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A DUAL ITERATIVE SUBSTRUCTURING METHOD WITH A SMALL PENALTY PARAMETER

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Abstract. A dual substructuring method with a penalty term was introduced in the previous works by the authors, which is a variant of the FETI-DP method. The proposed method imposes the continuity not only by using Lagrange multipliers but also by adding a penalty term which consists of a positive penalty parameter η and a measure of the jump across the interface. Due to the penalty term, the proposed iterative method has a better convergence property than the standard FETI-DP method in the sense that the condition number of the resulting dual problem is bounded by a constant independent of the subdomain size and the mesh size. In this paper, a further study for a dual iterative substructuring method with a penalty term is discussed in terms of its convergence analysis. We provide an improved estimate of the condition number which shows the relationship between the condition number and η as well as a close spectral connection of the proposed method with the FETI-DP method. As a result, a choice of a moderately small penalty parameter is guaranteed.

1. Introduction

In our previous works [12, 13], a dual iterative substructuring method with a penalty term was proposed for second order elliptic problems, which is based on a non-overlapping domain decomposition (DD). For non-overlapping DD methods as iterative solvers for seeking a finite element approximation of the weak solution of a concerning model, there are two properties to be considered: one is the convergence of the DD-based solution to the exact weak solution and the other is the convergence speed of the resulting iterative algorithm, which is determined by the condition number of the corresponding problem. Considering these properties, the key is how to enforce the continuity of finite element functions across the interface. Various methodologies have been introduced for handling the continuity constraint across the interface. The most popular methods, employed for different purposes are the Lagrangian

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method, the method of penalty functions, and the augmented Lagrangian method (e.g. [2, 4, 5, 6, 7, 11]).

The proposed dual iterative substructuring method is a variant of the dual-primal finite element tearing and interconnecting (FETI-DP) method [5]. The FETI-DP method is based on the Lagrangian method, which enforces the continuity across the interface by introducing Lagrange multipliers. Firstly, it is well-known that the DD-based solution, the primal finite element solution of the saddle-point problem in the FETI-DP method converges to the exact weak solution. Secondly, for the preconditioned FETI-DP with the optimal Dirichlet preconditioner, it was proved in [10, 14] that the condition number of the resulting dual problem grows at most as $O(1 + \ln(H/h))^2$, where H is the subdomain size and h is the mesh size.

On the other hand, the proposed method by the authors is based on the augmented Lagrangian method, which introduces a penalty term in addition to Lagrange multipliers in order to strengthen the continuity across the interface. The penalty term consists of a positive penalty parameter η and a measure of the jump across the interface. Let us look over what effect the choice of the penalty parameter has on two aforementioned properties for iterative DD solvers. Firstly, based on the fact that the addition of the penalty term to the saddle-point problem in the FETI-DP method makes no change in its primal finite element solution, it was shown in [12] that, for any choice of the penalty parameter, the finite element solution computed from the proposed method converges to the exact weak solution. Secondly, the added penalty term plays a major role in a faster convergence of the resulting iteration than that in the FETI-DP method.

The previous works in [12, 13] show that, for any fixed $\eta > 0$, the condition number of the dual problem is bounded above by $(1 + (C/\eta))C^*$, where C and C^* are constants independent of H and h. This estimate suggests us to choose a sufficiently large η for an optimal case. However the numerical results in [12, 13] show that there is a relatively small η that can be regarded as an optimal one in terms of condition number of the dual system. The study in this paper is motivated by such numerical observations.

This paper is devoted to enhancement of our previous work by a further study focusing on the case of small penalty parameters in terms of convergence analysis. We first take a careful look at the relationship between the standard FETI-DP operator and the proposed dual operator in algebraic form. Based on this observation, we provide an improved estimate of the condition number which shows the relationship between the condition number and η as well as a close spectral connection of the proposed method with the FETI-DP method. As a result, a choice of a moderately small penalty parameter is guaranteed.

This paper is organized as follows. In Sect. 2, we review a dual iterative substructuring method with a penalty term. Sect. 3 is devoted to analyzing the condition number of the resulting dual problem. Finally, we present numerical results in Sect. 4.

Throughout the paper, we denote by $\lambda^A_{\min}$ and $\lambda^A_{\max}$ the minimum eigenvalue and the maximum eigenvalue of a matrix A, respectively. To avoid the proliferation of constants, throughout the paper we will use $A \lesssim B$ and $A \gtrsim B$ to represent the statements that $A \leq (\text{constant})B$ and $A \geq (\text{constant})B$, respectively, where the positive constant is independent of the mesh size, the subdomain size, and the number of subdomains. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$.

2. Dual iterative substructuring with a penalty parameter

In this section, we first review a dual iterative substructuring method with a penalty term in our previous works. Then, we state how we can enhance these methods based on a better choice of a penalty parameter.

We consider the following Poisson model problem with the homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded polygonal or polyhedral domain and f is a given function in $L_2(\Omega)$. Let $\mathcal{T}_h$ denote a quasi-uniform triangulation on Ω. We are concerned with a discretized variational problem of (2.1) as follows: find $u_h \in \hat{X}_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in \hat{X}_h, \tag{2.2}$$

where

$$a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, dx, \quad (f, v_h) = \int_\Omega f v_h \, dx.$$

Here, the finite element space $\hat{X}_h$ is composed of the conforming $\mathbb{P}_1$ elements in $\mathbb{R}^2$ ($\mathbb{Q}_1$ elements in $\mathbb{R}^3$).

