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U-FLATNESS AND NON-EXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we define the modulus of n-dimensional U-flatness as the determinant of an $(n+1)\times(n+1)$ matrix. The properties of the modulus are investigated and the relationships between this modulus and other geometric parameters of Banach spaces are studied. Some results on fixed point theory for non-expansive mappings and normal structure in Banach spaces are obtained.

1. Introduction

Let X be a real Banach space with the dual space X^*. Denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. Recall that $\nabla_x \subset S_{X^*}$ denotes the set of norm 1 supporting functionals of $x \in S_X$.

Brodskiĭ and Mil'man [2] introduced the following geometric concepts in 1948:

Definition 1.1. Let X be a Banach space. A nonempty bounded and convex subset K of X is said to have *normal structure* if for every convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < \operatorname{diam} C.$$

A Banach space X is said to have

- *normal structure* if every bounded convex subset of X has normal structure;
- *weak normal structure* if every weakly compact convex set K of X has normal structure;
- *uniform normal structure* if there exists $0 < c < 1$ such that for every bounded closed convex subset C of K that contains more than one point there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\| : y \in C\} < c \cdot \operatorname{diam} C.$$

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Remark 1.2. The following facts are known.

- uniform normal structure $\Longrightarrow$ normal structure $\Longrightarrow$ weak normal structure.
- In the setting of reflexive spaces, normal structure $\Longleftrightarrow$ weak normal structure.

Kirk [9] proved that if a Banach space X has weak normal structure, then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

Let $\mathbb{N}$ be the set of all natural numbers and $n \in \mathbb{N}$.

For two sets of vectors $\{x_i\}_{i=1}^{n+1} \subseteq X$ and $\{f_i\}_{i=2}^{n+1} \subseteq X^*$, the following $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_{n+1}, f_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_{n+1}, f_{n+1} \rangle \end{bmatrix}$$

is denoted by $m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})$ [6].

Gao and Saejung [6] introduced the concept of volume by the convex hull of $x_1, x_2, \ldots, x_{n+1}$ in X of

$$v(x_1, x_2, \ldots, x_{n+1}) := \sup\{\det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1})\},$$

where the supremum is taken over all $f_i \in \nabla_{x_i}$, where $i = 2, 3, \ldots, n+1$.

Definition 1.3 ([6])**.** Let $\nu_X^n = \sup\{v(x_1, x_2, \ldots, x_{n+1}) : x_1, x_2, \ldots x_{n+1} \in S_X\}$ be the upper bound of all n-dimensional volume in X.

Definition 1.4 ([6])**.** Let X be a Banach space. Define

$$U_X^n(\varepsilon) = \inf \left\{ 1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\| : \begin{array}{c} x_1, x_2, \ldots, x_{n+1} \in S_X, \\ v(x_1, x_2, \ldots, x_{n+1}) \geq \varepsilon \end{array} \right\},$$

where $0 \leq \varepsilon \leq \nu_X^n$ to be the modulus of n-dimensional U-convexity of X.

The following results were proved [6]:

Proposition 1.5. *For a Banach space X with* $\dim(X) > n$*, we have* $\nu_X^n \geq 2$.

Lemma 1.6. $U_X^n(\varepsilon)$ *is a continuous function in* $[0, \nu_X^n)$.

Theorem 1.7. *If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X is reflexive.*

Theorem 1.8. *If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X has normal structure.*

2. Main results

We introduce the concept of the modulus of n-dimensional flatness as follows:

Definition 2.1. Let X be a Banach space and $0 \le \varepsilon \le \nu_X^n$. Then the modulus of n-dimensional U-flatness of X is defined as follows:

$$W_X^n(\varepsilon) = \sup\left\{1 - \frac{1}{n+1}\|x_1 + x_2 + \cdots + x_{n+1}\|\right\},$$

where the supremum is taken over all $\{x_i\}_{i=1}^{n+1} \subseteq S_X$ such that there exist $\{f_i\}_{i=2}^{n+1} \subseteq S_{X^*}$ with $f_i \in \nabla_{x_i}$ for all $i = 2, \ldots, n+1$ and $\det m(x_1, x_2, \ldots, x_{n+1}; f_2, f_3, \ldots, f_{n+1}) \le \varepsilon$.

Remark 2.2. $W_X^n(\varepsilon)$ is an increasing and continuous function on $[0, \nu_X^n)$.

Proof. The proof is the same as that of Corollary 5 of [10]. □

Remark 2.3. The name of the modulus, U-flatness, is defined by comparing with Definition 1.4.

Lemma 2.4 (Bishop-Phelps-Bollobás [1]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B_X$ and $h \in S_{X^*}$ with $1 - \langle z, h\rangle < \frac{\varepsilon^2}{4}$, then there exist $y \in S_X$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Lemma 2.5. *Let $A_{n\times n}$ be the following $n \times n$ matrix*

$$A_{n\times n} := \begin{bmatrix} 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n-2} & (-1)^{n-1} \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n+1} & (-1)^{n-1} & (-1)^{n} \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Then $\det(A_{n\times n}) = \frac{1}{2^{n-1}}$.

