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THE CONDITIONAL BOREL-CANTELLI LEMMA AND APPLICATIONS

Qianmin Chen and Jicheng Liu

Abstract. In this paper, we establish some conditional versions of the first part of the Borel-Cantelli lemma. As its applications, we study strong limit results of $\mathscr{F}$-independent random variables sequences, the convergence of sums of $\mathscr{F}$-independent random variables and the conditional version of strong limit results of the concomitants of order statistics.

1. Introduction

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\mathscr{F}$ be a sub-σ-algebra of $\mathscr{A}$. A sequence of events $\{A_n, n \geq 1\}$ is said to be conditionally independent given $\mathscr{F}$ if

$$P\left(\bigcap_{i=1}^{n} A_i \middle| \mathscr{F}\right) = \prod_{i=1}^{n} P(A_i|\mathscr{F}) \quad \text{a.s.}. \tag{1.1}$$

A sequence of dependent random variables $\{X_n, n \geq 1\}$ is said to be conditionally independent given $\mathscr{F}$ if

$$P\left(\bigcap_{i=1}^{n} (X_i \leq x_i) \middle| \mathscr{F}\right) = \prod_{i=1}^{n} P(X_i \leq x_i|\mathscr{F}) \quad \text{a.s.}. \tag{1.2}$$

If $\mathscr{F} = \{\emptyset, \Omega\}$, the conditional independence reduces to the ordinary (unconditional) independence. The ordinary independence does not imply conditional independence and the opposite implication is also not true.

In applied sciences, if you want to obtain a suitable model to analyze and study a real problem, conditioning is often used. For example, the martingales and Markov processes are well-known stochastic processes defined through conditional expectation. In particular, the past and the future of Markov processes are conditionally independent given the present. A more extensive enumeration of models such as statistical inference and engineering literature is given by Roussas [19] in which conditioning plays an essential role.

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The statistical perspective of conditional independence is that of a Bayesian. A problem begins with a parameter θ with its prior probability distribution that exists only in mind of the investigator. The statistical model that is most commonly in use is that of a sequence $\{X_n, n \geqslant 1\}$ of observable random variables that is independent and identically distributed for each given value of θ. A concrete example where conditional limit theorems are useful is the study of statistical inference for non-ergodic models as discussed in [4]. For instance, if one wants to estimate the mean off-spring for a Galton-Watson branching process, the asymptotic properties of the maximum likelihood estimator depend on the set of non-extinction.

In the past decade, a number of authors have obtained rich results of conditional dependence and relative limit results. For instance, Majerek et al. [13] for conditional strong law of large number, Yuan et al. [22, 23] for a conditional version of the classical central limit theorem, a conditional version of the extended Kolmogorov-Feller weak law of large numbers, Christofides and Hadjikyriakou [7] for conditional demimartingale, Liu and Prakasa Rao [11] for conditional Borel-Cantelli lemma, Ordonez Cabrera et al. [15] for conditionally negatively quadrant dependence, Wang and Wang [21] for conditional demimartingale and conditional N-demimartingale, Yuan and Lei (2013) for conditionally strong mixing.

The Borel-Cantelli lemma has been found to be extremely useful for the derivations of many theorems of probability. A classical form of Borel-Cantelli Lemma can be stated as follows. Let $\{A_n\}$ be a sequence of events on a probability space $(\Omega, \mathscr{F}, P)$. Then

(a) if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = 0$;

(b) if $\{A_n\}$ are independent and

$$\sum_{n=1}^{\infty} P(A_n) = \infty, \tag{1.3}$$

then $P(\limsup A_n) = 1$, where $\limsup A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$.

In many applications, the assumption of independence in (b) fails to hold and needs to be replaced by more relaxed assumptions. There are many attempts to weaken the independent condition in the second part of the Borel-Cantelli lemma [6, 9, 10, 14, 16, 17]. Under condition (1.3), the probability of event $\limsup A_n$ can be any value between 0 and 1. The first part of the Borel-Cantelli lemma was generalized in Barndorff-Nielsen (1961) as follows:

(a′) If $\liminf_{n\to\infty} P(A_n) = 0$ and $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, then

$$P(\limsup A_n) = 0.$$

In this paper, we will establish some conditional versions of the first part of the Borel-Cantelli lemma in Section 2, which extends the results of Barndorff-Nielsen (1961), Stepanov [20] and Majerek et al. [13] in the different directions. As its applications, we study the limit behaviors of random variables sequence, which include that strong limit results of independent random variables in Section 3, the convergence of sums of $\mathscr{F}$-independent random variables in Section 4 and limit results on the concomitants of order statistics in Section 5. Some of the unconditional versions could be found in Balakrishnan and Stepanov [3].

2. Conditional Borel-Cantelli lemma

Majerek has obtained a conditional version of the first part of the Borel-Cantelli Lemma.

Lemma 2.1. *Let $\{A_n, n \geq 1\}$ be a sequence of events such that*

$$\sum_{n=1}^{\infty} P(A_n|\mathscr{F}) < \infty \quad a.s..$$

Then $P(\limsup A_n) = 0$.

We get the following conditional versions of the first part of the Borel-Cantelli Lemma.