We start with recalling an iterative solver of (2.2) in [12, 13], which is a non-overlapping domain decomposition algorithm based on an augmented Lagrangian. We decompose Ω into non-overlapping subdomains $\{\Omega_j\}_{j=1}^{N_s}$ as open sets, where the boundary $\partial\Omega_j$ is aligned with $\mathcal{T}_h$ and the diameter of Ω_j is H_j. On each subdomain, the triangulation $\mathcal{T}_j$ is the triangulation of Ω_j inherited from $\mathcal{T}_h$ and matching grids are taken on the boundaries of neighboring subdomains across the interface Γ. Here the interface Γ is the union of the common interfaces among all subdomains, i.e., $\Gamma = \bigcup_{j<k} \Gamma_{jk}$, where Γ_{jk} denotes the common interface of two adjacent subdomains Ω_j and Ω_k.

Based on the non-overlapping subdomain decomposition, a partitioned problem is obtained as follows:

$$\min_{v \in \prod_{j=1}^{N_s} X_h^j} \left(\frac{1}{2} \sum_{j=1}^{N_s} \int_{\Omega_j} |\nabla v|^2 \, dx - (f, v) \right) \tag{2.3a}$$

$$\text{subject to } v^{(j)} = v^{(k)} \text{ on } \Gamma_{jk} \text{ for } \ j < k, \tag{2.3b}$$

where X_h^j is the restriction of $\hat{X}_h$ on a subdomain Ω_j and $v|_{\Omega_j}$ is denoted by $v^{(j)}$ for $v \in \prod_{j=1}^{N_s} X_h^j$. To make a localized minimization problem recover the original solution of (2.2), the continuity constraint (2.3b) needs to be satisfied on the interface Γ in an appropriate manner (e.g. [4, 5, 6, 7]).

We impose the continuity differently at vertices and the remaining interface nodes except vertices in terms of the choice of finite elements. The continuity at vertices is enforced strongly in a manner that subdomains sharing a vertex have the common value at the vertex. Let X_h^c denote the subspace of $\prod_{j=1}^{N_s} X_h^j$ obtained by enforcing the vertex continuity. On the other hand, adjacent subdomains are allowed to have different values on the interface nodes except vertices. The continuity on the interface except vertices remains as a constraint which requires the pointwise matching of finite elements on the interface. Then, the problem (2.3) is rewritten as a constrained minimization

$$\min_{v \in X_h^c} \left(\frac{1}{2} \sum_{j=1}^{N_s} \int_{\Omega_j} |\nabla v|^2 \, dx - (f, v) \right) \quad \text{subject to} \quad Bv = 0,$$

where B is a signed Boolean matrix which plays a role in making values defined individually on the interface pointwise-matched.

The FETI-DP method, one of the most advanced non-overlapping DD algorithms, enforces the pointwise matching constraints weakly by introducing Lagrange multipliers, that is, the FETI-DP method starts with the saddle-point problem

$$\mathcal{L}(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^M} \min_{v_h \in X_h^c} \mathcal{L}(v_h, \mu_h), \tag{2.4}$$

where a Lagrangian functional $\mathcal{L}$ is defined on $X_h^c \times \mathbb{R}^M$ as

$$\mathcal{L}(v, \mu) = \frac{1}{2} \sum_{j=1}^{N_s} \int_{\Omega_j} |\nabla v|^2 \, dx - (f, v) + \langle Bv, \mu \rangle.$$

Here, M represents the number of constraints used for imposing the pointwise matching on the interface and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^M$.

In the convergence studies [10, 14] for the FETI-DP method where the coarse problem is related to vertex continuity constraint at the subdomain corners, it is well-known that the condition number of the resulting dual problem from (2.4) grows asymptotically as $O(1 + \ln(H/h))^2$ in two dimensions (2D) and $O((H/h)(1 + \ln(H/h))^2)$ in three dimensions (3D) if it is accompanied by the Dirichlet preconditioner. In addition, a stronger coupling across the interface results in a better scalability in 3D in the sense that the condition number is estimated as $O(1 + \ln(H/h))^2$, where the extra degrees of freedom in a coarse problem are introduced in terms of the average continuity constraints over edges and/or faces in addition to the vertex continuity constraint. In a similar view, the authors [12, 13] introduced a stronger coupling than the continuity at the subdomain corners by the addition of a penalty term. The added penalty term results in enhancing the convergence to the extent of the constant condition number bound independent of both the subdomain size H and the mesh size h.

A penalty term $\eta\mathcal{J}$ is considered, which consists of a positive penalty parameter η and a measure of the jump on the interface. The addition of a penalty term $\eta\mathcal{J}$ to the Lagrangian $\mathcal{L}$ yields a saddle-point problem for an augmented Lagrangian functional $\mathcal{L}_\eta$ such as

$$\mathcal{L}_\eta(u_h, \lambda_h) = \max_{\mu_h \in \mathbb{R}^M} \min_{v_h \in X_h^c} \mathcal{L}_\eta(v_h, \mu_h), \tag{2.5}$$

where

$$\mathcal{L}_\eta(v, \mu) = \mathcal{L}(v, \mu) + \frac{1}{2}\eta\mathcal{J}(v, v).$$

In two dimensions, the penalty term $\mathcal{J}$ is a bilinear form on $X_h^c \times X_h^c$ defined as

$$\mathcal{J}(u, v) = \frac{1}{h}\sum_{j<k}\int_{\Gamma_{jk}} (u^{(j)} - u^{(k)})(v^{(j)} - v^{(k)})\, ds,$$

where $h = \max_{j=1,\dots,N_s} h_j$ with the mesh size h_j of $\mathcal{T}_j$. For three dimensional problems, in order to increase its practical efficiency, a modified penalty term was proposed in [13] by considering the interface except vertices as a union of two separate objects, faces and edges:

$$\mathcal{J}(u, v) = \mathcal{J}_{\mathcal{F}}(u, v) + \mathcal{J}_{\mathcal{E}}(u, v),$$

where

$$\mathcal{J}_{\mathcal{F}}(u, v) = \frac{1}{h}\sum_{j<k}\int_{\mathcal{F}_{jk}} (u_{\mathcal{F}_{jk}}^{(j)} - u_{\mathcal{F}_{jk}}^{(k)})(v_{\mathcal{F}_{jk}}^{(j)} - v_{\mathcal{F}_{jk}}^{(k)})\, dx$$

and

$$\mathcal{J}_{\mathcal{E}}(u, v) = \sum_{\mathcal{E}_l}\sum_{(m,n)\in I_{\mathcal{E}_l}}\int_{\mathcal{E}_l} (u^{(m)} - u^{(n)})(v^{(m)} - v^{(n)})\, ds.$$

