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ZERO-DENSITY ESTIMATES FOR EPSTEIN ZETA FUNCTIONS OF CLASS NUMBERS 2 OR 3

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Abstract. We investigate the zeros of Epstein zeta functions associated with positive definite quadratic forms with rational coefficients in the vertical strip $\sigma_1 < \Re s < \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$. When the class number h of the quadratic form is bigger than 1, Voronin gave a lower bound and Lee gave an asymptotic formula for the number of zeros. Recently Gonek and Lee improved their results by providing a new upper bound for the error term when $h > 3$. In this paper, we consider the cases $h = 2, 3$ and provide an upper bound for the error term, smaller than the one for the case $h > 3$.

1. Introduction

Define $Q(m,n)$ to be a positive definite quadratic form $am^2 + bmn + cn^2$ with $a, b, c \in \mathbb{Z}$ and let its discriminant be $D = b^2 - 4ac < 0$. Let $s = \sigma + it$ be a complex variable. The Epstein zeta function associated with Q is defined by

$$E(s,Q) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{Q(m,n)^s}$$

for $\sigma > 1$. It has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. Recently Gonek and Lee in [1] estimated $N(\sigma_1, \sigma_2; T)$, the number of zeros of $E(s,Q)$ in the rectangular region $\sigma_1 < \sigma \le \sigma_2$, $T < t \le 2T$ for $1/2 < \sigma_1 < \sigma_2 < 1$, when the class number $h(D)$ of $\mathbb{Q}(\sqrt{D})$ is bigger than 3, and obtained

$$N(\sigma_1, \sigma_2; T) = cT + O(T \exp(-b\sqrt{\log\log T})) \tag{1.1}$$

for some $b > 0$, where $c = c(\sigma_1, \sigma_2, Q) > 0$. The purpose of this paper is to find a stronger estimation of $N(\sigma_1, \sigma_2; T)$ for $1/2 < \sigma_1 < \sigma_2 < 1$ when $h(D) = 2$ or 3 as follows.

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Theorem 1. *Suppose that* $h(D) = 2$ *or* 3. *Then*

$$N(\sigma_1, \sigma_2; T) = cT + O\left(T \frac{\log\log T}{(\log T)^{\sigma_1/2}}\right)$$

holds for $1/2 < \sigma_1 < \sigma_2 < 1$, *where* $c = c(\sigma_1, \sigma_2, Q) > 0$.

We begin our estimation from the well-known identity

$$E(s, Q) = \frac{w_D}{h(D)} \sum_{\chi} \overline{\chi(\mathfrak{a}_Q)} L(s, \chi), \tag{1.2}$$

where w_D is the number of roots of unity in $\mathbb{Q}(\sqrt{D})$, the χ-sum is over all ideal class characters of $\mathbb{Q}(\sqrt{D})$, $\mathfrak{a}_Q$ is an integral ideal corresponding to Q and $L(s, \chi)$ is the Hecke L-function associated with χ. When $h(D) = 2$, let 1 and χ_1 be the ideal class characters of $\mathbb{Q}(\sqrt{D})$. By (1.2) we see that

$$E(s, Q) = \frac{w_D}{h(D)} (L(s, 1) + \overline{\chi_1(\mathfrak{a}_Q)} L(s, \chi_1)).$$

When $h(D) = 3$, let 1, χ_2 and $\overline{\chi_2}$ be the ideal class characters of $\mathbb{Q}(\sqrt{D})$. Since $L(s, \chi_2) = L(s, \overline{\chi_2})$, by (1.2) we find that

$$E(s, Q) = \frac{w_D}{h(D)} (L(s, 1) + 2\Re[\chi_2(\mathfrak{a}_Q)] L(s, \chi_2)).$$

When $h(D) > 3$, $E(s, Q)$ is a linear combination of more than two inequivalent Hecke L-functions. Therefore, $E(s, Q)$ is a linear combination of two inequivalent Hecke L-functions if and only if $h(D)$ is two or three.

Suppose that $h(D) = 2$ or 3 and write

$$E(s, Q) = c_1 L_1(s) + c_2 L_2(s),$$

where $c_1, c_2 \neq 0$ and $L_1(s)$ and $L_2(s)$ are two inequivalent Hecke L-functions. By Littlewood's lemma, we need to estimate the integral

$$\int_T^{2T} \log |E(s, Q)| dt$$

to prove Theorem 1 for $1/2 < \sigma < 1$. We split the integral into two pieces:

$$\int_T^{2T} \log |E(s, Q)| dt = \int_T^{2T} \log |c_1 L_2(s)| dt + \int_T^{2T} \log \left| \frac{L_1(s)}{L_2(s)} + \frac{c_2}{c_1} \right| dt. \tag{1.3}$$

The first integral on the right of (1.3) is computed using a short Dirichlet polynomial approximation of $\log |L_2(s)|$ and the second integral is estimated using a method in [2] and lemmas in [1]. These lead to the following theorem.