Lemma 2.2. *Let $\{A_n, n \geq 1\}$ be a sequence of events such that $P(A_n|\mathscr{F}) \to 0$. If*

$$\sum_{n=1}^{\infty} P(A_n A_{n+1}^c|\mathscr{F}) < \infty \quad a.s., \tag{2.1}$$

then $P(\limsup A_n|\mathscr{F}) = 0$ a.s.. In this case, $P(\limsup A_n) = 0$.

Proof. For any $m > n \geqslant 1$, we have

$$\begin{aligned} P\left(\bigcup_{k=n}^{m} A_k \middle| \mathscr{F}\right) &= P(A_m|\mathscr{F}) + P(A_{m-1}A_m^c|\mathscr{F}) + P(A_{m-2}A_{m-1}^c A_m^c|\mathscr{F}) \\ &\quad + \cdots + P(A_n A_{n+1}^c \cdots A_m^c|\mathscr{F}) \\ &\leq P(A_m|\mathscr{F}) + \sum_{k=n}^{m-1} P(A_k A_{k+1}^c|\mathscr{F}). \end{aligned}$$

Therefore

$$\begin{aligned} P(\limsup A_n|\mathscr{F}) &= P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \middle| \mathscr{F}\right) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k \middle| \mathscr{F}\right) \\ &= \lim_{n\to\infty}\lim_{m\to\infty} P\left(\bigcup_{k=n}^{m} A_k \middle| \mathscr{F}\right) \\ &\leq \lim_{m\to\infty} P(A_m|\mathscr{F}) + \lim_{n\to\infty}\sum_{k=n}^{\infty} P(A_k A_{k+1}^c|\mathscr{F}) \\ &= 0. \end{aligned}$$

In this case, $P(\limsup A_n) = E(P(\limsup A_n|\mathscr{F})) = 0$. □

Lemma 2.2 holds true if (2.1) is substituted with

$$\sum_{n=1}^{\infty} P(A_n^c A_{n+1}|\mathscr{F}) < \infty \text{ a.s.},$$

because for all $m > n \geq 1$

$$\begin{aligned} P\left(\bigcup_{k=n}^{m} A_k \middle| \mathscr{F}\right) &= P(A_n|\mathscr{F}) + P(A_n^c A_{n+1}|\mathscr{F}) + P(A_n^c A_{n+1}^c A_{n+2}|\mathscr{F}) \\ &\quad + \cdots + P(A_n^c A_{n+1}^c \cdots A_m|\mathscr{F}) \\ &\leq P(A_n|\mathscr{F}) + \sum_{k=n}^{m-1} P(A_k^c A_{k+1}|\mathscr{F}). \end{aligned}$$

Observe that

$$P(A_n A_{n+1}^c|\mathscr{F}) = P(A_n|\mathscr{F}) - P(A_n A_{n+1}|\mathscr{F}),$$

we know that even if

$$\sum_{n=1}^{\infty} P(A_n|\mathscr{F}) = \infty$$

and

$$\sum_{n=1}^{\infty} P(A_n A_{n+1}|\mathscr{F}) = \infty$$

hold true, as long as $\sum_{n=1}^{\infty}[P(A_n|\mathscr{F}) - P(A_n A_{n+1}|\mathscr{F})] < \infty$, then

$$P(\limsup A_n|\mathscr{F}) = 0.$$

It is easy to see that condition $\sum_{n=1}^{\infty} P(A_n|\mathscr{F}) < \infty$ is weaker than $\sum_{n=1}^{\infty} P(A_n) < \infty$, thus $\sum_{n=1}^{\infty}[P(A_n|\mathscr{F}) - P(A_n A_{n+1}|\mathscr{F})] < \infty$ is weaker than $\sum_{n=1}^{\infty}[P(A_n) - P(A_n A_{n+1})] < \infty$, namely we get the same result as Barndorff-Nielsen (1961) and Stepanov [20] under a weaker condition.

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables. If for all small $\epsilon > 0$,*

$$P(|X_n - \mu| \geq \epsilon|\mathscr{F}) \to 0 \text{ as } n \to \infty$$

and

$$\sum_{n=1}^{\infty}[P(|X_n - \mu| \geq \epsilon|\mathscr{F}) - P(|X_n - \mu| \geq \epsilon, |X_{n+1} - \mu| \geq \epsilon|\mathscr{F})] < \infty,$$

where μ is a constant, then $X_n \xrightarrow{a.s.} \mu$ as $n \to \infty$.

Proof. Let event $A_n = [|X_n - \mu| \geq \epsilon]$, then applying Lemma 2.2 to get $P(\limsup A_n) = 0$, thus $X_n \xrightarrow{a.s.} \mu$ as $n \to \infty$. □

In fact, the following more general result can be obtained.

Lemma 2.4. *Let $\{A_n, n \geq 1\}$ be a sequence of events such that $P(A_n|\mathscr{F}) \to 0$. If for some $m > 0$,*

$$\sum_{n=1}^{\infty} P(A_n^c \cdots A_{n+m-1}^c A_{n+m}|\mathscr{F}) < \infty,$$

then $P(\limsup A_n|\mathscr{F}) = 0$.

