otimes H^*) \circ (H^* \otimes \delta_H \otimes H^* \otimes H^*) \\ &\quad \circ (a_H \otimes H^* \otimes H^*), \\ \varepsilon_{H^*} &= b_H \circ (\eta_H \otimes H^*), \\ \delta_{H^*} &= (H^* \otimes H^* \otimes (b_H \circ (\mu_H \otimes H^*))) \circ (H^* \otimes a_H \otimes H \otimes H^*) \circ (a_H \otimes H^*) \end{aligned}$$

and the antipode is $(H^* \otimes b_H) \circ (H^* \otimes \lambda_H \otimes H^*) \circ (a_H \otimes H^*)$.

The groups $G(H^*)$ and $\text{Aut}_{G^s(\mathcal{C}, H)}(\mathbb{H})$ are isomorphic. The proof is equal to the one given in Proposition 3.7 of [14]. If $\alpha \in \text{Aut}_{G^s(\mathcal{C}, H)}(\mathbb{H})$, the morphism $z_\alpha = (H^* \otimes (\varepsilon_H \circ \alpha)) \circ a_H$ is in $G(\mathbf{H}^*)$. Then, we define the map $\text{Aut}_{G^s(\mathcal{C}, H)}(\mathbb{H}) \rightarrow G(\mathbf{H}^*)$ by $z(\alpha) = z_\alpha$. On the other hand, if $h \in G(\mathbf{H}^*)$, then $x_h = (H \otimes b_H) \circ (\delta_H \otimes h) : H \rightarrow H$ is a morphism of Galois \mathbf{H} -objects and then, by Remark 1.4, it is an isomorphism, that is $x_h \in \text{Aut}_{G^s(\mathcal{C}, H)}(\mathbb{H})$. The map $x : G(\mathbf{H}^*) \rightarrow \text{Aut}_{G^s(\mathcal{C}, H)}(\mathbb{H})$ defined by $x(h) = x_h$ is the inverse of z . Therefore,

$$K_1 G^s(\mathcal{C}, H) \cong G(\mathbf{H}^*).$$

Finally, $N^s(\mathcal{C}, H)$ is the subcategory of $G^s(\mathcal{C}, H)$ whose objects are the strong Galois H -objects with normal basis, note that $\mathcal{H} = \{\mathbb{H}\}$ it is also cofinal in $N^s(\mathcal{C}, H)$ and then

$$K_1 N^s(\mathcal{C}, H) \cong G(\mathbf{H}^*).$$

3. Invertible comodules with geometric normal basis

This section is devoted to study the connections between Galois H -objects and invertible comodules with geometric normal basis. First of all, we introduce the notion of invertible comodule with geometric normal basis which is a generalization to the non associative setting of the one defined by Caenepeel in [6].

Definition 3.1. Let H be a cocommutative faithfully flat Hopf quasigroup. A right H -comodule $\mathbf{M} = (M, \rho_M)$ is called invertible with geometric normal basis if there exist a faithfully flat unital magma S and an isomorphism $h_M : S \otimes M \rightarrow S \otimes H$ of right H -comodules such that h_M is almost linear, that is

$$(18) \quad h_M = (\mu_S \otimes H) \circ (S \otimes (h_M \circ (\eta_S \otimes M))).$$

A morphism between two invertible right H -comodules with normal basis is a morphism of right H -comodules.

Note that, if S is a monoid, h_M is a morphism of left S -modules, for $\varphi_{S \otimes M} = \mu_S \otimes M$ and $\varphi_{S \otimes H} = \mu_S \otimes H$, if and only if (18) holds. Then in the Hopf algebra setting this definition is the one introduced by Caenepeel in [6].

Example 3.2. Let H be a cocommutative faithfully flat Hopf quasigroup and let $\mathbb{A} = (A, \rho_A)$ be a Galois H -object. Then $\mathbf{A} = (A, \rho_A)$ is an invertible right H -comodule with geometric normal basis because $h_A = \gamma_A$ is an isomorphism of right H -comodules and trivially γ_A is almost linear. In particular, $\mathbf{H} = (H, \delta_H)$ is an example of invertible right H -comodule with geometric normal basis.

