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SPIN-STRUCTURES ON REAL BOTT MANIFOLDS

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Abstract. Real Bott manifolds is a class of flat manifolds with holonomy group $\mathbb{Z}_2^k$ of diagonal type. In this paper we formulate necessary and sufficient conditions of the existence of a Spin-structure on real Bott manifolds. It extends results of [9].

1. Introduction

Let M^n be a flat manifold of dimension n, i.e., a compact connected Riemannian manifold without boundary with zero sectional curvature. From the theorem of Bieberbach ([2], [17]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

(1) $$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

where $\mathbb{Z}^n$ is a maximal torsion free abelian subgroup of rank n and G is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $\mathrm{Isom}(\mathbb{R}^n) = \mathrm{O}(n) \times \mathbb{R}^n = E(n)$. In that case $p : \Gamma \to G$ is a projection on the first component of the semidirect product $O(n) \ltimes \mathbb{R}^n$ and $\pi_1(M_n) = \Gamma$ is a subgroup of $O(n) \ltimes \mathbb{R}^n$. Conversely, given a short exact sequence of the form (1), it is known that the group Γ is (isomorphic to) the fundamental group of a flat manifold. In this case Γ is called a Bieberbach group. We can define a holonomy representation $\phi : G \to \mathrm{GL}(n, \mathbb{Z})$ by the formula:

(2) $$\phi(g)(e) = \tilde{g}e(\tilde{g})^{-1}$$

for all $e \in \mathbb{Z}^n$, $g \in G$ and where $p(\tilde{g}) = g$. In this article we shall consider Bieberbach groups of rank n with holonomy group $\mathbb{Z}_2^k$, $1 \leq k \leq n-1$, and $\phi(\mathbb{Z}_2^k) \subset D \subset \mathrm{GL}(n, \mathbb{Z})$. Here D is the group of matrices with ± 1 on the diagonal.

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Let

$$M_n \overset{\mathbb{R}P^1}{\to} M_{n-1} \overset{\mathbb{R}P^1}{\to} \cdots \overset{\mathbb{R}P^1}{\to} M_1 \overset{\mathbb{R}P^1}{\to} M_0 = \{\bullet\} \tag{3}$$

be a sequence of real projective bundles such that $M_i \to M_{i-1}$, $i = 1, 2, \ldots, n$, is a projective bundle of a Whitney sum of a real line bundle L_{i-1} and the trivial line bundle over M_{i-1}. The sequence (3) is called the real Bott tower and the top manifold M_n is called the real Bott manifold, [4].

Let γ_i be the canonical line bundle over M_i and we set $x_i = w_1(\gamma_i)$ (w_1 is the first Stiefel-Whitney class). Since $H^1(M_{i-1}, \mathbb{Z}_2)$ is additively generated by $x_1, x_2, \ldots, x_{i-1}$ and L_{i-1} is a line bundle over M_{i-1}, we can uniquely write

$$w_1(L_{i-1}) = \sum_{l=1}^{i-1} a_{li} x_l, \tag{4}$$

where $a_{li} \in \mathbb{Z}_2$ and $i = 2, 3, \ldots, n$.

From above we obtain the matrix $A = [a_{li}]$ which is an $n \times n$ strictly upper triangular matrix whose diagonal entries are 0 and remaining entries are either 0 or 1. One can observe (see [12]) that the tower (3) is completely determined by the matrix A and therefore we may denote the real Bott manifold M_n by $M(A)$. From [12, Lemma 3.1] we can consider $M(A)$ as the orbit space $M(A) = \mathbb{R}^n/\Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & \ldots & 0 \\ 0 & 1 & 0 & . & . & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & (-1)^{a_{i,i+1}} & \ldots & 0 \\ . & . & . & . & . & \ldots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & (-1)^{a_{i,n}} \end{bmatrix}, \begin{pmatrix} 0 \\ . \\ 0 \\ \frac{1}{2} \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix} \right) \in E(n), \tag{5}$$

where $(-1)^{a_{i,i+1}}$ is in the $(i+1, i+1)$ position and $\frac{1}{2}$ is the ith coordinate of the last column, $i = 1, 2, \ldots, n-1$. $s_n = \big(I, \big(0, 0, \ldots, 0, \frac{1}{2}\big)\big) \in E(n)$. From [12, Lemmas 3.2 and 3.3] $s_1^2, s_2^2, \ldots, s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. In other words $M(A)$ is a flat manifold with holonomy group $\mathbb{Z}_2^k$ of diagonal type. Here k is a number of non zero rows of a matrix A.