Here, $\mathcal{F}_{jk}$ denotes the common face of Ω_j and Ω_k, $\mathcal{E}_l$ an edge shared by neighboring subdomains where l is an index of an edge, and $I_{\mathcal{E}_l}$ the set of indices of subdomain pairs which share an edge $\mathcal{E}_l$. In the face part $\mathcal{J}_{\mathcal{F}}$, $u_{\mathcal{F}_{jk}}^{(j)}$ is a part of u, which is related to the contribution to $u^{(j)}$ on $\mathcal{F}_{jk}$ only from the face nodal basis functions except the edge nodal basis functions.

The problem (2.5) is expressed in the algebraic form

$$\begin{bmatrix} A_\eta & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix}, \tag{2.6}$$

where

$$A_\eta = A + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix} \quad \text{with } A = \begin{bmatrix} A_{\Pi\Pi} & A_{\Pi\Delta} \\ A_{\Pi\Delta}^T & A_{\Delta\Delta} \end{bmatrix}, \tag{2.7}$$

$$B^T = \begin{bmatrix} 0 \\ B_\Delta^T \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_\Pi \\ \boldsymbol{u}_\Delta \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} f_\Pi \\ f_\Delta \end{bmatrix}, \quad \boldsymbol{u}_\Pi = \begin{bmatrix} \boldsymbol{u}_I \\ \boldsymbol{u}_c \end{bmatrix}. \tag{2.8}$$

Here λ indicates the Lagrange multipliers introduced for imposing the continuity constraint across the interface, Π the degrees of freedom associated with both the interior nodes (I) and the subdomain corners (c), and Δ the remaining part of the degrees of freedom on the interface: those related to the edge nodes in 2D and those associated with the face nodes and the edge nodes in 3D. The matrix J results from the penalty term $\mathcal{J}$, which is written as

$$J = B_\Delta^T D_M B_\Delta, \tag{2.9}$$

where a block diagonal matrix D_M will be detailed in Sect. 3. Eliminating $\boldsymbol{u}_\Pi$ and $\boldsymbol{u}_\Delta$ successively, we have a dual system

$$F_\eta \lambda = d_\eta, \tag{2.10}$$

where

$$F_\eta = B_\Delta S_\eta^{-1} B_\Delta^T, \quad d_\eta = B_\Delta S_\eta^{-1}(f_\Delta - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} f_\Pi)$$

with

$$S_\eta = S + \eta J = (A_{\Delta\Delta} - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} A_{\Pi\Delta}) + \eta J. \tag{2.11}$$

For the proposed dual iterative substructuring method which results in the dual problem (2.10), we are concerned with two key properties: one is the convergence of the primal solution u_h of the saddle-point problem (2.5) from which (2.10) is originated, to the exact weak solution of (2.1) and the other is the condition number of F_η which determines the convergence rate of iterations on (2.10). In this context, we now discuss the choice of a penalty parameter in the proposed dual iterative substructuring method.

Let us first look over what effect the choice of the penalty parameter has on the convergence of the finite element solution to the weak solution of (2.1). In finite element formulations based on penalty methods for (2.3) (cf. [1, 4, 15]), the choice of a sufficiently large penalty parameter is required for the stability of a concerning finite element formulation, which is necessary for the convergence of the finite element solution to the exact weak solution of (2.1). On the other hand, the penalty parameter η plays a different role in the saddle-point formulation (2.5) based on an augmented Lagrangian functional because Lagrange multipliers as well as a penalty term are introduced to enforce the continuity across the interface. More precisely, such a role difference was confirmed in [12] by the fact that the primal solution u_h of the saddle-point problem (2.5) is exactly equal to the finite element solution of (2.2) regardless of the choice of η. Hence there is no need to consider a right choice of η in the aspect of the convergence of a finite element solution to the solution of (2.1).

Let us next discuss the choice of the penalty parameter in terms of the condition number of F_η. The convergence study in [12, 13] shows that the dual system (2.10) has a constant condition number bound independent of H and h where a sufficiently large penalty parameter is taken. On the contrary, we have observed through numerical results that there might be an estimated parameter $\eta^* < 10$ with which the proposed dual iterative algorithm is almost optimal in terms of its condition number. Based on such observation, we shall focus on the case of small penalty parameters throughout the following sections.

3. Estimate of condition number

In this section, we find the relationship between the standard FETI-DP operator and the proposed dual operator in algebraic form. Based on the relationship, we carry out convergence analysis in terms of the condition number of the dual system F_η. As results, it is confirmed why a fast convergence of the iteration is attained even if a moderately small η is taken.