Theorem 2. *Suppose that* $h(D) = 2$ *or* 3. *Then*

$$\frac{1}{T} \int_T^{2T} \log |E(s, Q)| dt = \mathcal{M}(\sigma) + O\left(\frac{(\log\log T)^2}{(\log T)^{\sigma}}\right)$$

holds for $1/2 < \sigma < 1$, *and* $\mathcal{M}(\sigma)$ *has a continuous second derivative.*

Note that $\mathcal{M}(\sigma)$ is the Jensen function of $E(s,Q)$. The assertion in Theorem 2 that $\mathcal{M}(\sigma)$ has a continuous second derivative, is proved in [3]. The proof of Theorem 2 provides a representation for $\mathcal{M}(\sigma)$, namely

$$\mathcal{M}(\sigma) = \mathbb{E}[\log |c_1 L_1(\sigma, X) + c_2 L_2(\sigma, X)|].$$

Here

$$L_j(\sigma, X) := \prod_{\mathfrak{p}} \left(1 - \frac{\chi_j(\mathfrak{p}) X(p)}{\mathfrak{N}(\mathfrak{p})^\sigma}\right)^{-1}$$

is a random model of the Hecke L-functions $L_j(s) = L(s, \chi_j)$ for $j = 1, 2$, where p is the unique rational prime dividing $\mathfrak{N}(\mathfrak{p})$, and the $X(p)$ are uniformly and independently distributed on the unit circle $\mathbb{T}$.

We shall briefly explain why the estimation in Theorem 1 is better than (1.1). When $h(D) = 2$ or 3, we will prove that the second integral on the right of (1.3) is essentially

$$\int_{B_2(T)} \log \left| \frac{L_1(s)}{L_2(s)} + \frac{c_2}{c_1} \right| dt,$$

where $B_2(T)$ is the inverse image of the union of two rectangular regions in $\mathbb{C}$ under $\log L_1(s)/L_2(s)$ and it contains no singular points of the integrand. (See (3.5) and (3.3) for the definition of $B_2(T)$.) Then the above integral can be estimated using the discrepancy lemma (Lemma 6) and the distribution of the random model $\log L_1(\sigma, X)/L_2(\sigma, X)$. However, it is not possible with two rectangular regions in the cases for bigger $h(D)$. Indeed, Gonek and Lee in [1] required $(\log T)^A$ many rectangular regions to prove (1.1) for $h(D) > 3$ and the argument was elaborate.

We introduce the lemmas we will require in Section 2. Since Theorem 1 is a consequence of Theorem 2, we first prove Theorem 2 in Section 3 and then Theorem 1 in Section 4.

2. Lemmas

We state lemmas from [1] without proofs.

Lemma 1 (Lemma 2.2 of [1])**.** *Let $L(s) = L(s, \chi)$ be a Hecke L-function attached to an ideal class character of the quadratic field $\mathbb{Q}(\sqrt{D})$. For $\sigma > 1$ write*

$$\log L(s) = \sum_{p,n} \frac{a(p^n)}{p^{ns}},$$

and for $Y \geq 2$ and any s let

$$R_Y(s) = \sum_{p^n \leq Y} \frac{a(p^n)}{p^{ns}}.$$

Suppose that $1/2 < \sigma < 1$ and $B_1 > 0$ are fixed, and that $Y = (\log T)^{B_2}$ with $B_2 > 2(B_1 + 1)/(\sigma - 1/2)$. Then

$$\log L(s) = R_Y(s) + O\big((\log T)^{-B_1}\big)$$

for all $t \in [T, 2T]$ *except on a set of measure* $\ll T^{1-d(\sigma)}$, *where* $d(\sigma) > 0$.