Proof. Since

$$\begin{aligned}
&P\left(\bigcup_{k=n}^{j} A_k \middle| \mathscr{F}\right) \\
&= P(A_n|\mathscr{F}) + P(A_n^c A_{n+1}|\mathscr{F}) + \cdots + P(A_n^c \cdots A_{n+m-1}^c A_{n+m}|\mathscr{F}) \\
&\quad + P(A_n^c \cdots A_{n+m}^c A_{n+m+1}|\mathscr{F}) + \cdots + P(A_n^c \cdots A_{j-1}^c A_j|\mathscr{F}) \\
&\leq \sum_{k=n}^{n+m-1} P(A_k|\mathscr{F}) + \sum_{k=n}^{j-m} P(A_k^c \cdots A_{k+m-1}^c A_{k+m}|\mathscr{F}),
\end{aligned}$$

we have

$$\begin{aligned}
&P(\limsup A_n|\mathscr{F}) \\
&= P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \middle| \mathscr{F}\right) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k \middle| \mathscr{F}\right) \\
&= \lim_{n\to\infty}\lim_{j\to\infty} P\left(\bigcup_{k=n}^{j} A_k \middle| \mathscr{F}\right) \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{n+m-1} P(A_k|\mathscr{F}) + \lim_{n\to\infty}\sum_{k=n}^{\infty} P(A_k^c \cdots A_{k+m-1}^c A_{k+m}|\mathscr{F}) \\
&= 0.
\end{aligned}$$

□

3. Strong limit results for maxima

In this section, we will give several examples to show the importance of Theorem 2.3.

Example 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of dependent random variables defined for any $n \geq 1$ by the Clayton copula

$$F(x_1, x_2, \ldots, x_n) = \left[\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} - (n-1)\right]^{-1},$$

where $0 < x_i < 1$, $1 \le i \le n$. We know that X_1 is uniform distribution on $[0,1]$. Let $A = \{\omega : X_1 \in (0,1/2)\}$ and $\mathscr{F} = \{\varnothing, \Omega, A, A^c\}$, then

$$F(x_1, x_2, \ldots, x_n|A) = \begin{cases} 2\left[\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} - (n-1)\right]^{-1}, & x_1 \in (0,1/2), \\ 2\left[2 + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} - (n-1)\right]^{-1}, & x_1 \in [1/2,1), \end{cases}$$

$$F(x_1, x_2, \ldots, x_n|A^c) = \begin{cases} 0, & x_1 \in (0,1/2), \\ 2\left[\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n} - (n-1)\right]^{-1}, & x_1 \in [1/2,1). \end{cases}$$

Let $M_n = \max\{X_1, \ldots, X_n\}$, then for any $x \in (0,1)$ we have

$$\begin{aligned} P(M_n \le x|A) &= \begin{cases} 2\left[n\left(\dfrac{1}{x} - 1\right) + 1\right]^{-1}, & x \in (0,1/2), \\ 2\left[(n-1)\left(\dfrac{1}{x} - 1\right) + 2\right]^{-1}, & x \in [1/2,1) \end{cases} \\ &\sim \begin{cases} 2\left(-\dfrac{1}{\ln x}\right)n^{-1}, & x \in (0,1/2), \\ 2\left(-\dfrac{1}{\ln x}\right)(n-1)^{-1}, & x \in [1/2,1) \end{cases} \end{aligned}$$

and

$$\begin{aligned} P(M_n \le x|A^c) &= \begin{cases} 0, & x \in (0,1/2), \\ 2\left[n\left(\dfrac{1}{x} - 1\right) + 1\right]^{-1}, & x \in [1/2,1) \end{cases} \\ &\sim \begin{cases} 0, & x \in (0,1/2), \\ 2\left(-\dfrac{1}{\ln x}\right)n^{-1}, & x \in [1/2,1). \end{cases} \end{aligned}$$

Obviously we get $P(M_n \le x|\mathscr{F}) \to 0$ as $n \to \infty$, then we have

$$\begin{aligned} P(|M_n - 1| \ge \epsilon|\mathscr{F}) &= P(M_n \ge 1 + \epsilon|\mathscr{F}) + P(M_n \le 1 - \epsilon|\mathscr{F}) \\ &= 0 + P(M_n \le 1 - \epsilon|\mathscr{F}) \to 0. \end{aligned}$$

Because $\sum_{n=1}^{\infty} P(M_n \le x|\mathscr{F}) = \infty$, we can not apply Lemma 2.1. If $x \in (0,1/2)$, let us consider

$$\begin{aligned} &\sum_{n=1}^{\infty}[P(M_n \le x|\,A) - P(M_n \le x, M_{n+1} \le x|\,A)] \\ &= 2\sum_{n=1}^{\infty} \frac{x^{-1} - 1}{(n(x^{-1} - 1) + 1)((n+1)(x^{-1} - 1) + 1)} \\ &\sim \frac{2}{(-\ln x)} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \end{aligned}$$

Similarly, if $x \in (0,1)$, we get

$$\sum_{n=1}^{\infty}[P\left(M_n \le x|\,A\right) - P\left(M_n \le x, M_{n+1} \le x|\,A\right)] < \infty$$

and

$$\sum_{n=1}^{\infty}[P\left(M_n \le x|\,A^c\right) - P\left(M_n \le x, M_{n+1} \le x|\,A^c\right)] < \infty,$$

thus

$$\sum_{n=1}^{\infty}[P\left(M_n \le x|\,\mathscr{F}\right) - P\left(M_n \le x, M_{n+1} \le x|\,\mathscr{F}\right)] < \infty$$

holds true. By Theorem 2.3, we obtain $M_n \xrightarrow{\text{a.s.}} 1$.