Proof. It follows from mathematical induction:

By repeatedly using add $\frac{1}{2}$ times the first row to second row, then use the first row to estimate the determinant, we get the result. □

Lemma 2.6. *Let $B_{(n+1)\times(n+1)}$ be the following $(n+1) \times (n+1)$ matrix*

$$B_{(n+1)\times(n+1)} := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n} & (-1)^{n+1} & (-1)^{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

Then $\det(B_{(n+1)\times(n+1)}) = \frac{2n+1}{2^n}$.

Proof. It follows from mathematical induction and the preceding lemma:

Let $n = 1, B_{2\times 2} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & 1 \end{bmatrix}, \det(B_{2\times 2}) = \frac{3}{2}$.

If for n, $\det(B_{n\times n}) = \frac{2n-1}{2^{n-1}}$, then for $n+1$, by using the first column to estimate the matrix, we have

$$\begin{aligned}\det(B_{(n+1)\times(n+1)}) &= \det(A_{n\times n}) + \frac{1}{2}\det(B_{n\times n}) \\ &= \frac{1}{2^{n-1}} + \frac{2n-1}{2^n} = \frac{2n+1}{2^n}.\end{aligned}$$

□

Theorem 2.7 ([7])**.** *Let X be a Banach space. Then X is not reflexive if and only if for any $0 < \delta < 1$ there are a sequence $\{x_n\} \subseteq S_X$ and a sequence $\{f_n\} \subseteq S_{X^*}$ such that*

(a) $\langle x_m, f_n\rangle = \delta$ *whenever* $n \le m$; *and*
(b) $\langle x_m, f_n\rangle = 0$ *whenever* $n > m$.

Theorem 2.8. *If X is a Banach space with $W_X^n(\frac{2n+1}{2^n}) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $n \in \mathbb{N}$ be given. Let $y_i = (-1)^{i+1}\frac{x_i + x_{i+1}}{2}$ for $i = 1, \ldots, n+1$ and $g_i = (-1)^{i+1} f_i \in S_{X^*}$ for $i = 2, \ldots, n+1$. Then, we have

$$\delta \le \langle y_i, g_i\rangle = \left\langle (-1)^{i+1}\frac{x_i + x_{i+1}}{2}, (-1)^{i+1} f_i \right\rangle \le \frac{1}{2}\|x_i + x_{i+1}\| = \|y_i\| \le 1,$$

and

$$\det m(y_1, y_2, y_3 \ldots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4 \ldots, g_{n-1}, g_n, g_{n+1})$$

$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \cdots & \langle y_{n-1}, g_2\rangle & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \cdots & \langle y_{n-1}, g_3\rangle & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \langle y_1, g_{n-1}\rangle & \langle y_2, g_{n-1}\rangle & \langle y_3, g_{n-1}\rangle & \cdots & \langle y_{n-1}, g_{n-1}\rangle & \langle y_n, g_{n-1}\rangle & \langle y_{n+1}, g_{n-1}\rangle \\ \langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \cdots & \langle y_{n-1}, g_n\rangle & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \cdots & \langle y_2, g_{n+1}\rangle & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{\delta}{2} & \delta & -\delta & \cdots & (-1)^{n-1}\delta & (-1)^n\delta & (-1)^{n+1}\delta \\ 0 & -\frac{\delta}{2} & \delta & \cdots & (-1)^{n-2}\delta & (-1)^{n-1}\delta & (-1)^n\delta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \delta & -\delta & \delta \\ 0 & 0 & 0 & \cdots & -\frac{\delta}{2} & \delta & -\delta \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\delta}{2} & \delta \end{bmatrix}$$

$$= \delta^n \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{1}{2} & 1 & -1 & \cdots & (-1)^{n-1} & (-1)^n & (-1)^{n+1} \\ 0 & -\frac{1}{2} & 1 & \cdots & (-1)^{n-2} & (-1)^{n-1} & (-1)^n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -\frac{1}{2} & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

By Lemmas 2.5 and 2.6, we have

$$\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = \delta^n \frac{2n+1}{2^n}.$$

On the other hand, since

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|(-1)^{n+2} x_{n+2} + x_1\|}{2(n+1)} \le \frac{1}{n+1},$$

we have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \ge 1 - \frac{1}{n+1}.$$

Since δ can be chosen arbitrarily closed to 1, let $\delta = 1 - \frac{\varepsilon^2}{4}$ where ε can be chosen arbitrarily closed to 0.

Let $z_1 = y_1$. Next, let $i = 2, 3, \ldots, n+1$. From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist $z_i \in S_X$ and $h_i \in \nabla_{z_i}$ such that $\|y_i - z_i\| < \varepsilon$ and $\|g_i - h_i\| < \varepsilon$.