Proposition 3.3. Let H be a cocommutative faithfully flat Hopf quasigroup and \mathbf{M}, \mathbf{N} be invertible right H -comodules with geometric normal basis. Then the right H -comodule $\mathbf{M} \bullet \mathbf{N} = (M \bullet N, \rho_{M \bullet N})$, where $M \bullet N$ and $\rho_{M \bullet N}$ are defined as in Proposition 1.7, is a right H -comodule with geometric normal basis.

Proof. Let S, R and h_M, h_N be the faithfully flat unital magmas and the isomorphisms of right H -comodules associated to \mathbf{M} and \mathbf{N} respectively. Then $T = S \otimes R$ is faithfully flat. On the other hand,

$$T \otimes M \bullet N \xrightarrow{T \otimes i_{M \bullet N}} T \otimes M \otimes N \begin{array}{c} \xrightarrow{T \otimes \rho_{M \otimes N}^1} \\ \xrightarrow{T \otimes \rho_{M \otimes N}^2} \end{array} T \otimes M \otimes N \otimes H$$

and

$$T \otimes H \xrightarrow{T \otimes \delta_H} T \otimes H \otimes H \begin{array}{c} \xrightarrow{T \otimes \rho_{H \otimes H}^1} \\ \xrightarrow{T \otimes \rho_{H \otimes H}^2} \end{array} T \otimes H \otimes H \otimes H$$

are equalizer diagrams and for the morphism

$$g_{M \otimes N} = (S \otimes c_{R, H} \otimes H) \circ (h_M \otimes h_N) \circ (S \otimes c_{R, M} \otimes N) : S \otimes R \otimes M \otimes N \rightarrow S \otimes R \otimes H \otimes H$$

we have that

$$\begin{aligned}
& (S \otimes R \otimes \rho_{H \otimes H}^1) \circ g_{M \otimes N} \circ (S \otimes R \otimes i_{M \bullet N}) \\
&= (S \otimes c_{H,R} \otimes c_{H,H}) \circ (S \otimes H \otimes c_{H,R} \otimes H) \circ (((S \otimes \delta_H) \circ h_M) \otimes h_N) \\
&\quad \circ (S \otimes c_{R,M} \otimes N) \circ (S \otimes R \otimes i_{M \bullet N}) \\
&= (S \otimes c_{H,R} \otimes c_{H,H}) \circ (S \otimes H \otimes c_{H,R} \otimes H) \circ (((h_M \otimes H) \circ (S \otimes \rho_M)) \otimes h_N) \\
&\quad \circ (S \otimes c_{R,M} \otimes N) \circ (S \otimes R \otimes i_{M \bullet N}) \\
&= (((S \otimes c_{H,R} \otimes H) \circ (h_M \otimes h_N)) \otimes H) \circ (S \otimes c_{R,M} \otimes N \otimes H) \\
&\quad \circ (S \otimes R \otimes ((M \otimes c_{H,N}) \circ (\rho_M \otimes N) \circ i_{M \bullet N})) \\
&= (((S \otimes c_{H,R} \otimes H) \circ (h_M \otimes h_N)) \otimes H) \circ (S \otimes c_{M,R} \otimes ((M \otimes \rho_N) \circ i_{M \bullet N})) \\
&= (S \otimes R \otimes \rho_{H \otimes H}^2) \circ g_{M \otimes N} \circ (S \otimes R \otimes i_{M \bullet N}),
\end{aligned}$$

where the first and the third equalities follow by the naturality of c , the second and the fifth ones by the comodule morphism condition for h_M and h_N respectively and finally the fourth one by the properties of $i_{M \bullet N}$.