Lemma 2 (Lemma 3.1 of [1]). *Let* $1/2 < \sigma \le 2$ *be fixed. There exists a constant* $C > 0$ *depending at most on* J *such that for every positive integer* k *we have*

$$\frac{1}{T}\int_T^{2T} \Big| \log\Big|\sum_{j\le J} c_j L_j(s)\Big|\Big|^{2k} dt \ll (Ck)^{4k}.$$

Lemma 3 (Lemma 3.2 of [1]). *Let* $1/2 < \sigma \le 2$ *be fixed. There exist an absolute constant* $C_1 > 0$ *and a constant* $C_2 > 0$ *depending on* σ *such that for every positive integer* $k \le \log T/(C_2 \log\log T)$, *we have*

$$\frac{1}{T}\int_T^{2T} |\log L_j(s)|^{2k} dt \ll (C_1 k)^k$$

and

$$\frac{1}{T}\int_T^{2T} \big|\log L_i(s) - \log L_j(s)\big|^{2k} dt \ll (C_1 k)^k.$$

Lemma 4 (Lemma 3.3 of [1]). *Let* $1/2 < \sigma \le 2$ *be fixed. For every integer* $k > 0$ *we have*

$$\mathbb{E}\bigg(\Big| \log\Big|\sum_{j\le J} c_j L_j(\sigma, X)\Big|\Big|^{2k}\bigg) \ll (Ck)^{2k}$$

and

$$\mathbb{E}\bigg(\Big| \log c_j L_j(\sigma, X)\Big|^{2k}\bigg) \ll (Ck)^k,$$

where $C > 0$ *is a constant depending at most on* J.

Lemma 5. *Let* a *be a fixed nonzero complex number and* σ, A *be positive constants. Then we have*

$$\int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} \frac{e^w}{|e^{w+iy}-a|^2} dy dw = O_a((\log\log T)^2)$$

as $T \to \infty$.

Proof.

$$\begin{aligned}
&\int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} \frac{e^w}{|e^{w+iy}-a|^2} dy dw \\
&= \int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} \frac{e^{-w}}{|1-ae^{-w-iy}|^2} dy dw \\
&= \int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} e^{-w} \sum_{m,n=0}^{\infty} (ae^{-w})^m (\bar{a}e^{-w})^n e^{-i(m-n)y} dy dw \\
&= 2A|a|^{-1} \log\log T \sum_{m=0}^{\infty} \frac{1}{2m+1} e^{-(2m+1)(\log T)^{-\sigma}} + O\bigg(\frac{\log\log T}{(\log T)^A}\bigg)
\end{aligned}$$

$$+ O\Big(\sum_{m \neq n} \frac{1}{|m-n|(m+n+1)} e^{-(m+n)(\log T)^{-\sigma}} \Big).$$

It is now an easy exercise to prove that it is $O((\log\log T)^2)$. □

3. Proof of Theorem 2

We observe from (1.3) that we need to estimate the two integrals

$$(3.1) \qquad \begin{aligned} I_1 &= \int_T^{2T} \log|L_2(s)| dt, \\ I_2 &= \int_T^{2T} \log\left|\frac{L_1(s)}{L_2(s)} - a\right| dt \end{aligned}$$

for $1/2 < \sigma < 1$ and $a = -c_2/c_1 \neq 0$.

We first estimate I_1. Let

$$\log L_2(s) = \sum_p \sum_{n=1}^{\infty} \frac{a_2(p^n)}{p^{ns}}$$

for $\sigma > 1$. Then by Lemma 1 there exists a set $S_1(T) \subset [T, 2T]$ such that

$$\mathrm{meas}([T,2T] \setminus S_1(T)) \ll T^{1-d(\sigma)}$$

for some $d(\sigma) > 0$, and

$$\log|L_2(s)| = \Re\Big[\sum_{p^n \le (\log T)^{B_2}} \frac{a_2(p^n)}{p^{ns}} \Big] + O((\log T)^{-B_1})$$

for all $t \in S_1(T)$ and for $B_2 > 2(B_1+1)/(\sigma - 1/2)$. Thus, we have

$$\begin{aligned} I_1 = &\int_{S_1(T)} \Re\Big[\sum_{p^n \le (\log T)^{B_2}} \frac{a_2(p^n)}{p^{ns}} \Big] dt + \int_{[T,2T]\setminus S_1(T)} \log|L_2(s)| dt \\ &+ O(T(\log T)^{-B_1}) \\ = &\int_T^{2T} \Re\Big[\sum_{p^n \le (\log T)^{B_2}} \frac{a_2(p^n)}{p^{ns}} \Big] dt - \int_{[T,2T]\setminus S_1(T)} \Re\Big[\sum_{p^n \le (\log T)^{B_2}} \frac{a_2(p^n)}{p^{ns}} \Big] dt \\ &+ \int_{[T,2T]\setminus S_1(T)} \log|L_2(s)| dt + O(T(\log T)^{-B_1}) \\ = &\ I_{1,1} - I_{1,2} + I_{1,3} + O(T(\log T)^{-B_1}). \end{aligned}$$