This implies that

$$|\langle z_i, h_j \rangle - \langle y_i, g_j \rangle| \le |\langle z_i - y_i, g_j \rangle| + |\langle y_i, h_j - g_j \rangle| + |\langle z_i - y_i, h_j - g_j \rangle| \le 3\varepsilon.$$

It follows then that

$$\det m(z_1, z_2, \ldots, z_{n+1}; h_2, h_3, \ldots, h_{n+1}) = \left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon,$$

where c is a bounded constant. Moreover,

$$1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \ge 1 - \frac{1+\varepsilon}{n+1}.$$

From the definition of $W_X^n(\varepsilon)$, we have

$$W_X^n\left(\left(1 - \frac{\varepsilon^2}{4}\right)^n \frac{2n+1}{2^n} + c\varepsilon\right) \ge 1 - \frac{1+\varepsilon}{n+1}.$$

Since ε can be arbitrarily close to 0, the theorem is proved. □

Let $C_{(n+1)\times(n+1)}$ be the following $(n+1)\times(n+1)$ matrix:

$$C_{(n+1)\times(n+1)} := \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{2}{3} & 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} \\ \frac{1}{3} & -\frac{2}{3} & 1 & -1 & \cdots & (-1)^{n} & (-1)^{n+1} & (-1)^{n+2} \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}.$$

Then $\det(C_{2\times 2}) = \frac{5}{3}$, and $\det(C_{3\times 3}) = \frac{7}{9}$.

Theorem 2.9. *If X is a Banach space with $W_X^n(\det C_{(n+1)\times(n+1)}) < \frac{2}{3}$ for some $n \in \mathbb{N}$, then X is reflexive. In particular, for $n = 1$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is reflexive*; *and for $n = 2$ we have if $W_X^2(\frac{7}{9}) < \frac{2}{3}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $n \in \mathbb{N}$ be given. Let $y_i = (-1)^{i+1}\frac{x_i+x_{i+1}+x_{i+2}}{3}$ for $i = 1, \dots, n+1$ and $g_i = (-1)^{i+1} f_i \in S_{X^*}$ for $i = 2, \dots, n+1$. Then, we have

$$\begin{aligned} \delta \le \langle y_i, g_i\rangle &= \left\langle (-1)^{i+1}\frac{x_i + x_{i+1} + x_{i+2}}{3}, (-1)^{i+1} f_i \right\rangle \\ &\le \frac{1}{3}\|x_i + x_{i+1} + x_{i+2}\| = \|y_i\| \le 1, \end{aligned}$$

and

$$\begin{aligned} & m(y_1, y_2, y_3, y_4, \dots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \dots, g_{n-1}, g_n, g_{n+1}) \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \langle y_4, g_2\rangle & \cdots & \langle y_{n-1}, g_2\rangle & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \langle y_4, g_3\rangle & \cdots & \langle y_{n-1}, g_3\rangle & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\ \langle y_1, g_4\rangle & \langle y_2, g_4\rangle & \langle y_3, g_4\rangle & \langle y_4, g_4\rangle & \cdots & \langle y_{n-1}, g_4\rangle & \langle y_n, g_4\rangle & \langle y_{n+1}, g_4\rangle \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \langle y_1, g_{n-1}\rangle & \langle y_2, g_{n-1}\rangle & \langle y_3, g_{n-1}\rangle & \langle y_4, g_{n-1}\rangle & \cdots & \langle y_{n-1}, g_{n-1}\rangle & \langle y_n, g_{n-1}\rangle & \langle y_{n+1}, g_{n-1}\rangle \\ \langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \langle y_4, g_n\rangle & \cdots & \langle y_{n-1}, g_n\rangle & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \langle y_4, g_{n+1}\rangle & \cdots & \langle y_{n-1}, g_{n+1}\rangle & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\ &= \delta^n \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -\frac{2}{3} & 1 & -1 & 1 & \cdots & (-1)^{n-1} & (-1)^{n} & (-1)^{n+1} \\ \frac{1}{3} & -\frac{2}{3} & 1 & -1 & \cdots & (-1)^{n} & (-1)^{n+1} & (-1)^{n+2} \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 & \cdots & (-1)^{n+1} & (-1)^{n+2} & (-1)^{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{2}{3} & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{3} & -\frac{2}{3} & 1 \end{bmatrix}. \end{aligned}$$

We have

$$\begin{aligned}&\det m(y_1, y_2, y_3, y_4, \dots, y_{n-1}, y_n, y_{n+1}; g_2, g_3, g_4, \dots, g_{n-1}, g_n, g_{n+1})\\ &= \delta^n \det C_{(n+1)\times(n+1)}.\end{aligned}$$

On the other hand, for $n \geq 2$,

$$\begin{aligned}\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} &= \frac{\|x_1 + x_3 - x_4 + \cdots + (-1)^n x_{n+1} + (-1)^{n+2} x_{n+3}\|}{3(n+1)}\\ &\leq \frac{n+1}{3(n+1)}\delta = \frac{1}{3}\delta,\end{aligned}$$

and for $n = 1$,

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} = \frac{\|x_1 - x_4\|}{6} \leq \frac{1}{3}\delta.$$

We have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq 1 - \frac{1}{3}\delta \geq \frac{2}{3}\delta$$

for all $n \in \mathbb{N}$.