Let us denote by $D(A)$ a block diagonal matrix such that

$$D(A) = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix}.$$

In 2D, the matrix $J = B_\Delta^T D_M B_\Delta$ in (2.9) is detailed as

$$J = B_\Delta^T D\Big(\frac{1}{h} M_e\Big) B_\Delta, \tag{3.1}$$

where B_Δ is in a block form as

$$B_\Delta = [B_{e_1}, \dots, B_{e_{N_E}}]$$

for a block B_{e_l} related to subdomains sharing an edge $\mathcal{E}_l$ and M_e is the one-dimensional mass matrix on each edge. In 3D, the pointwise matching operator B_Δ is divided into two parts related to face nodes and edge nodes

$$B_\Delta = \begin{bmatrix} B_{\Delta,f} & 0 \\ 0 & B_{\Delta,e} \end{bmatrix},$$

where $B_{\Delta,f}$ is in a block form as

$$B_{\Delta,f} = [B_{f_1}, \dots, B_{f_{N_F}}]$$

for a block B_{f_l} related to subdomains sharing a face $\mathcal{F}_l$ and $B_{\Delta,e}$ is defined in a similar manner to B_Δ in 2D. Then, the matrix J is expressed in the form

$$J = \begin{bmatrix} B_{\Delta,f}^T & 0 \\ 0 & B_{\Delta,e}^T \end{bmatrix} \begin{bmatrix} D\Big(\frac{1}{h} M_f\Big) & 0 \\ 0 & D(M_e) \end{bmatrix} \begin{bmatrix} B_{\Delta,f} & 0 \\ 0 & B_{\Delta,e} \end{bmatrix}, \tag{3.2}$$

where M_f stands for the 2D mass matrix on each face.

We have the following condition number estimate of the concerned dual system based on a key relationship between two matrices F_η and F, where F is the standard FETI-DP operator as $F = B_\Delta S^{-1} B_\Delta^T$.

Theorem 1. *For any $\eta > 0$, we have*

$$\kappa(F_\eta) \le \frac{C_{F,D_M}}{\eta + C_{F,D_M}} \kappa(F) + \frac{\eta}{\eta + C_{F,D_M}} \kappa(D_M),$$

where $C_{F,D_M} = (\lambda_{\max}^F \lambda_{\min}^{D_M})^{-1}$.

Proof. For matrices A and B, we first note an inverse property in [8] as follows

$$(A+B)^{-1} = A^{-1} - A^{-1}B(A+B)^{-1}. \tag{3.3}$$

Using (2.11) and (3.3), we have that

$$\begin{aligned} F_\eta - F &= B_\Delta (S_\eta^{-1} - S^{-1}) B_\Delta^T \\ &= B_\Delta(-S^{-1}\eta J S_\eta^{-1}) B_\Delta^T \\ &= -\eta F D_M F_\eta. \end{aligned}$$

The last equality is obtained by (2.9). Thus, the proposed dual operator F_η is associated with the standard FETI-DP operator F as

$$\begin{aligned} F_\eta^{-1} &= F^{-1}(I + \eta F D_M) \\ &= F^{-1} + \eta D_M. \end{aligned} \tag{3.4}$$

Let λ^{F_η} be an arbitrary eigenvalue of F_η. Since F_η is invertible, $(\lambda^{F_\eta})^{-1}$ is also an eigenvalue of F_η^{-1}. Then from the fact that F_η is symmetric positive definite (cf. [12, 13]), it follows that

$$\kappa(F_\eta) = \frac{(\lambda_{\min}^{F_\eta})^{-1}}{(\lambda_{\max}^{F_\eta})^{-1}} = \frac{\lambda_{\max}^{F_\eta^{-1}}}{\lambda_{\min}^{F_\eta^{-1}}} = \kappa(F_\eta^{-1}).$$

Consequently, (3.4) yields

$$\begin{aligned} \kappa(F_\eta) &= \kappa(F_\eta^{-1}) \\ &\le \frac{(\lambda_{\min}^{F})^{-1} + \eta\lambda_{\max}^{D_M}}{(\lambda_{\max}^{F})^{-1} + \eta\lambda_{\min}^{D_M}} \\ &= \frac{C_{F,D_M}}{\eta + C_{F,D_M}}\kappa(F) + \frac{\eta}{\eta + C_{F,D_M}}\kappa(D_M), \end{aligned} \tag{3.5}$$

where $C_{F,D_M} = (\lambda_{\max}^{F}\lambda_{\min}^{D_M})^{-1}$. □

Remark 1. Theorem 1 shows the relationship between $\kappa(F_\eta)$ and η as well as the connection of $\kappa(F_\eta)$ with $\kappa(F)$. In particular, $\kappa(F_\eta)$ becomes close to $\kappa(F)$ as η decreases to zero, while the convergence studies in the previous works [12, 13] rule out the case with decreasing η to 0. In addition, it follows from (3.5) that

$$\kappa(F_\eta) \le \kappa(D_M) + \frac{C_{F,D_M}(\kappa(F) - \kappa(D_M))}{\eta + C_{F,D_M}}, \tag{3.6}$$

which implies that the result shown in Figure 1 in [12] is in agreement with (3.6) when $\kappa(F) > \kappa(D_M)$.

To derive a more precise estimate of $\kappa(F_\eta)$ from (3.6), we will find bounds of the extreme eigenvalues of matrices F and D_M. First, keeping in mind that D_M is a block diagonal matrix whose main diagonal blocks are mass matrices on edges or faces as shown in (3.1) and (3.2), we get the following estimate for the extreme eigenvalues of D_M by a scaling argument (cf. Lemma B.31 in [16]).

Proposition 2. *For the matrix D_M, we have that*

$$\lambda_{\min}^{D_M} \gtrsim h^{d-2}, \quad \lambda_{\max}^{D_M} \lesssim h^{d-2} \quad \text{for } d = 2, 3.$$

Next, the following proposition is obtained by using the specific type of Poincaré inequality in Lemma 5.1 of [3] and the trace theorem in $H^{1/2}(\partial\Omega_j)$-norm (cf. Lemma 4.6 of [16]).