In fact, Theorem 2.3 allows us to obtain further conclusions, such as the properties of

$$M_n^{n^\alpha},\ 0 < \alpha < 1.$$

First, observe that

$$P\left(M_n^{n^\alpha} \le x|A\right) = \begin{cases} 2[n(x^{-n^{-a}} - 1) + 1]^{-1} & x \in (0,1/2), \\ 2[(n-1)(x^{-n^{-a}} - 1) + 2]^{-1} & x \in [1/2,1) \end{cases}$$

$$\sim \begin{cases} 2\left(-\dfrac{1}{\ln x}\right) n^{\alpha-1} & x \in (0,1/2), \\ 2\left(-\dfrac{1}{\ln x}\right) \dfrac{1}{n^{1-\alpha} - n^{-\alpha}} & x \in [1/2,1). \end{cases}$$

Obviously we get $P(M_n^{n^\alpha} \le x|A) \to 0$ as $n \to \infty$. And if $x \in (0,1/2)$, we have

$$\sum_{n=1}^{\infty} P\left(M_n^{n^\alpha} \le x, M_{n+1}^{(n+1)^\alpha} \le x|A\right) = \sum_{n=1}^{\infty} \frac{2}{n(x^{-n^{-\alpha}} - 1) + x^{-(n+1)^{-a}}} = \infty,$$

$$\sum_{n=1}^{\infty}\left[P\left(M_n^{n^\alpha} \le x|A\right) - P\left(M_n^{n^\alpha} \le x, M_{n+1}^{(n+1)^\alpha} \le x|A\right)\right]$$
$$= 2\sum_{n=1}^{\infty} \frac{x^{-(n+1)^{-\alpha}} - 1}{(n(x^{-n^{-\alpha}} - 1) + 1)(n(x^{-n^{-\alpha}} - 1) + x^{-(n+1)^{-a}})}$$
$$\sim \frac{2}{(-\ln x)} \sum_{n=1}^{\infty} \frac{1}{n^{2-\alpha}} < \infty.$$

Similarly, if $x \in (0,1)$, we get

$$\sum_{n=1}^{\infty}\left[P\left(M_n^{n^\alpha} \le x|A\right) - P\left(M_n^{n^\alpha} \le x, M_{n+1}^{(n+1)^\alpha} \le x|A\right)\right] < \infty,$$

$$P(M_n^{n^\alpha} \le x|A^c) \to 0 \ \text{ as } n \to \infty,$$

and

$$\sum_{n=1}^{\infty}\left[P\left(M_n^{n^\alpha}\le x|A^c\right)-P\left(M_n^{n^\alpha}\le x, M_{n+1}^{(n+1)^\alpha}\le x|A^c\right)\right]<\infty,$$

thus

$$\sum_{n=1}^{\infty}\left[P\left(M_n^{n^\alpha}\le x|\mathscr{F}\right)-P\left(M_n^{n^\alpha}\le x, M_{n+1}^{(n+1)^\alpha}\le x|\mathscr{F}\right)\right]<\infty$$

and

$$P(M_n^{n^\alpha}\le x|\mathscr{F})\to 0 \text{ as } n\to\infty$$

hold true. By Theorem 2.3, we obtain $M_n^{n^\alpha}\xrightarrow{\text{a.s.}}1, 0<\alpha<1$.

Next, we will find a function $\varphi_n(M_n)$ which replace $M_n^{n^\alpha}$ in Example 3.1. Before our discussion, the following key lemma is needed.

Lemma 3.2. *Let $\{A_n, n\ge 1\}$ be a sequence of events such that $P(A_n|\mathscr{F})\to 0$. If for any $k\ge 1$,*

$$\lim_{n\to\infty}\frac{P(A_n^c\cdots A_{n+k}^c A_{n+k+1}|\mathscr{F})}{P(A_n^c\cdots A_{n+k-1}^c A_{n+k}|\mathscr{F})}\le q\in[0,1),$$

then $P(\limsup A_n|\mathscr{F})=0$.

Proof. For all large enough n and small ϵ,

$$\begin{aligned}\sum_{k=0}^{\infty}P(A_n^c\cdots A_{n+k-1}^c A_{n+k}|\mathscr{F}) &= P(A_n|\mathscr{F})+\sum_{k=1}^{\infty}P(A_n^c\cdots A_{n+k-1}^c A_{n+k}|\mathscr{F})\\ &\le P(A_n|\mathscr{F})+P(A_n^c A_{n+1}|\mathscr{F})\frac{q+\epsilon}{1-q-\epsilon}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}P(\limsup A_n|\mathscr{F}) &= P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k\middle|\mathscr{F}\right)=\lim_{n\to\infty}P\left(\bigcup_{k=n}^{\infty}A_k\middle|\mathscr{F}\right)\\ &= \lim_{n\to\infty}[P(A_n|\mathscr{F})+P(A_n^c A_{n+1}|\mathscr{F})+\cdots\\ &\quad +P(A_n^c\cdots A_{n+k-1}^c A_{n+k}|\mathscr{F})+\cdots]\\ &= \lim_{n\to\infty}\sum_{k=0}^{\infty}P(A_n^c\cdots A_{n+k-1}^c A_{n+k}|\mathscr{F})\\ &\le \lim_{n\to\infty}\left[P(A_n|\mathscr{F})+P(A_n^c A_{n+1}|\mathscr{F})\frac{q+\epsilon}{1-q-\epsilon}\right].\end{aligned}$$