The first integral $I_{1,1}$ is

$$I_{1,1} = \Re\Big[\sum_{p^n \le (\log T)^{B_2}} \frac{a_2(p^n)}{p^{n\sigma}} \frac{p^{-2nTi} - p^{-nTi}}{-in\log p} \Big] \ll (\log T)^{B_2(1-\sigma)}.$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
&|I_{1,2}| \\
&\le (\text{meas}([T,2T]\setminus S_1(T)))^{1/2}\left(\int_{[T,2T]\setminus S_1(T)}\left|\Re\left[\sum_{p^n\le(\log T)^{B_2}}\frac{a_2(p^n)}{p^{ns}}\right]\right|^2 dt\right)^{1/2} \\
&\le (\text{meas}([T,2T]\setminus S_1(T)))^{1/2}\left(\int_T^{2T}\left|\sum_{p^n\le(\log T)^{B_2}}\frac{a_2(p^n)}{p^{ns}}\right|^2 dt\right)^{1/2} \\
&\ll T^{1-d(\sigma)/2}.
\end{aligned}$$

Similarly by the Cauchy-Schwarz inequality and Lemma 3

$$|I_{1,3}| \le (\text{meas}([T,2T]\setminus S_1(T)))^{1/2}\left(\int_T^{2T}|\log|L_2(s)||^2 dt\right)^{1/2} \ll T^{1-d(\sigma)/2}.$$

Thus,

$$I_1 = O(T(\log T)^{-B_1})$$

for any fixed $B_1 > 0$. Since

$$\mathbb{E}(\log|L_2(\sigma,X)|) = \mathbb{E}\left(\Re\sum_p\sum_{n=1}^{\infty}\frac{a_2(p^n)X(p)^n}{p^{n\sigma}}\right) = 0,$$

we can also write

$$I_1 = \mathbb{E}(\log|L_2(\sigma,X)|) + O(T(\log T)^{-B_1}). \tag{3.2}$$

We next estimate I_2. Define

$$\begin{aligned}
B_1(T) = \{t\in[T,2T] : |\log|L_1(s)/L_2(s)|| \le A\log\log T,\\
|\arg L_1(s)/L_2(s)| \le A\log\log T\}
\end{aligned} \tag{3.3}$$

for a large constant $A > 0$. By Lemma 3

$$\begin{aligned}
&\text{meas}([T,2T]\setminus B_1(T)) \\
&\le \int_{[T,2T]\setminus B_1(T)}\frac{|\log|L_1(s)/L_2(s)||^{2k} + |\arg L_1(s)/L_2(s)|^{2k}}{(A\log\log T)^{2k}}dt \\
&\le \int_T^{2T}\frac{|\log|L_1(s)/L_2(s)||^{2k} + |\arg L_1(s)/L_2(s)|^{2k}}{(A\log\log T)^{2k}}dt \\
&\ll T\left(\frac{Ck}{A\log\log T}\right)^{2k}
\end{aligned}$$

for some $C > 0$. Choosing $k = A\log\log T/(Ce)$, we obtain

$$\text{meas}([T,2T]\setminus B_1(T)) \ll T(\log T)^{-2A/(Ce)}. \tag{3.4}$$

By the Cauchy-Schwarz inequality, (3.4) and Lemma 2

$$\int_{[T,2T]\setminus B_1(T)}\log\left|\frac{L_1(s)}{L_2(s)} - a\right|dt$$

$$\le (\text{meas}([T,2T]\setminus B_1(T)))^{1/2}\bigg(\int_T^{2T}\bigg|\log\bigg|\frac{L_1(s)}{L_2(s)}-a\bigg|\bigg|^2 dt\bigg)^{1/2}$$
$$\ll T(\log T)^{-A/(Ce)}.$$

Thus,

$$I_2=\int_{B_1(T)}\log\bigg|\frac{L_1(s)}{L_2(s)}-a\bigg|dt+O(T(\log T)^{-A/(Ce)}).$$

Next we want to remove a set of t near logarithmic singularities of the integrand out of the integral. Define

$$B_2(T)=\{t\in B_1(T):|\log|L_1(s)/L_2(s)|-\log|a||\ge(\log T)^{-\sigma}\}. \tag{3.5}$$

To show that the set $B_1(T)\setminus B_2(T)$ has small measure, we define two functions

$$\mathbb{L}(s)=(\Re\log L_1(s)/L_2(s),\Im\log L_1(s)/L_2(s)),$$
$$\mathbb{L}(\sigma,X)=(\Re\log L_1(\sigma,X)/L_2(\sigma,X),\Im\log L_1(\sigma,X)/L_2(\sigma,X)),$$

and two measures

$$\Psi_T(\mathcal{B})=\frac{1}{T}\text{meas}\{t\in[T,2T]:\mathbb{L}(s)\in\mathcal{B}\},$$
$$\Psi(\mathcal{B})=\mathbb{P}(\mathbb{L}(\sigma,X)\in\mathcal{B})$$

for Borel sets $\mathcal{B}\in\mathbb{R}^2$ and for a fixed $1/2<\sigma<1$. Then we have the following lemma.