The theorem can be proved by using the Bishop-Phelps-Bollobás result (Lemma 2.4), and same idea in the proof of Theorem 2.8. □

We consider $n = 1$.

Theorem 2.10. *If X is a Banach space with $W_X^1(\frac{2m+1}{m+1}) < \frac{m}{m+1}$ for some $m \in \mathbb{N}$, then X is reflexive. In particular, for $m = 2$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given. Let $\{x_i\} \subseteq S_X$ and $\{f_i\} \subseteq S_{X^*}$ be two sequences satisfying the two conditions in Theorem 2.7.

Let $m \in \mathbb{N}$ be given. Let

$$y_1 = \frac{x_1 + x_2 + \cdots + x_m + x_{m+1}}{m+1}, y_2 = -\frac{x_2 + x_3 + \cdots + x_{m+1} + x_{m+2}}{m+1}$$

and $g_2 = -f_2 \in S_{X^*}$.

Consider the 2-dimensional subspace of X spanned by y_1 and y_2.

We have

$$\det m(y_1, y_2; g_2) = \det \begin{bmatrix} 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ -\frac{m}{m+1} & 1 \end{bmatrix} \delta = \frac{2m+1}{m+1}\delta,$$

and

$$\left\|\frac{y_1 + y_2}{2}\right\| = \left\|\frac{x_1 - x_{m+2}}{2(m+1)}\right\| \leq \frac{1}{m+1}\delta.$$

This is

$$1 - \left\|\frac{y_1 + y_2}{2}\right\| \geq \frac{m}{m+1}\delta.$$

Similar to the proof of Theorem 2.8 we have

$$W_X^1\left(\frac{2m+1}{m+1}\right) \geq \frac{m}{m+1}.$$

This completes the proof. □

In 2008, Saejung proved the following result:

Lemma 2.11 ([11]). *If X is a Banach space with B_{X^*} is weak* sequentially compact and it fails to have weak normal structure, then for any $\varepsilon > 0$ and $n \in \mathbb{N}$ there are $\{x_1, x_2, \ldots, x_n\} \subseteq S_X$ and $\{f_1, f_2, \ldots, f_n\} \subseteq S_{X^*}$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$ *for all* $i \neq j$;
(b) $\langle x_i, f_i\rangle = 1$ *for all* $1 \leq i \leq n$; *and*
(c) $|\langle x_i, f_j\rangle| < \varepsilon$ *for all* $i \neq j$.

Theorem 2.12. *If X is a Banach space with B_{X^*} is weak* sequentially compact and $W_X^n(1) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$, then X has weak normal structure.*

Proof. Suppose that X does not have weak normal structure. Let $0 < \varepsilon < 1$ be given. Then there are $\{x_i\}_{i=1}^{n+1} \subseteq S_X$ and $\{f_i\}_{i=1}^{n+1} \subseteq S_{X^*}$ satisfying the three conditions in Lemma 2.11.

For convenience, let $|\langle x_i, f_j\rangle| = \varepsilon_{i,j}$. Then $\varepsilon_{i,j} \leq \varepsilon$ for all $i \neq j$.

Let $y_i = \frac{x_{i+1}-x_i}{\|x_{i+1}-x_i\|} \in S_X$ for $i = 1, \ldots, n+1$ and $g_i = f_{i+1} \in S_{X^*}$ for $i = 2, \ldots, n+1$. Then

$$\|y_i - (x_{i+1} - x_i)\| \leq \varepsilon$$

for $i = 1, \ldots, n+1$. Moreover,

$$\begin{aligned} &\|y_1 + y_2 + \cdots + y_i + \cdots + y_{n+1}\| \\ \leq\ &\|(x_2 - x_1) + (x_3 - x_2) + \cdots + (x_{i+1} - x_i) + \cdots + (x_{n+2} - x_{n+1})\| + (n+1)\varepsilon \\ =\ &\|x_{n+2} - x_1\| + (n+1)\varepsilon. \end{aligned}$$