We have $\lim_{n\to\infty}P(A_n|\mathscr{F})=0$ and $P(A_n^c A_{n+1}|\mathscr{F})\le P(A_n|\mathscr{F})$, thus

$$\lim_{n\to\infty}P(A_n^c A_{n+1}|\mathscr{F})=0,$$

obviously $\lim_{n\to\infty}P(\limsup A_n|\mathscr{F})=0$. □

Let $\{X_n, n \geq 1\}$ be a sequence of $\mathscr{F}$-independent random variables with distribution $F(x)$. Let $M_n = \max\{X_1, \ldots, X_n\}$, $r_F = \sup\{x \in R : F(x) < 1\}$, where $r_F \leq \infty$. If $r_F < \infty$, let $A_n = \{|M_n - r_F| \geq \epsilon\} = \{M_n \leq r_F - \epsilon\}$, then $P(A_n|\mathscr{F}) = [P(X_1 \leq r_F - \epsilon|\mathscr{F})]^n$, obviously $0 < P(X_1 \leq r_F - \epsilon|\mathscr{F}) < 1$, thus $\sum_{k=1}^{\infty} P(A_n|\mathscr{F}) < \infty$. By Lemma 2.1, we have

$$M_n \xrightarrow{\text{a.s.}} r_F.$$

If $r_F = \infty$, by definition $r_F = +\infty$, thus $P(M_n < 0|\mathscr{F}) = [P(X_n < 0|\mathscr{F})]^n$, then $\lim\limits_{n\to\infty} P(M_n < 0) = 0$. For convenient, let $X_n \geq 0$. Let $m_n = 1/M_n$, $A_n = \{|m_n| \geq \epsilon\} = \{m_n \geq \epsilon\} = \{M_n \leq 1/\epsilon\}$, because $1/\epsilon < +\infty$, just like above, we have $m_n \xrightarrow{\text{a.s.}} 0$, namely

$$M_n \xrightarrow{\text{a.s.}} r_F.$$

Thus we have a simple conclusion as follow, for any $x \in \{x \in R : F(x) < 1\}$, if $\sum_{n=1}^{\infty} P(X_n \leq x|\mathscr{F}) < \infty$, then

$$X_n \xrightarrow{\text{a.s.}} r_F.$$

Let $\varphi_n(x)$ be a measurable function of n and x, which is decreasing in n and increasing in x. Next ,we study the limit behavior of $\varphi_n(M_n)$ and obtain the following result.

Theorem 3.3. *Let a function $\varphi_n(x)$ be such that $P(\varphi_n(M_n) \leq x|\mathscr{F}) \to 0$. Then $\varphi_n(M_n) \xrightarrow{a.s.} r_F$.*

Proof. Let $x_n = \varphi_n^{-1}(x)$, we have

$$P(\varphi_n(M_n) \leq x|\mathscr{F}) = P(M_n \leq x_n|\mathscr{F}).$$

Thus $P(M_n \leq x_n|\mathscr{F}) \to 0$, then $P(M_n \leq x_n) = E[P(M_n \leq x_n|\mathscr{F})] \to 0$, namely $P(x_n \leq M_n) \to 1$. By definition $M_n \leq r_F$, thus $P(x_n \leq M_n) \leq P(x_n \leq r_F) \leq 1$, hence $\lim\limits_{n\to\infty} P(x_n \leq r_F) = 1$. We obtain $x_n \to r_F$ because x_n is increasing and with supremum r_F. Let $A_n = \{\varphi_n(M_n) \leq x\}$, then

$$\begin{aligned}
&\frac{P(A_n^c \cdots A_{n+k}^c A_{n+k+1}|\mathscr{F})}{P(A_n^c \cdots A_{n+k-1}^c A_{n+k}|\mathscr{F})} \\
&= \frac{P(x_n < M_n \leq x_{n+1}, \ldots, x_{n+k} < M_{n+k} \leq x_{n+k+1}, M_{n+k+1} \leq x_{n+k+1}|\mathscr{F})}{P(x_n < M_n \leq x_{n+1}, \ldots, x_{n+k-1} < M_{n+k-1} \leq x_{n+k}, M_{n+k} \leq x_{n+k}|\mathscr{F})} \\
&= \frac{P(x_{n+k} < X_{n+k} \leq x_{n+k+1}|\mathscr{F})P(X_{n+k+1} \leq x_{n+k+1}|\mathscr{F})}{P(X_{n+k} \leq x_{n+k}|\mathscr{F})} \to 0.
\end{aligned}$$

The conditions of Lemma 3.2 are fulfilled. The result follows. □

Example 3.4. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that

$$F(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \left(1 - \frac{1}{x_i}\right),$$

where $x_i \geq 1,\ 1 \leq i \leq n$. Let $A = \{\omega : X_1 \in [m, \infty), m > 1\}$ and $\mathscr{F} = \{\emptyset, \Omega, A, A^c\}$, then

$$F(x_1, x_2, \dots, x_n|A) = \begin{cases} 0, & x_1 \in [1, m), \\ \prod_{i=1}^{n}(1 - \frac{1}{x_i})/\frac{1}{m}, & x_1 \in [m, \infty), \end{cases}$$

and

$$F(x_1, x_2, \dots, x_n|A^c) = \begin{cases} \prod_{i=1}^{n}(1 - \frac{1}{x_i})/(1 - \frac{1}{m}), & x_1 \in [1, m), \\ \prod_{i=2}^{n}(1 - \frac{1}{x_i}), & x_1 \in [m, \infty). \end{cases}$$