Lemma 6 (Theorem 1.2 of [1]). *Let $1/2<\sigma<1$ be fixed. Then*

$$\sup_{\mathcal{R}}|\Psi_T(\mathcal{R})-\Psi(\mathcal{R})|\ll(\log T)^{-\sigma},$$

where $\mathcal{R}$ runs over all rectangular regions in $\mathbb{R}^2$ (possibly unbounded) with sides parallel to the coordinate axes.

By Lemma 6 and the absolute continuity of Ψ, we have that

$$\begin{aligned}\frac{1}{T}\text{meas}(B_1(T)\setminus B_2(T))&=\Psi_T(\mathcal{R}_1)=\Psi(\mathcal{R}_1)+O((\log T)^{-\sigma})\\&=O((\log T)^{-\sigma}),\end{aligned} \tag{3.6}$$

where

$$\mathcal{R}_1=(\log|a|-(\log T)^{-\sigma},\log|a|+(\log T)^{-\sigma})\times[-A\log\log T,A\log\log T].$$

By Hölder's inequality with $\mathscr{L}=\log\log T$, (3.6) and Lemma 2, we find that

$$\int_{B_1(T)\setminus B_2(T)}\log\bigg|\frac{L_1(s)}{L_2(s)}-a\bigg|dt$$
$$\le(\text{meas}(B_1(T)\setminus B_2(T)))^{1-1/(2\mathscr{L})}\bigg(\int_T^{2T}\bigg|\log\bigg|\frac{L_1(s)}{L_2(s)}-a\bigg|\bigg|^{2\mathscr{L}}dt\bigg)^{1/(2\mathscr{L})}$$
$$\ll T\frac{(\log\log T)^2}{(\log T)^{\sigma}}.$$

Hence

$$I_2 = \int_{B_2(T)} \log\left|\frac{L_1(s)}{L_2(s)} - a\right| dt + O\left(T\frac{(\log\log T)^2}{(\log T)^\sigma}\right).$$

We split $B_2(T)$ into two sets

$B_{2,1}(T) = \{t \in B_1(T) : \log|a| + (\log T)^{-\sigma} \le \log|L_1(s)/L_2(s)| \le A\log\log T\}$,

$B_{2,2}(T) = \{t \in B_1(T) : -A\log\log T \le \log|L_1(s)/L_2(s)| \le \log|a| - (\log T)^{-\sigma}\}$,

and define

$$I_{2,j} = \int_{B_{2,j}(T)} \log\left|\frac{L_1(s)}{L_2(s)} - a\right| dt$$

for $j = 1, 2$. Then

$$I_2 = I_{2,1} + I_{2,2} + O\left(T\frac{(\log\log T)^2}{(\log T)^\sigma}\right).$$

We now estimate $I_{2,1}$. Define

$$\mathcal{R}_2 = [\log|a| + (\log T)^{-\sigma}, A\log\log T] \times [-A\log\log T, A\log\log T].$$

Then we see that

$$B_{2,1}(T) = \{t \in [T, 2T] : \mathbb{L}(s) \in \mathcal{R}_2\}$$

and

$$\frac{1}{T}I_{2,1} = \frac{1}{T}\int_{B_{2,1}(T)} \log\left|\frac{L_1(s)}{L_2(s)} - a\right| dt = \int_{\mathcal{R}_2} \log|e^{u+iv} - a| d\Psi_T(u,v).$$

We split the integral into two

$$\frac{1}{T}I_{2,1} = \int_{\mathcal{R}_2} \log|e^{u+iv} - a| d\Psi(u,v) + \int_{\mathcal{R}_2} \log|e^{u+iv} - a| dD_T(u,v),$$

where $D_T = \Psi_T - \Psi$. We want to show that the last integral is small. We divide it into three pieces as

$$\begin{aligned}
&\int_{\mathcal{R}_2} \log|e^{u+iv} - a| dD_T(u,v)\\
&= \int_{\mathcal{R}_2} \log|e^{u+iv} - a| - \log|e^{A\log\log T + iv} - a| dD_T(u,v)\\
&\quad + \int_{\mathcal{R}_2} \log|e^{A\log\log T + iv} - a| - \log|e^{A\log\log T + iA\log\log T} - a| dD_T(u,v)\\
&\quad + \int_{\mathcal{R}_2} \log|e^{A\log\log T + iA\log\log T} - a| dD_T(u,v)\\
&= J_1 + J_2 + J_3.
\end{aligned}$$