Proposition 3. *For any $v_j \in X_h^j$ such that $v_j = 0$ at vertices of a subdomain $\Omega_j \subset \mathbb{R}^d$, we have*

$$|v_j|^2_{H^1(\Omega_j)} \gtrsim \begin{cases} \left(H_j\left(1+\ln\frac{H_j}{h_j}\right)\right)^{-1} \|v_j\|^2_{L^2(\partial\Omega_j)} & \text{for } d=2 \\ \left(H_j\left(1+\frac{H_j}{h_j}\right)\right)^{-1} \|v_j\|^2_{L^2(\partial\Omega_j)} & \text{for } d=3. \end{cases}$$

Next, we are concerned with the Schur complement matrix S to estimate the extreme eigenvalues of F.

Lemma 4. *For $S = A_{\Delta\Delta} - A_{\Pi\Delta}^T A_{\Pi\Pi}^{-1} A_{\Pi\Delta}$, we have*

$$\underline{C}\boldsymbol{v}_\Delta^T\boldsymbol{v}_\Delta \lesssim \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta \lesssim \overline{C}\boldsymbol{v}_\Delta^T\boldsymbol{v}_\Delta,$$

where

$$\underline{C} = \begin{cases} \min_{j=1,\ldots,N_s}\left(\frac{H_j}{h_j}\left(1+\ln\frac{H_j}{h_j}\right)\right)^{-1} & \text{for } d=2 \\ \min_{j=1,\ldots,N_s}\left(\frac{H_j}{h_j^2}\left(1+\frac{H_j}{h_j}\right)\right)^{-1} & \text{for } d=3 \end{cases}$$

and

$$\overline{C} = h^{d-2} \quad \text{for } d = 2, 3.$$

Proof. Let us denote by $\tilde{a}(\cdot,\cdot)$ the localized bilinear form defined as

$$\tilde{a}(u,v) = \sum_{j=1}^{N_s}\int_{\Omega_j} \nabla u\cdot\nabla v\,dx, \quad \forall (u,v) \in X_h^c \times X_h^c.$$

For an arbitrary $v \in X_h^c$, we can write by three functions in X_h^c as

$$v = v_I + v_c + v_\Delta,$$

where v_I vanishes at all the nodes except for the interior nodes to subdomains, and similarly v_c and v_Δ are defined in terms of the subdomain corners and the remaining nodes on the interface, respectively. Let us express v in a vector form as

$$\boldsymbol{v} = [\boldsymbol{v}_I, \boldsymbol{v}_c, \boldsymbol{v}_\Delta]^T.$$

First, it is noted that for any v_Δ,

$$\begin{aligned} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta &= \min\left\{\boldsymbol{w}^T A \boldsymbol{w} \;:\; \boldsymbol{w}_\Delta = \boldsymbol{v}_\Delta,\ \forall \boldsymbol{w} = [\boldsymbol{w}_I, \boldsymbol{w}_c, \boldsymbol{w}_\Delta]^T\right\} \\ &\le \min\left\{\boldsymbol{w}^T A \boldsymbol{w} \;:\; \boldsymbol{w}_\Delta = \boldsymbol{v}_\Delta,\ \forall \boldsymbol{w} = [\boldsymbol{w}_I, \boldsymbol{0}, \boldsymbol{w}_\Delta]^T\right\} \\ &= \sum_{j=1}^{N_s}\int_{\Omega_j} |\nabla \mathcal{H}_j v_\Delta|^2\,dx, \end{aligned}$$

where $\mathcal{H}_j v_\Delta$ is the discrete harmonic extension of $v_\Delta|_{\partial\Omega_j}$ into Ω_j.

We now consider a lower bound. For $\boldsymbol{v}_\Pi = [\boldsymbol{v}_I, \boldsymbol{v}_c]^T$ satisfying

$$A_{\Pi\Pi}\boldsymbol{v}_\Pi + A_{\Pi\Delta}\boldsymbol{v}_\Delta = 0,$$

we have

$$\boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta = \tilde{a}(v_I + v_\Delta, v_I + v_\Delta) + 2\tilde{a}(v_I + v_\Delta, v_c) + \tilde{a}(v_c, v_c). \tag{3.7}$$

By deriving a strengthened Cauchy-Schwarz inequality in a similar way to Lemma 4.3 in [13], it is shown that there exist a constant γ such that

$$2\tilde{a}(v_I + v_\Delta, v_c) \geq -\gamma(\tilde{a}(v_I + v_\Delta, v_I + v_\Delta) + \tilde{a}(v_c, v_c)), \tag{3.8}$$

where $0 < \gamma < 1$ is independent of H and h. Combination of (3.7) and (3.8) gives

$$\begin{aligned} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta &\geq (1-\gamma)\tilde{a}(v_I + v_\Delta, v_I + v_\Delta) + (1-\gamma)\tilde{a}(v_c, v_c) \\ &\geq (1-\gamma)\tilde{a}(v_I + v_\Delta, v_I + v_\Delta) \\ &= (1-\gamma)\sum_{j=1}^{N_s}\int_{\Omega_j} |\nabla v_j|^2\, dx, \end{aligned}$$

where $v_j = (v_I + v_\Delta)|_{\Omega_j}$ vanishes at subdomain vertices. Then, the lower bound is completed by Proposition 3 and the following property

$$\|v_\Delta|_{\Omega_j}\|^2_{L^2(\partial\Omega_j)} \approx h_j^{d-1}\boldsymbol{v}_{\Delta_j}^T \boldsymbol{v}_{\Delta_j}, \tag{3.9}$$

where $\boldsymbol{v}_{\Delta_j}$ is the vector of coefficients of the finite element function $v_\Delta|_{\Omega_j}$ in X_h^j.