Obviously $\{X_n, n \geq 1\}$ is $\mathscr{F}$-independent. Let $M_n = \max\{X_1, \dots, X_n\}$, then we have

$$P(M_n \leq x|A) = \begin{cases} 0, & x \in [1, m), \\ m(1 - \frac{1}{x})^n, & x \in [m, \infty) \end{cases}$$

and

$$P(M_n \leq x|A^c) = \begin{cases} (1 - \frac{1}{x})^n/(1 - \frac{1}{m}), & x \in [1, m), \\ (1 - \frac{1}{x})^{n-1}, & x \in [m, \infty). \end{cases}$$

M_n tends in probability and with probability one to infinity because $\sum_{n=1}^{\infty} P(M_n \leq x|\mathscr{F}) < \infty$ and $r_F = +\infty$. Let us study the following problem. For which sequence $a_n \to 0$ the sequence $a_n M_n$ continues to tend to infinity with probability one?

For large enough N, if $n \geq N$, then $\frac{x}{a_n} \geq m$, observe that

$$\begin{aligned} P(a_n M_n \leq x|A) &= P\left(M_n \leq \frac{x}{a_n}|A\right) \\ &= m\left(1 - \frac{a_n}{x}\right)^n \sim m e^{\frac{-na_n}{x}}, \end{aligned}$$

then we have

$$\sum_{n=N}^{\infty} P(a_n M_n \leq x|A) \sim m \sum_{n=N}^{\infty} e^{\frac{-na_n}{x}}.$$

Let event $A_n = \{a_n M_n \leq x\}$, if

$$\sum_{n=1}^{\infty} e^{\frac{-na_n}{x}} < \infty, \tag{3.1}$$

then

$$\sum_{n=1}^{\infty} P(a_n M_n \leq x|A) < \infty.$$

Similarly, the inequality holds true for A^c, thus

$$\sum_{n=1}^{\infty} P(a_n M_n \le x|\mathscr{F}) < \infty.$$

By Lemma 2.1, $P(\limsup A_n) = 0$, that is, $a_n M_n \xrightarrow{a.s.} \infty$. We know that

$$P(a_n M_n \le x|A) = m\left(1 - \frac{a_n}{x}\right)^n \sim m e^{\frac{-n a_n}{x}}.$$

If $na_n \to \infty$, then $P(a_n M_n \le x|A) \to 0$. Similarly, if $na_n \to \infty$, then $P(a_n M_n \le x|A^c) \to 0$, thus $P(a_n M_n \le x|\mathscr{F}) \to 0$. It follows from Theorem 3.3 that $a_n M_n \xrightarrow{a.s.} \infty$ is valid, and the condition is weaker than (3.1).

4. Convergence of sums of $\mathscr{F}$-independent random variables

In this section, we study the convergence of sums of $\mathscr{F}$-independent random variables. For this, we need the following inequality.

Theorem 4.1 (Conditional Kolmogorov's inequality)**.** *If $\{X_n, n \ge 1\}$ is a sequence of $\mathscr{F}$-independent random variables belonging to $L^2_{\mathscr{F}}$, then for an arbitrary $\mathscr{F}$-measurable random variable $\epsilon > 0$ a.s., we have*

$$1 - \frac{(\epsilon + c)^2}{\sum_{k=1}^n \sigma^2_{\mathscr{F}} X_k} \le P[\max_{1 \le k \le n} |S_k - E^{\mathscr{F}} S_k| \ge \epsilon|\mathscr{F}] \le \frac{1}{\epsilon^2} \sum_{k=1}^n \sigma^2_{\mathscr{F}} X_k \ \ a.s.,$$

where $S_n = X_1 + X_2 + \cdots + X_n,\ |X_k| \le c \le \infty$.

Proof. The proof of the right part has been given by Majerek in [13], we only prove the left part. Obviously, if $c = \infty$ the conclusion is true, thus we can suppose $c < \infty$. In the following, for the sake of convenience, let $E^{\mathscr{F}} S_k = 0, E^{\mathscr{F}} X_k = 0$. Let $A_0 = \Omega$, $B_1 = \{|S_1| \ge \epsilon\}$,

$$A_k = \{\max_{1 \le j \le k} |S_j| < \epsilon\},\ k = 1, 2, \ldots, n$$

and

$$B_k = A_{k-1} - A_k = \{|S_j| < \epsilon, 1 \le j \le k-1; |S_k| \ge \epsilon\},\ k = 2, 3, \ldots, n.$$

Because $B_k \in \sigma\{X_1, X_2, \ldots, X_k\}$, $S_k I_{B_k}$ is $\mathscr{F}$-independent given $S_n - S_k$. Thus, we have

$$\begin{aligned} E^{\mathscr{F}} S_n^2 I_{B_k} &= E^{\mathscr{F}} (S_k - (S_n - S_k))^2 I_{B_k} \\ &= E^{\mathscr{F}} S_k^2 I_{B_k} + E^{\mathscr{F}} (S_n - S_k)^2 I_{B_k}. \end{aligned} \tag{4.1}$$