We first note that

$$J_3 = \log|e^{A\log\log T + iA\log\log T} - a| D_T(\mathcal{R}_2) \ll \frac{\log\log T}{(\log T)^\sigma}.$$

Next we may write J_2 as

$$J_2 = \int_{\mathcal{R}_2} \int_v^{A \log\log T} \Re\left[\frac{-ie^{A\log\log T + iw}}{e^{A\log\log T+iw} - a}\right] dw\ dD_T(u,v).$$

By changing the order of the integrals, we find that

$$\begin{aligned} J_2 &= \int_{-A\log\log T}^{A\log\log T} \int_{\mathcal{R}_2(w)} \Re\left[\frac{-ie^{A\log\log T + iw}}{e^{A\log\log T+iw} - a}\right]\ dD_T(u,v)\ dw \\ &= \int_{-A\log\log T}^{A\log\log T} D_T(\mathcal{R}_2(w))\ \Re\left[\frac{-ie^{A\log\log T + iw}}{e^{A\log\log T+iw} - a}\right] dw, \end{aligned}$$

where

$$\mathcal{R}_2(w) = [\log|a| + (\log T)^{-\sigma}, A\log\log T] \times [-A\log\log T, w].$$

By Lemma 6

$$J_2 \ll \frac{\log\log T}{(\log T)^\sigma}.$$

Similarly, we may write J_1 as

$$J_1 = -\int_{\mathcal{R}_2} \int_u^{A\log\log T} \Re\left[\frac{e^{w+iv}}{e^{w+iv} - a}\right] dw\ dD_T(u,v).$$

In this case we divide it into two pieces as

$$\begin{aligned} J_1 = &-\int_{\mathcal{R}_2} \int_u^{A\log\log T} \Re\left[\frac{e^{w+iv}}{e^{w+iv}-a}\right] - \Re\left[\frac{e^{w+iA\log\log T}}{e^{w+iA\log\log T} - a}\right] dw\ dD_T(u,v) \\ &- \int_{\mathcal{R}_2} \int_u^{A\log\log T} \Re\left[\frac{e^{w+iA\log\log T}}{e^{w+iA\log\log T} - a}\right] dw\ dD_T(u,v) \\ &= J_4 + J_5. \end{aligned}$$

We can bound J_5 by the same method of bounding J_2 and obtain

$$J_5 \ll \frac{\log\log T}{(\log T)^\sigma}.$$

By changing the integrand of J_4 into an integral, we have

$$J_4 = \int_{\mathcal{R}_2} \int_u^{A\log\log T} \int_v^{A\log\log T} \Re\left[\frac{-aie^{w+iy}}{(e^{w+iy}-a)^2}\right] dydw\ dD_T(u,v).$$

We change the order of integrals and then apply Lemma 6 to prove

$$\begin{aligned} J_4 &= \int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} D_T(\mathcal{R}_2(w,y)) \Re\left[\frac{-aie^{w+iy}}{(e^{w+iy}-a)^2}\right] dydw \\ &\ll \frac{1}{(\log T)^\sigma} \int_{\log|a|+(\log T)^{-\sigma}}^{A\log\log T} \int_{-A\log\log T}^{A\log\log T} \frac{e^w}{|e^{w+iy}-a|^2} dydw, \end{aligned}$$

where

$$\mathcal{R}_2(w,y) = [\log|a| + (\log T)^{-\sigma}, w] \times [-A\log\log T, y].$$

By Lemma 5

$$J_4 \ll \frac{(\log\log T)^2}{(\log T)^\sigma}.$$

By combining the above inequalities, we deduce that

$$\frac{1}{T}I_{2,1} = \int_{\mathcal{R}_2} \log|e^{u+iv} - a| d\Psi(u,v) + O\biggl(\frac{(\log\log T)^2}{(\log T)^\sigma}\biggr).$$