We next estimate an upper bound by using the $H^1 - H^{1/2}$ norms relationship for the discrete harmonic extension (Lemma 4.10 in [16]), the inverse inequality, and (3.9). As a result, we have that

$$\begin{aligned} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta &\leq \sum_{j=1}^{N_s}\int_{\Omega_j} |\nabla \mathcal{H}_j v_\Delta|^2\, dx \\ &\lesssim \sum_{j=1}^{N_s} |v_\Delta|_{\Omega_j}|^2_{H^{1/2}(\partial\Omega_j)} \\ &\lesssim \sum_{j=1}^{N_s} h_j^{-1}\|v_\Delta|_{\Omega_j}\|^2_{L^2(\partial\Omega_j)} \\ &\lesssim h^{d-2}\boldsymbol{v}_\Delta^T \boldsymbol{v}_\Delta. \end{aligned}$$

□

Lemma 5. *For $F = B_\Delta S^{-1} B_\Delta^T$, we have*

$$\overline{C}^{-1}\langle\lambda, \lambda\rangle \lesssim \langle F\lambda, \lambda\rangle \lesssim \underline{C}^{-1}\langle\lambda, \lambda\rangle \quad \forall\ \lambda \in \mathbb{R}^M,$$

where the constants $\underline{C}$ and $\overline{C}$ are given in Lemma 4.

Proof. Let $\overline{B}$ denote $B_\triangle B_\triangle^T$. Based on the fact that $F = B_\triangle S^{-1} B_\triangle^T$, it follows from Lemma 4 that for any $\lambda \in \mathbb{R}^M$,

$$\lambda_{\min}^{S^{-1}} \lambda_{\min}^{\overline{B}} \langle \lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq \lambda_{\max}^{S^{-1}} \lambda_{\max}^{\overline{B}} \langle \lambda, \lambda \rangle,$$

that is,

$$\overline{C}^{-1} \lambda_{\min}^{\overline{B}} \langle \lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq \underline{C}^{-1} \lambda_{\max}^{\overline{B}} \langle \lambda, \lambda \rangle.$$

Hence, it is sufficient to estimate the extreme eigenvalues of $\overline{B}$. In 2D, noting that $\overline{B} = B_\triangle B_\triangle^T = 2I$, it is obvious that $\lambda_{\min}^{\overline{B}} = \lambda_{\max}^{\overline{B}} = 2$. Similarly, $B_{\triangle,f}$ in 3D satisfies the same relationship, that is, $B_{\triangle,f} B_{\triangle,f}^T = 2I$. To estimate the extreme eigenvalues of $B_{\triangle,e} B_{\triangle,e}^T$, we take a close look at the structure of $B_{\triangle,e}$. Recall that $B_{\triangle,e}$ is in a block form as

$$B_{\triangle,e} = [B_{e_1}, \ldots, B_{e_{N_E}}],$$

where B_{e_l} is a block related to subdomains sharing an edge $\mathcal{E}_l$. Thus, we have

$$\lambda_{\min}^{\overline{B}} = \min\{\lambda_{\min}^{\overline{B}_1}, \ldots, \lambda_{\min}^{\overline{B}_{N_E}}, 2\}, \quad \lambda_{\max}^{\overline{B}} = \max\{\lambda_{\max}^{\overline{B}_1}, \ldots, \lambda_{\max}^{\overline{B}_{N_E}}, 2\},$$

where $\overline{B}_l = B_{e_l} B_{e_l}^T$. Assuming that an edge $\mathcal{E}_l$ is shared by N_{e_l} subdomains, the $\overline{B}_l$ is an $(N_{e_l} - 1) \times (N_{e_l} - 1)$ block matrix in the form

$$\begin{pmatrix} 2I & -I & 0 & \cdots & 0 \\ -I & 2I & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & 2I & -I \\ 0 & \cdots & 0 & -I & 2I \end{pmatrix}.$$

Then, it is clear that $\lambda_{\max}^{\overline{B}} < 4$. On the other hand, using the well-known property of eigenvalues of tridiagonal Toeplitz matrix in [9], we have

$$\lambda_{\min}^{\overline{B}_l} = 2 - 2\cos\left(\frac{\pi}{N_{e_l}}\right) \quad \forall\, l = 1, \ldots, N_E.$$

Moreover, due to the shape-regularity of a partition $\{\Omega_j\}_j$ of Ω, there exists a constant $N_{\max}$ such that

$$N_{e_l} \leq N_{\max} \quad \forall\, l = 1, \ldots, N_E \tag{3.10}$$

so that

$$\lambda_{\min}^{\overline{B}} = 2 - 2\cos\left(\frac{\pi}{N_{\max}}\right).$$

Consequently, the proof is completed by the fact that extreme eigenvalues of $B_\triangle B_\triangle^T$ have constant bounds independent of H and h. $\square$

Remark 2. Proposition 2 implies that

$$\kappa(D_M) \lesssim 1 \quad \text{in } \mathbb{R}^d (d = 2, 3).$$

On the other hand, it follows from Lemmas 4 and 5 that

$$\kappa(F) \lesssim \begin{cases} \max_{j=1,\dots,N_s} \left(\frac{H_j}{h_j} \left(1 + \ln \frac{H_j}{h_j} \right) \right) & \text{in } \mathbb{R}^2 \\ \max_{j=1,\dots,N_s} \left(\frac{H_j}{h_j} \left(1 + \frac{H_j}{h_j} \right) \right) & \text{in } \mathbb{R}^3. \end{cases}$$

Then it is noted that either $\kappa(F) \leq \kappa(D_M)$ or $\kappa(F) > \kappa(D_M)$ holds according to the size of H/h. In the case of small H/h such that

$$\kappa(F) \leq \kappa(D_M),$$

it follows from Theorem 1 that, for any $\eta > 0$,

$$\kappa(F_\eta) \leq \kappa(D_M) \lesssim 1. \tag{3.11}$$

In the case of large H/h such that

$$\kappa(F) > \kappa(D_M),$$

see the following theorem.