Next, we consider the following matrix:

$$\begin{aligned} &m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) \\ =\ &\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \langle y_3, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \langle y_1, g_3\rangle & \langle y_2, g_3\rangle & \langle y_3, g_3\rangle & \cdots & \langle y_n, g_3\rangle & \langle y_{n+1}, g_3\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_n\rangle & \langle y_2, g_n\rangle & \langle y_3, g_n\rangle & \cdots & \langle y_n, g_n\rangle & \langle y_{n+1}, g_n\rangle \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \langle y_3, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\ =\ &\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \frac{\varepsilon_{2,3}-\varepsilon_{1,3}}{\|x_2-x_1\|} & \frac{1-\varepsilon_{2,3}}{\|x_3-x_2\|} & \frac{\varepsilon_{4,3}-1}{\|x_4-x_3\|} & \cdots & \frac{\varepsilon_{n+1,3}-\varepsilon_{n,3}}{\|x_{n+1}-x_n\|} & \frac{\varepsilon_{n+2,3}-\varepsilon_{n+1,3}}{\|x_{n+2}-x_{n+1}\|} \\ \frac{\varepsilon_{2,4}-\varepsilon_{1,4}}{\|x_2-x_1\|} & \frac{\varepsilon_{3,4}-\varepsilon_{2,4}}{\|x_3-x_2\|} & \frac{1-\varepsilon_{3,4}}{\|x_4-x_3\|} & \cdots & \frac{\varepsilon_{n+1,4}-\varepsilon_{n,4}}{\|x_{n+1}-x_n\|} & \frac{\varepsilon_{n+2,4}-\varepsilon_{n+1,4}}{\|x_{n+2}-x_{n+1}\|} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\varepsilon_{2,n+1}-\varepsilon_{1,n+1}}{\|x_2-x_1\|} & \frac{\varepsilon_{3,n+1}-\varepsilon_{2,n+1}}{\|x_3-x_2\|} & \frac{\varepsilon_{4,n+1}-\varepsilon_{3,n+1}}{\|x_4-x_3\|} & \cdots & \frac{1-\varepsilon_{n,n+1}}{\|x_{n+1}-x_n\|} & \frac{\varepsilon_{n+2,n+1}-1}{\|x_{n+2}-x_{n+1}\|} \\ \frac{\varepsilon_{2,n+2}-\varepsilon_{1,n+2}}{\|x_2-x_1\|} & \frac{\varepsilon_{3,n+2}-\varepsilon_{2,n+2}}{\|x_3-x_2\|} & \frac{\varepsilon_{4,n+2}-\varepsilon_{3,n+2}}{\|x_4-x_3\|} & \cdots & \frac{\varepsilon_{n+1,n+2}-\varepsilon_{n,n+2}}{\|x_{n+1}-x_n\|} & \frac{1-\varepsilon_{n+1,n+2}}{\|x_{n+2}-x_{n+1}\|} \end{bmatrix}. \end{aligned}$$

It follows then that

$$\det m(y_1, y_2, \ldots, y_{n+1}; g_2, g_3, \ldots, g_{n+1}) = 1 + c\varepsilon,$$

where c is a bounded constant.

On the other hand, since

$$\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \leq \frac{\|x_{n+2} - x_1\|}{n+1} + \varepsilon \leq \frac{1+\varepsilon}{n+1} + \varepsilon,$$

we have

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \geq 1 - \frac{1+\varepsilon}{n+1} - \varepsilon.$$

Let $z_1 = y_1$. Next, let $i = 2, 3, \ldots, n+1$.

From Bishop-Phelps-Bollobás result (Lemma 2.4), there exist $z_i \in S_X$ and $h_i \in \nabla_{z_i}$ such that

$$\|y_i - z_i\| < \varepsilon \text{ and } \|g_i - h_i\| < \varepsilon.$$

In particular,

$$|\langle z_i, h_j\rangle - \langle y_i, g_j\rangle| \leq |\langle z_i - y_i, g_j\rangle| + |\langle y_i, h_j - g_j\rangle| + |\langle z_i - y_i, h_j - g_j\rangle| \leq 3\varepsilon.$$

This implies that

$$\begin{aligned}
&\det m(z_1, z_2, \ldots, z_{n+1}, h_2, h_3, \ldots, h_{n+1}) \\
&= \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \langle z_1, h_2\rangle & \langle z_2, h_2\rangle & \langle z_3, h_2\rangle & \cdots & \langle z_n, h_2\rangle & \langle z_{n+1}, h_2\rangle \\ \langle z_1, h_3\rangle & \langle z_2, h_3\rangle & \langle z_3, h_3\rangle & \cdots & \langle z_n, h_3\rangle & \langle z_{n+1}, h_3\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle z_1, h_n\rangle & \langle z_2, h_n\rangle & \langle z_3, h_n\rangle & \cdots & \langle z_n, h_n\rangle & \langle z_{n+1}, h_n\rangle \\ \langle z_1, h_{n+1}\rangle & \langle z_2, h_{n+1}\rangle & \langle z_3, h_{n+1}\rangle & \cdots & \langle z_n, h_{n+1}\rangle & \langle z_{n+1}, h_{n+1}\rangle \end{bmatrix} \\
&= 1 + d\varepsilon,
\end{aligned}$$

where d is a bounded constant. Hence

$$1 - \frac{\|z_1 + z_2 + \cdots + z_{n+1}\|}{n+1} \geq 1 - \frac{1+\varepsilon}{n+1} - 2\varepsilon.$$

Since ε can be arbitrarily small, it follows from the definition of $W_X^n(\cdot)$ that

$$W_X^n(1) \geq 1 - \frac{1}{n+1}.$$

This completes the proof. □

Theorem 2.13. *If X is a Banach space satisfying one of the following two conditions*:

- $W_X^n(1) < 1 - \frac{1}{n+1}$ *for some $n \in \mathbb{N}$ with $n \geq 2$; or*
- $W_X^1(1) < \frac{1}{2}$ *and $W_X^1(\frac{5}{3}) < \frac{2}{3}$ for $n = 1$.*

Then X has normal structure.