We have the following two lemmas.

Lemma 1.1 ([12], Lemma 2.1)**.** *The cohomology ring $H^*(M(A), \mathbb{Z}_2)$ is generated by degree one elements $x_1, \ldots, x_n$ as a graded ring with n relations*

$$x_j^2 = x_j \sum_{i=1}^{n} a_{ij} x_i,$$

for $j = 1, \ldots, n$.

Lemma 1.2 ([12], Lemma 2.2)**.** *The real Bott manifold $M(A)$ is orientable if and only if the sum of entries is* $0 \pmod 2$ *for each row of the matrix A.*

There are a few ways to decide whether there exists a Spin-structure on an oriented flat manifold M^n. We start with:

Definition 1.1 ([7])**.** An oriented flat manifold M^n has a Spin-structure if and only if there exists a homomorphism $\epsilon\colon \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \epsilon = p$, where $\lambda_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the covering map.

There is an equivalent condition for existence of Spin-structure. It is well known ([7]) that the closed oriented differential manifold M has a Spin-structure if and only if the second Stiefel-Whitney class vanishes.

The kth Stiefel-Whitney class [13, page 3, (2.1)] is given by the formula

$$w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, \dots, y_n) \in H^k(M(A); \mathbb{Z}_2), \tag{6}$$

where σ_k is the kth elementary symmetric function, $B(p)$ is a map induced by p on the classification space and

$$y_i := w_1(L_{i-1}) \tag{7}$$

for $i = 2, 3, \dots, n$. Hence,

$$w_2(M(A)) = \sum_{1<i<j\leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2). \tag{8}$$

Definition 1.2 ([4], page 4)**.** A binary square matrix A is a Bott matrix if $A = PBP^{-1}$ for a permutation matrix P and a strictly upper triangular binary matrix B.

Our paper is a sequel of [9]. There are given some conditions of the existence of Spin-structures.

Theorem 1.1 ([9], page 1021)**.** *Let A be a matrix of an orientable real Bott manifold $M(A)$.*

(1) *Let $l \in \mathbb{N}$ be an odd number. If there exist $1 \leq i < j \leq n$ and rows $A_{i,*}$, $A_{j,*}$ such that*

$$\sharp\{m : a_{i,m} = a_{j,m} = 1\} = l$$

and

$$a_{ij} = 0,$$

then $M(A)$ has no Spin-structure.

(2) *If $a_{ij} = 1$ and there exist $1 \leq i < j \leq n$ and rows*

$$A_{i,*} = (0, \dots, 0, a_{i,i_1}, \dots, a_{i,i_{2k}}, 0, \dots, 0),$$
$$A_{j,*} = (0, \dots, 0, a_{j,i_{2k+1}}, \dots, a_{j,i_{2k+2l}}, 0, \dots, 0)$$

such that $a_{i,i_1} = \cdots = a_{i,i_{2k}} = 1$, $a_{i,m} = 0$ for $m \notin \{i_1, \dots, i_{2k}\}$, $a_{j,i_{2k+1}} = \cdots = a_{j,i_{2k+2l}} = 1$, $a_{j,r} = 0$ for $r \notin \{i_{2k+1}, \dots, i_{2k+2l}\}$ and l, k are odd, then $M(A)$ has no Spin-structure.

In this paper we extend this theorem and we formulate necessary and sufficient conditions of the existence of a Spin-structure on real Bott manifolds. Here is our main result for Bott manifolds with holonomy group $\mathbb{Z}_2^k$.

Theorem 1.2. *Let A be a Bott matrix. Then the real Bott manifold $M(A)$ has a Spin-structure if and only if for all $1 \leq i < j \leq n$ manifolds $M(A_{ij})$ have a Spin-structure, where A_{ij} is the $\mathbb{Z}_2$-matrix consisting of ith and jth rows of A.*

In fact our condition reduces problem of existence of Spin-structure to the case of Bott manifold with holonomy $\mathbb{Z}_2^2$.