Observe that on the set B_k, $|S_k| \le |S_{k-1}| + |X_k| \le \epsilon + c$ and $A_n^c = \bigcup_{k=1}^n B_k$, applying equation (4.1), we obtain

$$E^{\mathscr{F}} S_n^2 I_{A_n^c} = \sum_{k=1}^n E^{\mathscr{F}} S_k^2 I_{B_k} + \sum_{k=1}^n E^{\mathscr{F}} (S_n - S_k)^2 I_{B_k}$$

$$\leq (c+\epsilon)^2 \sum_{k=1}^{n} P(B_k|\mathscr{F}) + \sum_{k=1}^{n}\sum_{j=k+1}^{n} E^{\mathscr{F}} X_j^2 P(B_k|\mathscr{F})$$

$$\leq [(c+\epsilon)^2 + \sum_{k=1}^{n} E^{\mathscr{F}} X_k^2] P(A_n^c|\mathscr{F}). \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} E^{\mathscr{F}} S_n^2 I_{A_n^c} &= E^{\mathscr{F}} S_n^2 - E^{\mathscr{F}} S_n^2 I_{A_n} \\ &\geq \sum_{k=1}^{n} E^{\mathscr{F}} X_k^2 - \epsilon^2 P(A_n|\mathscr{F}) \\ &= \sum_{k=1}^{n} E^{\mathscr{F}} X_k^2 - \epsilon^2 + \epsilon^2 P(A_n^c|\mathscr{F}). \end{aligned} \tag{4.3}$$

By equations (4.2) and (4.3), we obtain the conclusion

$$P(A_n^c|\mathscr{F}) \geq \frac{\sum_{k=1}^{n} E^{\mathscr{F}} X_k^2 - \epsilon^2}{(\epsilon+c)^2 + \sum_{k=1}^{n} E^{\mathscr{F}} X_k^2 - \epsilon^2} \geq 1 - \frac{(\epsilon+c)^2}{\sum_{k=1}^{n} \sigma_{\mathscr{F}}^2 X_k}. \qquad \square$$

A sufficient condition for the convergence of series $\sum_{n=1}^{\infty} X_n$ has been given in [12] as follow.

Theorem 4.2. *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathscr{F}$-independent random variables such that $E^{\mathscr{F}} X_n = 0$, if*

$$\sum_{n=1}^{\infty} E^{\mathscr{F}} X_n^2 < \infty,$$

then $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Theorem 4.3. *Let $\{X_n, n \geq 1\}$ be a sequence of $\mathscr{F}$-independent random variables such that $|X_n| \leq c$, c is a constant.*

(i) *If $E^{\mathscr{F}} X_n = 0$ and $\sum_{n=1}^{\infty} E^{\mathscr{F}} X_n^2 = \infty$, then $\sum_{n=1}^{\infty} X_n$ diverges almost surely.*

(ii) *If $\sum_{n=1}^{\infty} X_n$ converges almost surely, where $\sigma_{\mathscr{F}}^2 X_n$ is a conditional variance of X_n, then $\sum_{n=1}^{\infty} E^{\mathscr{F}} X_n$ and $\sum_{n=1}^{\infty} \sigma_{\mathscr{F}}^2 X_n$ converge almost surely.*

Proof. (i) By the left of Kolmogorov's inequality, we have

$$P[\max_{1\leq k\leq m} |X_{n+1} + \cdots + X_{n+k}| \geq \epsilon|\mathscr{F}] \geq 1 - \frac{(\epsilon+c)^2}{\sum_{k=n+1}^{n+m} E^{\mathscr{F}} X_n^2} \to 1 \text{ as } m \to \infty,$$

thus for any $n \geq 1$, we obtain $P[\sup_{k\geq 1} |X_{n+1} + \cdots + X_{n+k}| \geq \epsilon|\mathscr{F}] = 1$, in this case $P[\sup_{k\geq 1} |X_{n+1} + \cdots + X_{n+k}| \geq \epsilon] = 1$, obviously $\sum_{n=1}^{\infty} X_n$ diverges almost surely.

(ii) Let X_n' be a new sequence of random variables such that $\{X_n, X_n'; n \geq 1\}$ are $\mathscr{F}$-independent, and for a fixed n, X_n and X_n' are identically distributed.

Let $X''_n = X_n - X'_n$, thus X''_n is a sequence of $\mathscr{F}$-independent random variables such that $|X''_n| \le 2c, E^{\mathscr{F}} X''_n = 0, \sigma^2_{\mathscr{F}} X''_n = 2\sigma^2_{\mathscr{F}} X_n$.

Because $\sum_{n=1}^{\infty} X_n$ and $\sum_{n=1}^{\infty} X'_n$ converge almost surely, $\sum_{n=1}^{\infty} X''_n$ converges almost surely. By (i), we have $\sum_{n=1}^{\infty} \sigma^2_{\mathscr{F}} X''_n < \infty$, thus $\sum_{n=1}^{\infty} \sigma^2_{\mathscr{F}} X_n < \infty$. By Theorem 4.2, $\sum_{n=1}^{\infty}(X_n - E^{\mathscr{F}} X_n)$ converges almost surely, thus $\sum_{n=1}^{\infty} E^{\mathscr{F}} X_n$ converges almost surely. □

5. Limit results for concomitants of order statistics

In this section, we will apply our conditional versions of Borel-Cantelli lemma to discuss the strong limit results for concomitants of order statistics, which are the conditional versions of the results in [3].