Proof. Since X is reflexive, it follows that B_{X^*} is weak* sequentially compact. Moreover, $\frac{2n+1}{2^n} < 1$ for $n \in \mathbb{N}$ and $n \geq 3$, and $\frac{7}{9} < 1$ for $n = 2$. The first result is a direct consequence of Theorems 2.8, 2.9 and 2.12. The second result is a direct consequence of Theorems 2.10 and 2.12. □

Definition 2.14 ([4, 5])**.** Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$,

$$(1-\varepsilon)\|y\| \le \|Ty\| \le (1+\varepsilon)\|y\|.$$

We say that X is *super-reflexive* if any space Y which is finitely representable in X is reflexive.

Theorem 2.15. *If X is a Banach space with $W_X^n(\frac{2n+1}{2^n}) < 1 - \frac{1}{n+1}$ for some $n \in \mathbb{N}$ and $n \ge 2$, or $W_X^1(\frac{2m+1}{m+1}) < \frac{m}{m+1}$ for $n = 1$ and some $m \in \mathbb{N}$, then X is super-reflexive. In particular, for $m = 2$ we have if $W_X^1(\frac{5}{3}) < \frac{2}{3}$, then X is super-reflexive.*

Proof. We only prove the first part (for $n \ge 2$). The proof of second part (for $n = 1$) is same.

The proof is similar to that of Theorem 2.12 in [6]. Suppose that X is not super-reflexive. Then there exists a nonreflexive Banach space Y such that Y can be finitely representable. From Remark 2.2 and Theorem 2.8, for each n there exists some positive function $f(\varepsilon)$ which goes to 0 as ε goes to 0, satisfies $W_Y^n(\frac{2n+1}{2^n} - \varepsilon) > 1 - \frac{1}{n+1} - f(\varepsilon)$. Therefore, there exist $\{y_i\}_{i=1}^{n+1} \subseteq S_Y$ and $\{g_i\} \in \nabla_{y_i} \subseteq S_{Y^*}$ for $i = 2, \dots, n+1$ such that

$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \le \frac{2n+1}{2^n} - \varepsilon,$$

and

$$1 - \frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} > 1 - \frac{1}{n+1} - f(\varepsilon).$$

Let $N = \operatorname{span}\{y_1, y_2, \dots, y_{n+1}\}$, and $T : N \to M \subseteq X$ be an isomorphism with range M.

Let us consider the conjugate mapping T^* of T. Let $g_{i|N}$ be the restricting g_i on N. Then $\langle Ty_j, (T^*)^{-1} g_{i|N}\rangle = \langle y_j, g_i\rangle$ for $1 \le i, j \le n+1$.

We have

$$1 - \varepsilon \le \|T\| \le 1 + \varepsilon,$$

$$1 - \varepsilon \le \|T^*\| \le 1 + \varepsilon,$$

and

$$1 - \varepsilon \le \|(T^*)^{-1}\| \le 1 + \varepsilon.$$

Let $x_i = Ty_i$ and $f_i = (T^*)^{-1} g_{i|N}$ for $i = 1, \dots, n+1$. Then

$$\langle x_j, f_i\rangle = \langle Ty_j, (T^*)^{-1} g_{i|N}\rangle = \langle y_j, g_i\rangle.$$

If $i = j$, then $\langle x_i, f_i\rangle = \langle y_i, g_i\rangle = 1$, so $f_i \in \nabla_{x_i}$ and we have

$$\begin{aligned}
&\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle x_1, f_2\rangle & \langle x_2, f_2\rangle & \cdots & \langle x_n, f_2\rangle & \langle x_{n+1}, f_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle x_1, f_{n+1}\rangle & \langle x_2, f_{n+1}\rangle & \cdots & \langle x_n, f_{n+1}\rangle & \langle x_{n+1}, f_{n+1}\rangle \end{bmatrix} \\
= &\det \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \langle y_1, g_2\rangle & \langle y_2, g_2\rangle & \cdots & \langle y_n, g_2\rangle & \langle y_{n+1}, g_2\rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle y_1, g_{n+1}\rangle & \langle y_2, g_{n+1}\rangle & \cdots & \langle y_n, g_{n+1}\rangle & \langle y_{n+1}, g_{n+1}\rangle \end{bmatrix} \\
\le &\frac{2n+1}{2^n} - \varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} &= \frac{\|T(y_1 + y_2 + \cdots + y_{n+1})\|}{n+1} \\
&\le (1+\varepsilon)\frac{\|y_1 + y_2 + \cdots + y_{n+1}\|}{n+1} \\
&\le \frac{1+\varepsilon}{n+1} + (1+\varepsilon)f(\varepsilon).
\end{aligned}$$