The structure of the paper is as follows. In Section 2 we prove a formula about the second Stiefel-Whitney class of the real Bott manifolds. This is the main tool in the proof of our main result. Section 3 has a very technical character. In this section we shall give a complete characterization of the existence of the Spin-structure on manifolds $M(A_{ij})$, $1 \leq i < j \leq n$.

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2. Proof of the Main Theorem

We have the following lemma.

Lemma 2.1. *Let A be an $n \times n$ the Bott matrix. Then,*

$$w_2(M(A)) = \sum_{1 \leq i < j \leq k} w_2(M(A_{ij})).$$

Proof. From Lemma 1.1 and ([3]) we have that the second cohomology group of $H^2(M(A), \mathbb{Z}_2)$ has a basis

$$\mathcal{B} = \{x_i x_j : 1 \leq i < j \leq n\}.$$

Moreover, also from Lemma 1.1 x_j^2 can be expressed by a linear combination of $x_k x_j$ for $k < j$. Note that this combination always contains an x_j-term. Hence, we get that $w_2(M(A))$ is a sum of linear elements

$$w_2(M(A)) = \sum_{k<j} x_k x_j.$$

Each term $x_k x_j$ of this sum is an element from basis $\mathcal{B}$ and it is equal to the second Stiefel-Whitney class of the real Bott manifold $M(A_{kj})$, so we get

$$w_2(M(A)) = \sum_{k<j} x_k x_j = w_2(M(A_{kj})).$$

Thus, the second Stiefel-Whitney class of the real Bott manifold $M(A)$ is equal to the sum of second Stiefel-Whitney classes of elementary components $M(A_{kj})$, $k < j$. □

From proof of Lemma 2.1 we obtain the proof of Main Theorem 1.2.

Proof of Theorem 1.2. Let us recall the manifold M has a Spin-structure if and only if $w_2(M) = 0$. At the beginning let us assume, that for each pair $1 \leq i < j \leq n$, we have $w_2(M(A_{ij})) = 0$. Then from Lemma 2.1 we have

$$w_2(M(A)) = \sum_{1 \leq i < j \leq k} w_2(M(A_{ij})) = 0,$$

so the real Bott manifold $M(A)$ has a Spin-structure.

On the other hand, if the manifold $M(A)$ admits the Spin-structure, then

$$0 = w_2(M(A)) = \sum_{1 \leq i < j \leq k} w_2(M(A_{ij})).$$

Since the second Stiefel-Whitney classes of $M(A_{ij})$ are non negative so for all $1 \leq i < j \leq n$ we get

$$w_2(M(A_{ij})) = 0. \qquad \square$$

In the next section of our paper we concentrate on calculations of Spin-structure on manifolds A_{ij}.

3. Existence of Spin-structure on manifolds $M(A_{ij})$

From now, let A be a matrix of an orientable real Bott manifold $M(A)$ of dimension n with two non-zero rows. From Lemma 1.2 we have that the number of entries 1, in each row, is an odd number and we have following three cases:

CASE I. There are no columns with double entries 1,
CASE II. The number of columns with double entries 1 is an odd number,
CASE III. The number of columns with double entries 1 is an even number.

We give conditions for an existence of the Spin-structure on $M(A_{ij})$. In the further part of the paper we adopt the notation $0_p = (\underbrace{0, \ldots, 0}_{p \text{ - times}})$. From the definition, rows of number i and j correspond to generators s_i, s_j which define a finite index abelian subgroup $H \subset \pi_1(M(A))$ (see [10]).

Theorem 3.1. 1. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above Case I. If there exist $1 \leq i < j \leq n$ such that*

$$A_{i,*} = (0_{i_1}, a_{i,i_1+1}, \ldots, a_{i,i_1+2k}, 0_{i_{2l}}, 0_{i_p}),$$
$$A_{j,*} = (0_{i_1}, 0_{i_{2k}}, a_{j,i_1+2k+1}, \ldots, a_{j,i_1+2k+2l}, 0_{i_p}),$$

where $a_{i,m} = 1$ for $m \in \{i_1 + 1, \ldots, i_1 + 2k\}$, $a_{j,r} = 1$ for $r \in \{i_1 + 2k + 1, \ldots, i_1 + 2k + 2l\}$, then $M(A)$ admits the Spin-structure if and only if either l is an even number or l is an odd number and $j \notin \{i_1 + 1, \ldots, i_1 + 2k\}$.

2. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above Case I. If there exist $1 \leq i < j \leq n$ such that*

$$A_{i,*} = (0_{i_1}, 0_{i_{2k}}, a_{i,i_{2k}+1}, \ldots, a_{i,i_{2k}+2l}, 0_{i_p}),$$
$$A_{j,*} = (0_{i_1}, a_{j,i_1+1}, \ldots, a_{j,i_1+2k}, 0_{i_{2l}}, 0_{i_p}),$$

where $a_{j,m} = 1$ for $m \in \{i_1+1, \dots, i_1+2k\}$, $a_{i,r} = 1$ for $r \in \{i_{2k}+1, \dots, i_{2k}+2l\}$, then $M(A)$ has the Spin-structure.

Proof. 1. From (7) we have

$$y_{i_1+1} = \cdots = y_{i_1+2k} = x_i,$$
$$y_{i_1+2k+1} = \cdots = y_{i_1+2k+2l} = x_j.$$

Using (8) and $x_i^2 = x_i \sum_{j=1}^{n} a_{ji} x_j$ we get

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_i^2 + 4klx_ix_j + l(2l-1)x_j^2 \\ &= k(2k-1)x_i^2 + l(2l-1)x_j^2 = l(2l-1)x_j^2 = lx_j^2. \end{aligned}$$

Summing up, we have to consider the following cases

(1) if $l = 2b$, then $w_2(M(A)) = 2bx_j^2 = 0$. Hence $M(A)$ has a Spin-structure,

(2) if $l = 2b+1$, then

$$\begin{aligned} w_2(M(A)) &= (2b+1)x_j^2 = x_j^2 \\ &= \begin{cases} 0, & \text{if } j \notin \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has a Spin-structure,} \\ x_ix_j, & \text{if } j \in \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has no Spin-structure.} \end{cases} \end{aligned}$$

2. From (7)

$$y_{i_1+1} = \cdots = y_{i_1+2k} = x_j,$$
$$y_{i_1+2k+1} = \cdots = y_{i_1+2k+2l} = x_i.$$

Moreover, from (8) and since $i_1 > j > i$

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_j^2 + 4klx_ix_j + l(2l-1)x_i^2 \\ &= k(2k-1)\underbrace{x_j^2}_{=0} + l(2l-1)\underbrace{x_i^2}_{=0} = 0. \end{aligned}$$

Hence $M(A)$ has the Spin-structure. □

Theorem 3.2. 1. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above Case II. If there exist $1 \le i < j \le n$ such that*

$$A_{i,*} = (0_{i_1}, a_{i,i_1+1}, \dots, a_{i,i_1+2k}, a_{i,i_1+2k+1}, \dots, a_{i,i_1+2k+2l}, 0_{i_{2m}}, 0_{i_p}),$$
$$A_{j,*} = (0_{i_1}, 0_{i_{2k}}, a_{j,i_1+2k+1}, \dots, a_{j,i_1+2k+2l}, a_{j,i_1+2k+2l+1}, \dots, a_{j,i_1+2k+2l+2m}, 0_{i_p}),$$

where $a_{i,r} = 1$ for $r \in \{i_1+1, \dots, i_1+2k+2l\}$, $a_{j,s} = 1$ for $s \in \{i_1+2k+1, \dots, i_1+2k+2l+2m\}$, then $M(A)$ has the Spin-structure if and only if either l and m are numbers of the same parity or l and m are numbers of different parity and $j \notin \{i_1+1, \dots, i_1+2k\}$.

2. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above case II. If there exist $1 \le i < j \le n$ such that*

$$A_{i,*} = (0_{i_1}, 0_{i_1+2k}, a_{i,i_1+2k+1}, \dots, a_{i,i_1+2k+2l}, a_{i,i_1+2k+2l+1}, \dots, a_{i,i_1+2k+2l+2m}, 0_{i_p}),$$
$$A_{j,*} = (0_{i_1}, a_{j,i_1+1}, \dots, a_{j,i_1+2k}, a_{j,i_1+2k+1}, \dots, a_{j,i_1+2k+2l}, 0_{i_{2m}}, 0_{i_p}),$$

where $a_{j,m} = 1$ *for* $m \in \{i_1+1, \dots, i_1+2k+2l\}$, $a_{i,r} = 1$ *for* $r \in \{i_1+2k+1, \dots, i_1+2k+2l+2m\}$, *then* $M(A)$ *has the Spin-structure.*

Proof. 1. From (7) we have

$$\begin{aligned} y_{i_1+1} = \dots = y_{i_1+2k} &= x_i, \\ y_{i_1+2k+1} = \dots = y_{i_1+2k+2l} &= x_i + x_j, \\ y_{i_1+2k+2l+1} = \dots = y_{i_1+2k+2l+2m} &= x_j. \end{aligned}$$

From (8) and $x_i^2 = x_i \sum_{j=1}^n a_{ji} x_j$ we get

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_i^2 + 4klx_i(x_i+x_j) + l(2l-1)(x_i+x_j)^2 + m(2m-1)x_j^2 \\ &= l(2l-1)x_j^2 + m(2m-1)x_j^2 = (l+m)x_j^2. \end{aligned}$$

We have to consider the following cases:

(1) If $l+m$ is an even number, then $w_2(M(A)) = 0$. Hence $M(A)$ has a Spin-structure.

(2) If $l+m$ is an odd number, then

$$\begin{aligned} w_2(M(A)) &= x_j^2 \\ &= \begin{cases} 0, & \text{if } j \notin \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has a Spin-structure,} \\ x_i x_j, & \text{if } j \in \{i_1+1, \dots, i_1+2k\}, M(A) \text{ has no Spin-structure.} \end{cases} \end{aligned}$$

2. Using (7) we get

$$\begin{aligned} y_{i_1+1} = \dots = y_{i_1+1} &= x_j, \\ y_{i_1+2k+1} = \dots = y_{i_1+2k+2l} &= x_i + x_j, \\ y_{i_1+2k+2l+1} = \dots = y_{i_1+2k+2l+2m} &= x_i. \end{aligned}$$

Moreover, from (8) and since $i_1 > j > i$

$$\begin{aligned} w_2(M(A)) &= k(2k-1)x_j^2 + l(2l-1)x_i^2 + 4klx_j(x_i+x_j) + 4kmx_ix_j \\ &\quad + 4lmx_i(x_i+x_j) + l(2l-1)(x_i+x_j)^2 + m(2m-1)x_i^2 \\ &= k(2k-1)\underbrace{x_j^2}_{=0} + l(2l-1)\underbrace{x_i^2}_{=0} + l(2l-1)\underbrace{x_j^2}_{=0} + m(2m-1)\underbrace{x_i^2}_{=0} \\ &= 0. \end{aligned}$$

Hence $M(A)$ has a Spin-structure. ☐

Theorem 3.3. 1. *Let* A *be a matrix of an orientable real Bott manifold* $M(A)$ *from the above Case III. If there exist* $1 \le i < j \le n$ *such that*

$$A_{i,*} = (0_{i_1}, a_{i,i_1+1}, \dots, a_{i,i_1+2k+1}, a_{i,i_1+2k+2}, \dots, a_{i,i_1+2k+2l+2}, 0_{i_{2m+1}}, 0_{i_p}),$$

$$A_{j,*} = (0_{i_1}, 0_{i_{2k+1}}, a_{j,i_{2k+2}}, \dots, a_{j,i_1+2k+2l+2}, a_{j,i_1+2k+2l+3}, \dots, a_{j,i_1+2k+2l+2m+3}, 0_{i_p}),$$

where $a_{i,r} = 1$ *for* $r \in \{i_1+1, \dots, i_1+2k+2l+2\}$, $a_{j,s} = 1$ *for* $s \in \{i_1+2k+2, \dots, i_1+2k+2l+2m+3\}$, *then* $M(A)$ *admits the Spin-structure if and only if* l *and* m *are numbers of the same parity and* $j \in \{i_1+1, \dots, i_1+2k+2\}$.