Similarly, we can also prove that

$$\frac{1}{T}I_{2,2} = \int_{\mathcal{R}_3} \log|e^{u+iv} - a| d\Psi(u,v) + O\biggl(\frac{(\log\log T)^2}{(\log T)^\sigma}\biggr),$$

where

$$\mathcal{R}_3 = [-A\log\log T, \log|a| - (\log T)^{-\sigma}] \times [-A\log\log T, A\log\log T].$$

Thus,

$$\frac{1}{T}I_2 = \int_{\mathcal{R}_2\cup\mathcal{R}_3} \log|e^{u+iv} - a| d\Psi(u,v) + O\biggl(\frac{(\log\log T)^2}{(\log T)^\sigma}\biggr).$$

To show that

$$\begin{aligned}\frac{1}{T}I_2 &= \int_{\mathbb{R}^2} \log|e^{u+iv} - a| d\Psi(u,v) + O\biggl(\frac{(\log\log T)^2}{(\log T)^\sigma}\biggr) \\ &= \mathbb{E}\biggl(\log\biggl|\frac{L_1(\sigma,X)}{L_2(\sigma,X)} - a\biggr|\biggr) + O\biggl(\frac{(\log\log T)^2}{(\log T)^\sigma}\biggr),\end{aligned} \tag{3.7}$$

we need to prove that the integral

$$J_4 = \int_{\mathcal{R}_1\cup(\mathbb{R}^2\setminus\mathcal{R}_4)} \log|e^{u+iv} - a| d\Psi(u,v)$$

is smaller than the O-term, where

$$\mathcal{R}_4 = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 = [-A\log\log T, A\log\log T]^2.$$

Observe that

$$\Psi(\mathcal{R}_1) = O((\log T)^{-\sigma})$$

by the absolute continuity of Ψ and

$$\begin{aligned}\Psi(\mathbb{R}^2 \setminus \mathcal{R}_4) &= \mathbb{P}(\mathbb{L}(\sigma,X) \in \mathbb{R}^2 \setminus \mathcal{R}_4) \\ &\le \mathbb{P}(|\Re\log L_1(\sigma,X)/L_2(\sigma,X)| > A\log\log T) \\ &\quad + \mathbb{P}(|\Im\log L_1(\sigma,X)/L_2(\sigma,X)| > A\log\log T).\end{aligned}$$

By Lemma 4 we deduce that

$$\begin{aligned}&\mathbb{P}(|\Re\log L_1(\sigma,X)/L_2(\sigma,X)| > A\log\log T) \\ &\le (A\log\log T)^{-2\mathscr{L}}\mathbb{E}(|\Re\log L_1(\sigma,X)/L_2(\sigma,X)|^{2\mathscr{L}}) \\ &\ll (A\log\log T)^{-2\mathscr{L}}(C\mathscr{L})^{\mathscr{L}}\end{aligned}$$

$$\begin{aligned}&\ll \left(\frac{C\log\log T}{A^2(\log\log T)^2}\right)^{\log\log T}\\&\ll (\log T)^{-\sigma}\end{aligned}$$

for some $C > 0$ and $\mathscr{L} = \log\log T$. Similarly, the same holds for

$$\mathbb{P}(|\Im \log L_1(\sigma, X)/L_2(\sigma, X)| > A\log\log T).$$

Hence,

$$\Psi(\mathbb{R}^2 \setminus \mathcal{R}_4) \ll (\log T)^{-\sigma}. \tag{3.8}$$

By Hölder's inequality, (3.8) and Lemma 4 we deduce that

$$\begin{aligned}J_4 &= \int_{\mathcal{R}_1\cup(\mathbb{R}^2\setminus\mathcal{R}_4)} \log|e^{u+iv} - a|d\Psi(u,v)\\
&\le (\Psi(\mathcal{R}_1 \cup (\mathbb{R}^2 \setminus \mathcal{R}_4))^{1-1/(2\mathscr{L})}\left(\int_{\mathcal{R}_1\cup(\mathbb{R}^2\setminus\mathcal{R}_4)} |\log|e^{u+iv} - a||^{2\mathscr{L}} d\Psi(u,v)\right)^{1/(2\mathscr{L})}\\
&\le (\Psi(\mathcal{R}_1 \cup (\mathbb{R}^2 \setminus \mathcal{R}_4))^{1-1/(2\mathscr{L})}\left(\int_{\mathbb{R}^2} |\log|e^{u+iv} - a||^{2\mathscr{L}} d\Psi(u,v)\right)^{1/(2\mathscr{L})}\\
&= (\Psi(\mathcal{R}_1 \cup (\mathbb{R}^2 \setminus \mathcal{R}_4))^{1-1/(2\mathscr{L})}\left(\mathbb{E}\left[\left|\log\left|\frac{L_1(\sigma,X)}{L_2(\sigma,X)} - a\right|\right|^{2\mathscr{L}}\right]\right)^{1/(2\mathscr{L})}\\
&\ll \frac{\log\log T}{(\log T)^{\sigma}}.\end{aligned}$$

Thus, this proves (3.7).