This implies that

$$1 - \frac{\|x_1 + x_2 + \cdots + x_{n+1}\|}{n+1} \ge 1 - \frac{1+\varepsilon}{n+1} - (1+\varepsilon)f(\varepsilon).$$

Since $f(\varepsilon)$ can be arbitrarily small, we have

$$W_X^n\left(\frac{2n+1}{2^n}\right) \ge 1 - \frac{1}{n+1}.$$

This completes the proof. □

We consider the uniform normal structure. To discuss this result, let us recall the concept of the "ultra"-technique.

Let $\mathcal{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to *converge to x with respect to $\mathcal{F}$*, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq \mathcal{U}$ or $I - A \subseteq \mathcal{U}$;
(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 2.16 ([3, 12]). Let $\mathcal{U}$ be an ultrafilter on I and let $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The *ultra-product* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultra-product. It follows from remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{2.1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultra-product. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 2.17 ([12]). *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; and in this case, the mapping J defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle \quad \textit{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 2.18. *Let X be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) = W^n_X(\varepsilon)$.*

Proof. Since X can be embedded into $X_{\mathcal{U}}$ isometrically, we may consider X as a subspace of $X_{\mathcal{U}}$. From the definition of $W^n_X(\varepsilon)$, we have $W^n_{X_{\mathcal{U}}}(\varepsilon) \geq W^n_X(\varepsilon)$.

We prove the reverse inequality.

For any very small $\eta > 0$, from the definition of $W^n_{X_{\mathcal{U}}}(\varepsilon)$, let $(x^1_i)_{\mathcal{U}}$, $(x^2_i)_{\mathcal{U}}$, $\dots$, $(x^n_i)_{\mathcal{U}}$, $(x^{n+1}_i)_{\mathcal{U}}$ belong to $S_{X_{\mathcal{U}}}$, and let $(f^2_i)_{\mathcal{U}} \in \nabla_{(x^2_i)_{\mathcal{U}}}$, $(f^3_i)_{\mathcal{U}} \in \nabla_{(x^3_i)_{\mathcal{U}}}$, $\dots$, $(f^n_i)_{\mathcal{U}} \in \nabla_{(x^n_i)_{\mathcal{U}}}$, $(f^{n+1}_i)_{\mathcal{U}} \in \nabla_{(x^{n+1}_i)_{\mathcal{U}}}$ be such that

$$m((x^1_i)_{\mathcal{U}}, (x^2_i)_{\mathcal{U}}, \dots, (x^n_i)_{\mathcal{U}}, (x^{n+1}_i)_{\mathcal{U}}; (f^2_i)_{\mathcal{U}}, (f^3_i)_{\mathcal{U}}, \dots, (f^n_i)_{\mathcal{U}}, (f^{n+1}_i)_{\mathcal{U}}) \leq \varepsilon,$$

and

$$1 - \frac{\|(x^1_i)_{\mathcal{U}} + (x^2_i)_{\mathcal{U}} + \cdots + (x^n_i)_{\mathcal{U}} + (x^{n+1}_i)_{\mathcal{U}}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - \eta.$$

Without loss of generality, we may assume by (2.1) that

$$1 - \eta < \|(x^j_i)_{\mathcal{U}}\| < 1 + \eta \text{ for } j = 1, \dots, n+1,$$

$$1 - \eta < \|(f^j_i)_{\mathcal{U}}\| < 1 + \eta \text{ for } j = 2, \dots, n+1,$$

and

$$1 - \eta < \langle (x^j_i)_{\mathcal{U}}, (f^j_i)_{\mathcal{U}}\rangle < 1 + \eta \text{ for } j = 2, \dots, n+1.$$

From the property of ultra-product, we know the subsets

$$P = \{i : m((x^1_i)_{\mathcal{U}}, (x^2_i)_{\mathcal{U}}, \dots, (x^n_i)_{\mathcal{U}}, (x^{n+1}_i)_{\mathcal{U}}; (f^2_i)_{\mathcal{U}}, (f^3_i)_{\mathcal{U}}, \dots, (f^n_i)_{\mathcal{U}}, (f^{n+1}_i)_{\mathcal{U}}) \leq \varepsilon\}$$

and

$$Q = \left\{ i : 1 - \frac{\|(x_i^1)_{\mathcal{U}} + (x_i^2)_{\mathcal{U}} + \cdots + (x_i^n)_{\mathcal{U}} + (x_i^{n+1})_{\mathcal{U}}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - \eta \right\}$$

are all in $\mathcal{U}$. So the intersection $P \cap Q$ is in $\mathcal{U}$ too, and hence is not empty.