Using the estimated extreme eigenvalues of D_M and F, we can characterize bounds of the condition number of the concerned dual system as follows.

Theorem 6. *For any H/h such that*

$$\kappa(F) > \kappa(D_M),$$

there is a constant C_{opt} independent of H and h such that

$$\kappa(F_\eta) < \kappa(D_M) + C_{opt} \quad \text{for any } \eta \geq C_{opt}.$$

Proof. Let us denote by $\mathcal{F}(\eta)$ the upper bound of $\kappa(F_\eta)$ in (3.6):

$$\mathcal{F}(\eta) = \kappa(D_M) + \frac{C_{F,D_M}(\kappa(F) - \kappa(D_M))}{\eta + C_{F,D_M}}.$$

For any H/h such that $\kappa(F) > \kappa(D_M)$, we have that

(i) $\mathcal{F}(\eta)$ is a strictly decreasing function over $(-C_{F,D_M}, \infty)$,

(ii) for $\eta^* = -C_{F,D_M} + \sqrt{C_{F,D_M}(\kappa(F) - \kappa(D_M))}$,

$$\mathcal{F}(\eta^*) = \kappa(D_M) + \sqrt{C_{F,D_M}(\kappa(F) - \kappa(D_M))}.$$

Since $C_{F,D_M} = \left(\lambda^F_{\max}\lambda^{D_M}_{\min}\right)^{-1}$, we have

$$C_{F,D_M}\kappa(F) = \left(\lambda^F_{\min}\lambda^{D_M}_{\min}\right)^{-1}.$$

Proposition 2 and Lemma 5 imply that there are constants C_1 and C_2 independent of H and h such that

$$\lambda^F_{\min} \geq C_1 h^{2-d}, \quad \lambda^{D_M}_{\min} \geq C_2 h^{d-2}. \tag{3.12}$$

Then using (3.12), we have that

$$C_{F,D_M}(\kappa(F)-\kappa(D_M)) < C_{F,D_M}\kappa(F) = \frac{1}{\lambda_{\min}^{D_M}\lambda_{\min}^{F}} \le \frac{1}{C_1 C_2}$$

which implies that there exists a constant C_{opt} independent of H and h such that

$$\sqrt{C_{F,D_M}(\kappa(F)-\kappa(D_M))} < C_{opt}. \tag{3.13}$$

Then combining (3.13) with (i) and (ii), we have that

$$\eta^* < C_{opt},$$
$$\mathcal{F}(C_{opt}) < \mathcal{F}(\eta^*) < \kappa(D_M) + C_{opt}.$$

Therefore, for any $\eta > C_{opt}$, it holds that

$$\kappa(F_\eta) < \kappa(D_M) + C_{opt}.$$

□

Remark 3. For the case of uniform triangulations in 2D, the constants C_1 and C_2 in (3.12) can be estimated as

$$C_1 = \frac{1}{3}, \quad C_2 = 2$$

and then C_{opt} can be chosen to be $\sqrt{\frac{3}{2}} \approx 1.2247$.

Remark 4. The subdomain problem for a large η is ill-conditioned with respect to η, which requires an extra inner preconditioner (cf. [12, 13]). On the other hand, a choice of moderately small η is guaranteed due to Theorem 6 so that the resultant dual problem can be solved without such preconditioners.

4. Numerical results

In this section, computational results are presented, which verify the theoretical bounds estimated in previous sections. We consider the model problem with the exact solution

$$u(x,y) = \begin{cases} y(1-y)\sin(\pi x) & \text{in 2D} \\ \sin(\pi x)\sin(\pi y)z(1-z) & \text{in 3D} \end{cases}$$

as follows

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = (0,1)^d, d = 2, 3$.

Let us point out that the dual problem in (2.6) is formulated for the condition number estimate. The problem (2.6) is reformulated for the implementation as follows. By rearranging $\boldsymbol{u}$ in (2.8) in order $\boldsymbol{u} = [\boldsymbol{u}_r, \boldsymbol{u}_c]^T$ where $\boldsymbol{u}_I$ and $\boldsymbol{u}_\Delta$ are assembled into $\boldsymbol{u}_r$, (2.6) is represented as

$$\begin{bmatrix} A_{rr,\eta} & A_{rc} & B_r^T \\ A_{rc}^T & A_{cc} & 0 \\ B_r & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_r \\ \boldsymbol{u}_c \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_r \\ \boldsymbol{f}_c \\ 0 \end{bmatrix}, \tag{4.1}$$

where

$$A_{rr,\eta} = A_{rr} + \begin{bmatrix} 0 & 0 \\ 0 & \eta J \end{bmatrix} \quad \text{with} \begin{bmatrix} A_{II} & A_{I\Delta} \\ A_{I\Delta}^T & A_{\Delta\Delta} \end{bmatrix} \tag{4.2}$$

and

$$B_r = \begin{bmatrix} 0 & B_\Delta \end{bmatrix}.$$

Then by eliminating $\boldsymbol{u}_r$ and $\boldsymbol{u}_c$ in order in (4.1), we get that

$$F_\eta \lambda = d_\eta, \tag{4.3}$$

where

$$F_\eta = F_{rr} + F_{rc} F_{cc}^{-1} F_{rc}^T, \quad d_\eta = d_r - F_{rc} F_{cc}^{-1} d_c, \tag{4.4}$$

$$F_{rr} = B_r (A_{rr,\eta})^{-1} B_r^T,\ F_{rc} = B_r (A_{rr,\eta})^{-1} A_{rc},\ F_{cc} = A_{cc} - A_{rc}^T (A_{rr,\eta})^{-1} A_{rc},$$

and

$$d_r = B_r^T (A_{rr,\eta})^{-1} \boldsymbol{f}_r, \quad d_c = \boldsymbol{f}_c - A_{rc}^T (A_{rr,\eta})^{-1} \boldsymbol{f}_r.$$