2. *Let A be a matrix of an orientable real Bott manifold $M(A)$ from the above case III. If there exist $1 \leq i < j \leq n$ such that*

$$A_{i,*} = (0_{,i_1}, 0_{i_{2l+1}}, a_{i,i_1+2k+2}, \dots, a_{i,i_1+2k+2l+2}, a_{i,i_1+2k+2l+3}, \dots, a_{i,i_1+2k+2l+2m+3}, 0_{i_p}),$$
$$A_{j,*} = (0_{i_1}, a_{j,i_1+1}, \dots, a_{j,i_1+2k+1}, a_{j,i_1+2k+2}, \dots, a_{j,i_1+2k+2l+2}, 0_{i_{2m}}, 0_{i_p}),$$

where $a_{j,m} = 1$ for $m \in \{i_1+1, \dots, i_1+2k+2l+2\}$, $a_{i,r} = 1$ for $r \in \{i_1+2k+2, \dots, i_1+2k+2l+2m+3\}$, then $M(A)$ has no Spin-structure.

Proof. 1. From (7)

$$y_{i_1+1} = \dots = y_{i_1+2k+1} = x_i,$$
$$y_{i_1+2k+2} = \dots = y_{i_1+2k+2l+2} = x_i + x_j,$$
$$y_{i_1+2k+2l+3} = \dots = y_{i_1+2k+2l+2m+3} = x_j.$$

From (8) and $x_i^2 = x_i \sum_{j=1}^n a_{ji} x_j$ we obtain

$$\begin{aligned} w_2(M(A)) = {} & k(2k+1)x_i^2 + (2k+1)(2l+1)x_i(x_i+x_j) + (2k+1)(2m+1)x_ix_j \\ & + l(2l+1)(x_i+x_j)^2 + (2l+1)(2m+1)x_j(x_i+x_j) \\ & + m(2m+1)x_j^2 \\ = {} & (l+m+1)x_j^2 + (2l+1)(2m+1)x_ix_j \\ = {} & (l+m+1)x_j^2 + x_ix_j. \end{aligned}$$

Now, if l and m are number of the same parity we have

$$w_2(M(A)) = x_ix_j + x_j^2$$
$$= \begin{cases} x_ix_j, & \text{if } j \notin \{i_1+1, \dots, i_1+2k+2\},\ M(A) \text{ has no Spin-structure}, \\ 0, & \text{if } j \in \{i_1+1, \dots, i_1+2k+2\},\ M(A) \text{ has a Spin-structure}. \end{cases}$$

2. From (7)

$$y_{i_1+1} = \dots = y_{i_1+2k+1} = x_j,$$
$$y_{i_1+2k+2} = \dots = y_{i_1+2k+2l+2} = x_i + x_j,$$
$$y_{i_1+2k+2l+3} = \dots = y_{i_1+2k+2l+2m+3} = x_i.$$

From (8) and since $i_1 > j > i$ we get

$$\begin{aligned} w_2(M(A)) = {} & k(2k+1)x_j^2 + m(2m+1)x_i^2 + (2k+1)(2l+1)x_j(x_i+x_j) \\ & + (2k+1)(2m+1)x_ix_j + l(2l+1)(x_i+x_j)^2 \\ & + (2l+1)(2m+1)x_i(x_i+x_j) + m(2m-1)x_i^2 \\ = {} & k(2k+1)\underbrace{x_j^2}_{=0} + l(2l+1)\underbrace{(x_i+x_j)^2}_{=0} + m(2m+1)\underbrace{x_i^2}_{=0} \\ & + x_j(x_i+x_j) + x_ix_j + x_i(x_i+x_j) = x_ix_j \neq 0, \end{aligned}$$

so $M(A)$ has no Spin-structure. □

At the end we illustrate our consideration by an example.

Example 3.1. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We have the following elementary components of A

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{12}}, \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{13}}, \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{14}},$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{23}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{24}}, \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A_{34}}.$$

From Theorems 3.1, 3.2, 3.3 we get that manifolds $M(A_{13})$, $M(A_{24})$ have no Spin-structure and all others elementary components have Spin-structure. So, from Theorem 1.2 for the manifold $M(A)$ we get

$$\omega_2(M(A)) = x_1 x_3 + x_2 x_4.$$

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