Let $i \in P \cap Q$ be fixed. Then

$$1 - \eta < \|x_i^j\| < 1 + \eta \text{ for } j = 1, \dots, n+1;$$

$$1 - \eta < \|f_i^j\| < 1 + \eta \text{ for } j = 2, \dots, n+1;$$

$$1 - \eta < \langle x_i^j, f_i^j \rangle < 1 + \eta \text{ for } j = 2, \dots, n+1;$$

$$m(x_i^1, x_i^2, \dots, x_i^n, x_i^{n+1}; f_i^2, f_i^3, \dots, f_i^n, f_i^{n+1}) \le \varepsilon;$$

and

$$1 - \frac{\|x_i^1 + x_i^2 + \cdots + x_i^n + x_i^{n+1}\|}{n+1} > W^n_{X_U}(\varepsilon) - \eta.$$

From Lemma 2.4, for $0 < \eta < 1$ (since η can be arbitrarily small, if necessary we can normalize vectors x_i^j and f_i^j to use Lemma 2.4) there are $\{y_j\}_{j=1}^{n+1} \subseteq S_X$ and $\{g_j\}_{j=2}^{n+1} \subseteq S_{X^*}$ such that

- $g_j \in \nabla_{y_j}$ for all $j = 2, \dots, n+1$;
- $\|x_i^j - y_j\| < \eta$ for all $j = 1, \dots, n+1$;
- $\|f_i^j - g_j\| < \eta$ for $j = 2, \dots, n+1$.

Similar to the proof of Theorem 2.8, we have

$$\det m(y_1, y_2, \dots, y_n, y_{n+1}; g_2, g_3, \dots, g_n, g_{n+1}) \le \varepsilon + c\eta,$$

and $1 - \frac{\|y_1 + y_2 + \cdots + y_n + y_{n+1}\|}{n+1} > W^n_{X_{\mathcal{U}}}(\varepsilon) - d\eta$, where c and d are constants.

Since $\eta > 0$ can be arbitrarily small, we have $W^n_X(\varepsilon) \ge W^n_{X_{\mathcal{U}}}(\varepsilon)$. □

Lemma 2.19 ([8]). *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 2.20. *Suppose that X is a Banach space satisfying one of the following conditions*:

- $W^n_X(1) < 1 - \frac{1}{n+1}$ *for some $n \in \mathbb{N}$ with $n \ge 2$; or*
- $W^1_X(1) < \frac{1}{2}$ *and* $W^1_X(\frac{5}{3}) < \frac{2}{3}$ *for $n = 1$.*

Then X has uniform normal structure.

Proof. It follows directly from Theorems 2.13, 2.15, 2.18 and Lemma 2.19. □

Example. Let H be a Hilbert space. We have $W^1_H(\varepsilon) = \frac{2-\sqrt{4-2\varepsilon}}{2}$ for $0 \le \varepsilon \le 2$.

Since $W^1_H(1) = \frac{2-\sqrt{2}}{2} = 0.29289\cdots < \frac{1}{2}$, and $W^1_H(\frac{5}{3}) = \frac{2-\sqrt{\frac{2}{3}}}{2} = 0.59175\cdots < \frac{2}{3}$, from Theorem 2.20, H has uniform normal structure.

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References

[1] B. Bollobás, *An extension to the theorem of Bishop and Phelps*, Bull. London Math. Soc. **2** (1970), 181–182.

[2] M. S. Brodskiĭ and D. P. Mil'man, *On the center of a convex set*, (Russian) Doklady Akad. Nauk SSSR (N.S.) **59** (1948), 837–840.

[3] D. Dacunha-Castelle and J. L. Krivine, *Applications des ultraproduits à l'étude des espaces et des algèbres de Banach*, (French) Studia Math. **41** (1972), 315–334.

[4] M. M. Day, *Normed Linear Spaces*, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 21. Springer-Verlag, New York-Heidelberg, 1973.

[5] J. Diestel, *Geometry of Banach spacesselected topics*, Lecture Notes in Mathematics, Vol. 485, Springer-Verlag, Berlin-New York, 1975.

[6] J. Gao and S. Saejung, *The n-dimensional U-convexity and geometry of Banach spaces*, J. Fixed Point Theory **16** (2015), no. 2, 381–392.

[7] R. C. James, *Weakly compact sets*, Trans. Amer. Math. Soc. **113** (1964), 129–140.

[8] M. A. Khamsi, *Uniform smoothness implies super-normal structure property*, Nonlinear Anal. **19** (1992), no. 11, 1063–1069.

[9] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly **72** (1965), 1004–1006.

[10] T.-C. Lim, *On moduli of k-convexity*, Abstr. Appl. Anal. **4** (1999), no. 4, 243–247.

[11] S. Saejung, *Convexity conditions and normal structure of Banach spaces*, J. Math. Anal. Appl. **344** (2008), no. 2, 851–856.

[12] B. Sims, *"Ultra"-techniques in Banach space theory*, Queen's Papers in Pure and Applied Mathematics, Vol. 60, Queen's University, Kingston, ON, 1982.

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