Remark 5. The dual operator (4.4) is different from that of the FETI-DP algorithm in types of subdomain problems. The proposed method needs to solve the coupled subdomain problems $A_{rr,\eta}$ due to the penalty term while the FETI-DP method solves A_{rr} as the localized subdomain problems. In addition, let us point out that the condition number of $A_{rr,\eta}$ grows as $O((\frac{H}{h})^d)$, not $O(h^{-d})$. In [13], the computational costs of the proposed method and the FETI-DP method are compared in terms of operation counts by considering differences in both their condition numbers and types of subdomain problems.

We solve the dual problem in (4.3) by the conjugate gradient method (CGM) with a constant initial guess ($\lambda_0 \equiv 1$). The stop criterion is the relative reduction of the initial residual by a chosen TOL

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq \text{TOL},$$

where r_k is the dual residual error on the kth CG iteration and $\text{TOL} = 10^{-8}$. The coupled subdomain problems $A_{rr,\eta}$ as part of the dual problem is also solved by CGM. Let us recall that subdomain problems need to be solved exactly. This means that the CG iteration for subdomain problems should be solved up to the level of finite element discretization error. In this view, the CG iteration is stopped when the relative primal residual error is less than 10^{-13}. Through numerical tests, Ω in 2D is decomposed into N_s square subdomains with $N_s = 1/H \times 1/H$. Each subdomain is partitioned into $2 \times H/h \times H/h$ uniform triangular elements. In 3D, Ω is decomposed into N_s cubic subdomains

TABLE 1. Condition number of F_η for a small η where $N_s = 4 \times 4$ in 2D

η	$\frac{H}{h}=4$		$\frac{H}{h}=8$		$\frac{H}{h}=16$		$\frac{H}{h}=32$	
	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #
0	7.2033	14	2.2901e+1	23	5.9558e+1	33	1.4707e+2	48
0.2	3.7811	12	5.6829	15	6.4744	18	6.7436	19
0.4	2.6637	10	3.3617	13	3.5166	13	3.6410	14
0.6	2.0733	9	2.3969	10	2.5127	11	2.5753	12
0.8	1.6990	8	1.9367	9	1.9974	10	2.0247	10
1	1.5030	7	1.6468	8	1.6801	9	1.6957	9
2	1.1304	5	1.1067	5	1.1053	5	1.1050	5
4	1.3353	6	1.4469	7	1.4625	8	1.4477	8
6	1.5050	7	1.7008	9	1.7470	9	1.7378	9
8	1.6130	7	1.8691	9	1.9404	10	1.9387	10
10	1.6875	7	1.9945	10	2.0799	11	2.0868	11
10^6	2.0938	3	2.7170	7	2.9243	13	2.9771	14

with $N_s = 1/H \times 1/H \times 1/H$ while each subdomain is partitioned into $H/h \times H/h \times H/h$ uniform cubic elements.

In Table 1 for the two-dimensional problem, the condition numbers of the dual system are presented in the cases with η in $[0, 10]$. In addition, for comparison with the case of a larger η, the result for $\eta = 10^6$ is presented. For each $\eta > 0$, the condition number $\kappa(F_\eta)$ is bounded by a constant even if H/h increases. In Table 1, any penalty parameter chosen in $(1/2, 10)$ improves the condition number regardless of the increase of H/h. In addition, the condition numbers for the case with $\eta \in (1/2, 10)$ are less than that for the case with a large η. According to the condition number and the iteration count, $\eta = 2$ is regarded as an optimal one. Table 2 for 3D shows similar results to 2D; $\eta = 1$ seems to be optimal as H/h increases.

5. Concluding remarks

In this paper we analyzed a dual iterative substructuring method with a penalty term in terms of condition number. Compared with the standard FETI-DP method for the Poisson model, the proposed method has the unique feature that (i) the condition number of the resultant dual problem is bounded by a constant independent of the mesh size and the subdomain size without any preconditioner and (ii) in both 2D and 3D problems, only the vertex continuity is introduced at the primal constraint associated with the coarse problem. As future works, we consider the extension of our result to the elliptic problems with variable coefficients and/or discontinuous coefficients.

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TABLE 2. Condition number of F_η for a small η where $N_s = 4 \times 4 \times 4$ in 3D

η	$\frac{H}{h} = 4$		$\frac{H}{h} = 8$		$\frac{H}{h} = 16$		$\frac{H}{h} = 32$	
	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #	$\kappa(F_\eta)$	iter. #
0	8.1805e+1	73	3.0183e+2	107	1.1892e+3	153	4.6946e+3	218
0.2	6.9551	22	6.8882	22	6.7708	21	6.6486	21
0.4	4.4201	18	4.6965	18	4.8197	18	4.8325	17
0.6	3.8658	16	4.3214	16	4.4810	17	4.4959	16
0.8	3.5613	15	4.0772	16	4.2515	16	4.2834	16
1	3.3611	15	3.9076	15	4.0901	15	4.1292	15
2	3.1992	14	4.0118	16	4.3020	16	4.3345	16
4	3.6343	15	4.8935	17	5.3381	18	5.4422	19
6	3.8905	15	5.4275	17	5.9726	19	6.1152	20
8	4.0564	15	5.7842	18	6.4011	20	6.5659	21
10	4.1740	15	6.0390	19	6.7099	21	6.8890	21
10^6	4.8585	7	7.5658	14	8.5609	16	8.8699	18

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