Therefore, there exists a unique morphism $h_{M \bullet N} : T \otimes M \bullet N \rightarrow T \otimes H$ such that

$$(19) \quad (T \otimes \delta_H) \circ h_{M \bullet N} = g_{M \otimes N} \circ (T \otimes i_{M \bullet N}).$$

Moreover, if we define the morphism

$$g'_{M \otimes N} = (S \otimes c_{M,R} \otimes N) \circ (h_M^{-1} \otimes h_N^{-1}) \circ (S \otimes c_{R,H} \otimes H) : S \otimes R \otimes H \otimes H \rightarrow S \otimes R \otimes M \otimes N$$

by the naturality of c , the comodule morphism condition for h_M^{-1} and h_N^{-1} and the cocommutativity of H , the following equalities hold

$$\begin{aligned}
& (S \otimes R \otimes \rho_{M \otimes N}^1) \circ g'_{M \otimes N} \circ (S \otimes R \otimes \delta_H) \\
&= (S \otimes c_{M,R} \otimes c_{H,N}) \circ (S \otimes M \otimes c_{H,R} \otimes N) \circ (((S \otimes \rho_M) \circ h_M^{-1}) \otimes h_N^{-1}) \\
&\quad \circ (S \otimes c_{H,R} \otimes H) \circ (S \otimes R \otimes \delta_H) \\
&= (S \otimes c_{M,R} \otimes c_{H,N}) \circ (S \otimes M \otimes c_{H,R} \otimes N) \circ (((h_M^{-1} \otimes H) \circ (S \otimes \delta_H)) \otimes h_N^{-1}) \\
&\quad \circ (S \otimes c_{H,R} \otimes H) \circ (S \otimes R \otimes \delta_H) \\
&= (g'_{M \otimes N} \otimes H) \circ (S \otimes R \otimes ((H \otimes \delta_H) \circ \delta_H)) \\
&= (S \otimes R \otimes \rho_{M \otimes N}^2) \circ g'_{M \otimes N} \circ (S \otimes R \otimes \delta_H).
\end{aligned}$$

As a consequence, there exists a unique morphism $h'_{M \bullet N} : T \otimes H \rightarrow T \otimes M \bullet N$ such that

$$(20) \quad (T \otimes i_{M \bullet N}) \circ h'_{M \bullet N} = g'_{M \otimes N} \circ (T \otimes \delta_H).$$

Thus, by (19) and (20)

$$\begin{aligned}
h_{M \bullet N} \circ h'_{M \bullet N} &= (T \otimes ((\varepsilon_H \otimes H) \circ \delta_H)) \circ h_{M \bullet N} \circ h'_{M \bullet N} \\
&= (T \otimes \varepsilon_H \otimes H) \circ g_{M \otimes N} \circ (T \otimes i_{M \bullet N}) \circ h'_{M \bullet N} \\
&= (T \otimes \varepsilon_H \otimes H) \circ g_{M \otimes N} \circ g'_{M \otimes N} \circ (T \otimes \delta_H) = id_{T \otimes H}
\end{aligned}$$

and

$$\begin{aligned} (T \otimes i_{M \bullet N}) \circ h'_{M \bullet N} \circ h_{M \bullet N} &= g'_{M \otimes N} \circ (T \otimes \delta_H) \circ h_{M \bullet N} \\ &= g'_{M \otimes N} \circ g_{M \otimes N} \circ (T \otimes i_{M \bullet N}) = T \otimes i_{M \bullet N} \end{aligned}$$

and then $h_{M \bullet N}$ is an isomorphism with inverse $h_{M \bullet N}^{-1} = h'_{M \bullet N}$.