Finally, by (1.3), (3.1), (3.2) and (3.7)

$$\begin{aligned}&\frac{1}{T}\int_T^{2T} \log|c_1L_1(s) + c_2L_2(s)|dt\\
&= \mathbb{E}(\log|c_1L_1(\sigma,X) + c_2L_2(\sigma,X)|) + O\left(\frac{(\log\log T)^2}{(\log T)^{\sigma}}\right).\end{aligned}$$

4. Proof of Theorem 1

By Littlewood's lemma we have

$$\begin{aligned}\int_\sigma^{\sigma_0}\Bigg(\sum_{\substack{\beta>u\\ T\le\gamma\le 2T}} 1\Bigg)du &= \frac{1}{2\pi}\int_T^{2T} \log|E(\sigma+it,Q)|dt\\
&\quad - \frac{1}{2\pi}\int_T^{2T} \log|E(\sigma_0+it,Q)|dt + O(\log T),\end{aligned}$$

where σ_0 is a real number such that $E(s,Q)$ has no zero in $\Re s \geq \sigma_0$. By Theorem 2

$$\int_\sigma^{\sigma_0}\Bigg(\sum_{\substack{\beta>u\\ T\leq\gamma\leq 2T}} 1\Bigg)du = \frac{T}{2\pi}\mathcal{M}(\sigma) - \frac{1}{2\pi}\int_T^{2T}\log|E(\sigma_0+it,Q)|dt + O\left(T\frac{(\log\log T)^2}{(\log T)^\sigma}\right).$$

Let $h>0$. Differencing this at σ and $\sigma+h$ and then dividing by h, we deduce that

$$\frac{1}{h}\int_\sigma^{\sigma+h}\Bigg(\sum_{\substack{\beta>u\\ T\leq\gamma\leq 2T}} 1\Bigg)du = \frac{T}{2\pi}\frac{\mathcal{M}(\sigma)-\mathcal{M}(\sigma+h)}{h} + O\bigg(\frac{T}{h}\frac{(\log\log T)^2}{(\log T)^\sigma}\bigg).$$

Since $\mathcal{M}(\sigma)$ is twice differentiable, for a sufficiently small $h>0$,

$$\frac{1}{h}\int_\sigma^{\sigma+h}\Bigg(\sum_{\substack{\beta>u\\ T\leq\gamma\leq 2T}} 1\Bigg)du = -\frac{T}{2\pi}\mathcal{M}'(\sigma) + O\bigg(hT + \frac{T}{h}\frac{(\log\log T)^2}{(\log T)^\sigma}\bigg).$$

The integrand is a decreasing function of u, so

$$\sum_{\substack{\beta>\sigma+h\\ T\leq\gamma\leq 2T}} 1 \leq -\frac{T}{2\pi}\mathcal{M}'(\sigma) + O\bigg(hT + \frac{T}{h}\frac{(\log\log T)^2}{(\log T)^\sigma}\bigg) \leq \sum_{\substack{\beta>\sigma\\ T\leq\gamma\leq 2T}} 1.$$

In the left inequality we replace σ by $\sigma-h$ and use $\mathcal{M}'(\sigma-h) = \mathcal{M}'(\sigma)+O(h)$. Then we find that

$$\sum_{\substack{\beta>\sigma\\ T\leq\gamma\leq 2T}} 1 = -\frac{T}{2\pi}\mathcal{M}'(\sigma) + O\bigg(hT + \frac{T}{h}\frac{(\log\log T)^2}{(\log T)^\sigma}\bigg).$$

Taking $h = (\log T)^{-\sigma/2}\log\log T$, we obtain

$$\sum_{\substack{\beta>\sigma\\ T\leq\gamma\leq 2T}} 1 = -\frac{T}{2\pi}\mathcal{M}'(\sigma) + O\bigg(T\frac{\log\log T}{(\log T)^{\sigma/2}}\bigg).$$

Therefore,

$$N(\sigma_1,\sigma_2;T) = \frac{T}{2\pi}(\mathcal{M}'(\sigma_2)-\mathcal{M}'(\sigma_1)) + O\bigg(T\frac{\log\log T}{(\log T)^{\sigma_1/2}}\bigg)$$

holds for $1/2<\sigma_1<\sigma_2<1$.

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