The morphism $h_{M \bullet N}$ is a morphism of right H -comodules because

$$\begin{aligned} &(h_{M \bullet N} \otimes H) \circ (T \otimes \rho_{M \bullet N}) \\ &= (T \otimes ((H \otimes \varepsilon_H) \circ \delta_H) \otimes H) \circ (h_{M \bullet N} \otimes H) \circ (T \otimes \rho_{M \bullet N}) \\ &= (T \otimes H \otimes \varepsilon_H \otimes H) \circ (g_{M \otimes N} \otimes H) \circ (T \otimes ((i_{M \bullet N} \otimes H) \circ \rho_{M \bullet N})) \\ &= (T \otimes H \otimes \varepsilon_H \otimes H) \circ (g_{M \otimes N} \otimes H) \circ (T \otimes ((M \otimes \rho_N) \circ i_{M \bullet N})) \\ &= (T \otimes H \otimes ((H \otimes \varepsilon_H) \circ \delta_H)) \circ g_{M \otimes N} \circ (T \otimes i_{M \bullet N}) \\ &= (T \otimes \delta_H) \circ h_{M \bullet N}, \end{aligned}$$

where the first equality follows by the counit property, the second and the last ones by (19), the third one the properties of $\rho_{M \bullet N}$ and the fourth one by the comodule condition for h_N .

Finally, we will prove that $h_{M \bullet N}$ is almost linear. Indeed:

$$\begin{aligned} &(\mu_T \otimes H) \circ (T \otimes (h_{M \bullet N} \circ (\eta_T \otimes M \bullet N))) \\ &= (\mu_T \otimes ((\varepsilon_H \otimes H) \circ \delta_H)) \circ (T \otimes (h_{M \bullet N} \circ (\eta_T \otimes M \bullet N))) \\ &= (\mu_{S \otimes R} \otimes \varepsilon_H \otimes H) \circ (S \otimes R \otimes (g_{M \otimes N} \circ (\eta_S \otimes \eta_R \otimes i_{M \bullet N}))) \\ &= (S \otimes \varepsilon_H \otimes R \otimes H) \circ (((\mu_S \otimes H) \circ (S \otimes (h_M \circ (\eta_S \otimes M)))) \circ ((\mu_R \otimes H) \\ &\quad \circ (R \otimes (h_N \circ (\eta_R \otimes N)))) \circ (S \otimes c_{R, M} \otimes N) \circ (S \otimes R \otimes i_{M \bullet N})) \\ &= (T \otimes \varepsilon_H \otimes H) \circ g_{M \otimes N} \circ (T \otimes i_{M \bullet N}) \\ &= (T \otimes ((\varepsilon_H \otimes H) \circ \delta_H)) \circ h_{M \bullet N} \\ &= h_{M \bullet N}. \end{aligned}$$

In the last equalities, the first and the sixth ones follow by the properties of the counit, the second and the fifth ones by (19), the third one is a consequence of the naturality of c and the fourth one relies on the almost linear condition for h_M and h_N . \square

As a direct consequence of this proposition we have the following theorem.

Theorem 3.4. *Let H be a cocommutative faithfully flat Hopf quasigroup. If we denote by $P_{\text{gnb}}(K, H)$ the category whose objects are the invertible right H -comodules with geometric normal basis and whose morphisms are the morphisms of right H -comodules between them, $P_{\text{gnb}}(K, H)$ with the product defined in the previous proposition is a symmetric monoidal category where the unit object is \mathbf{H} and the symmetry isomorphisms, the left, right and associative constraints are defined as in Proposition 1.13. Moreover, the set of isomorphism classes in $P_{\text{gnb}}(K, H)$ is a monoid that we will denote by $\text{Pic}_{\text{gnb}}(K, H)$.*

Remark 3.5. There is a monoid morphism $\omega : Gal_{\mathcal{C}}(H) \rightarrow P_{gnb}(K, H)$ defined by $\omega([\mathbb{A}]) = [\mathbf{A}]$. If $\omega([\mathbb{A}]) = [\mathbf{H}]$ we have that $A \cong H$ as right H -comodules. Then $[\mathbb{A}] \in N_{\mathcal{C}}(H)$. Also, if $[\mathbb{A}] \in Gal_{\mathcal{C}}^s(H)$ and $\omega([\mathbb{A}]) = [\mathbf{H}]$, $[\mathbb{A}] \in N_{\mathcal{C}}^s(H)$.

Acknowledgements. The authors were supported by Ministerio de Economía y Competitividad (Spain) and by Feder funds. Project MTM2013-43687-P: Homología, homotopía e invariantes categóricos en grupos y álgebras no asociativas.

References

- [1] J. N. Alonso Álvarez, J. M. Fernández Vilaboa, R. González Rodríguez, and C. Soneira Calvo, *Projections and Yetter-Drinfeld modules over Hopf (co)quasigroups*, J. Algebra **43** (2015), 153–199.
- [2] ———, *Cleft comodules over Hopf quasigroups*, Commun. Contemp. Math. **17** (2015), no. 6, 1550007, 20 pp.
- [3] H. Bass, *Lectures on topics in algebraic K-theory*, Notes by Amit Roy, Tata Institute of Fundamental Research Lectures on Mathematics, 41, Tata Institute of Fundamental Research, Bombay, 1967.
- [4] M. Beattie, *A direct sum decomposition for the Brauer group of H -module algebras*, J. Algebra **43** (1976), no. 2, 686–693.
- [5] R. H. Bruck, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.
- [6] S. Caenepeel, *A variation of Sweedler’s complex and the group of Galois objects of an infinite Hopf algebra*, Comm. Algebra **24** (1996), no. 9, 2991–3015.
- [7] S. U. Chase and M. E. Sweedler, *Hopf algebras and Galois theory*, Lecture Notes in Mathematics, **97** Springer-Verlag, Berlin-New York 1969.
- [8] J. Klim and S. Majid, *Hopf quasigroups and the algebraic 7-sphere*, J. Algebra **323** (2010), no. 11, 3067–3110.
- [9] M. P. López López, *Objetos de Galois sobre un álgebra de Hopf finita*, Álgebra **25**, Universidad de Santiago de Compostela, Departamento de Álgebra, Santiago de Compostela, 1980.
- [10] M. P. López López and E. Villanueva Nóvoa, *The antipode and the (co)invariants of a finite Hopf (co)quasigroup*, Appl. Categ. Structures **21** (2013), no. 3, 237–247.
- [11] J. M. Pérez-Izquierdo, *Algebras, hyperalgebras, nonassociative bialgebras and loops*, Adv. Math. **208** (2007), no. 2, 834–876.
- [12] J. M. Pérez-Izquierdo and I. P. Shestakov, *An envelope for Malcev algebras*, J. Algebra **272** (2004), no. 1, 379–393.
- [13] J. M. Fernández Vilaboa, *Grupos de Brauer y de Galois de un álgebra de Hopf en una categoría cerrada*, Álgebra **42**, Universidad de Santiago de Compostela, Departamento de Álgebra, Santiago de Compostela, 1985.
- [14] J. M. Fernández Vilaboa, E. Villanueva Novioa, and González Rodríguez, *Exact sequences for the Galois group*, Comm. Algebra **24** (1996), no. 11, 3413–3435.

JOSE N. ALONSO ÁLVAREZ
 DEPARTAMENTO DE MATEMÁTICAS
 FACULTAD DE ECONÓMICAS
 UNIVERSIDAD DE VIGO
 CAMPUS UNIVERSITARIO LAGOAS-MARCOSENDE, E-36280 VIGO, SPAIN
 E-mail address: jnalonso@uvigo.es

RAMON GONZÁLEZ RODRÍGUEZ
DEPARTAMENTO DE MATEMÁTICA APLICADA II
E.T.S.I. TELECOMUNICACIÓN UNIVERSIDAD DE VIGO
CAMPUS UNIVERSITARIO LAGOAS-MARCOSENDE, E-36310 VIGO, SPAIN
E-mail address: `rgon@dma.uvigo.es`

JOSE M. FERNÁNDEZ VILABOA
DEPARTAMENTO DE ÁLXEBRA
FACULTAD DE MATEMÁTICAS
UNIVERSIDAD DE SANTIAGO DE COMPOSTELA
E-15771 SANTIAGO DE COMPOSTELA, SPAIN
E-mail address: `josemanuel.fernandez@usc.es`