The theory of the concomitants of order statistics plays an important role in the process of solving practical problems. For example, Bhattacharya [5] has given an instance to show it's importance as follows: Studying two numerical characteristics X and Y defined for each individual in a population. If we want to select individuals according to their ranks in Y that can not be obtained, we can just do it by their ranks in a related variate X. In this case, the problems about concomitants of order statistics arise naturally.

The concept of concomitants of order statistics was introduced by David (1973) and Bhattacharya [5]. Suppose that $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n), \ldots$ are $\mathscr{F}$-independent and identically distributed random vectors with continuous bivariate distribution function $F(x, y)$, marginal distributions $H(x)$ and $G(y)$, and corresponding probability density functions $h(x)$ and $g(y)$, Let $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ be the order statistics of the sample $X_1, X_2, \ldots, X_n$ and $Y_{[1,n]}, Y_{[2,n]}, \ldots, Y_{[n,n]}$ be the corresponding concomitants of the defined order statistics, relating to the sample $Y_1, Y_2, \ldots, Y_n$. Suppose that $l_H = \inf\{x \in R : H(x) > 0\}$ and $r_H = \sup\{x \in R : H(x) < 1\}$ are the left and right extremities of H.

5.1. Limit results for $Y_{[n-k,n]}$ when k is fixed

Lemma 5.1. *Suppose*

$$\lim_{x \to r_H^-} \frac{G(y|\mathscr{F}) - F(x, y|\mathscr{F})}{1 - H(x|\mathscr{F})} = \beta(y|\mathscr{F}) \in [0, 1], \tag{5.1}$$

then, for $n \to \infty, k \ge 0$,

$$P(Y_{[n-k,n]} \le y|\mathscr{F}) \to \beta(y|\mathscr{F}).$$

Proof. Obviously, for a fixed k, $X_{n-k,n}$ is increasing in n and $X_{n-k,n} \le r_H$, then we have $X_{n-k,n} \to r_H, n \to \infty$. Thus for some small enough non-negative number ϵ,

$$\lim_{n \to \infty} P(X_{n-k,n} \ge r_H - \epsilon|\mathscr{F}) = 1.$$

We have

$$\lim_{n\to\infty} P(Y_{[n-k,n]} \le y|\mathscr{F}) = \lim_{n\to\infty} \frac{P(Y_{[n-k,n]} \le y, r_H - \epsilon \le X_{n-k,n} \le r_H|\mathscr{F})}{P(r_H - \epsilon \le X_{n-k,n} \le r_H|\mathscr{F})}$$
$$= \lim_{\epsilon\to 0} \frac{P(Y \le y, r_H - \epsilon \le X \le r_H|\mathscr{F})}{P(r_H - \epsilon \le X \le r_H|\mathscr{F})}$$
$$= \lim_{x\to r_H^-} \frac{F(dx, y|\mathscr{F})}{h(x|\mathscr{F})}.$$

Observing that $\lim_{x\to r_H^-} \frac{G(y|\mathscr{F})-F(x,y|\mathscr{F})}{1-H(x|\mathscr{F})} = \lim_{x\to r_H^-} \frac{F(dx,y|\mathscr{F})}{h(x|\mathscr{F})}$, then we obtain

$$\lim_{n\to\infty} P(Y_{[n-k,n]} \le y|\mathscr{F}) = \lim_{x\to r_H^-} \frac{G(y|\mathscr{F}) - F(x, y|\mathscr{F})}{1 - H(x|\mathscr{F})}.$$

□

Suppose that for some $c \in [-\infty, +\infty]$,

$$\beta(y|\mathscr{F}) = 0(y < c) \text{ and } \beta(y|\mathscr{F}) = 1(y > c).$$

Then the distribution $F(x, y)$ is called a c-stable maximum-concomitants conditional distribution. If such a c does not exist, then $F(x, y)$ is called an unstable maximum-concomitants conditional distribution.

Theorem 5.2. *Let for all $y < r_G$, $\beta(y|\mathscr{F}) = 0$ holds true, and*

$$\int_R \frac{G(y|\mathscr{F}) - F(x, y|\mathscr{F})}{(1 - H(x|\mathscr{F}))^2} [h(x|\mathscr{F}) - F(dx, y|\mathscr{F})] < \infty.$$

Then $Y_{[n-k,n]} \xrightarrow{a.s.} r_G$ as $n \to \infty$.

Proof. Suppose event $A_n = \{Y_{[n-k,n]} \le y\}$, then $P(A_n|\mathscr{F}) \to 0$. By symmetry and $\mathscr{F}$-independence, we have

$$P(Y_{[n-k,n]} \le y|\mathscr{F}) = \frac{n!}{(n-k-1)!k!} \int_R \int_{l_G}^{y} P(X_2 \le x, \ldots,$$
$$X_{n-k} \le x, X_{n-k+1} > x, \ldots, X_n > x|\mathscr{F}) F(dx, dy|\mathscr{F}).$$

It follows that

$$P(Y_{[n-k,n]} \le y|\mathscr{F}) \tag{5.2}$$
$$= \frac{n!}{(n-k-1)!k!} \int_R (1 - H(x|\mathscr{F}))^k H^{n-k-1}(x|\mathscr{F}) F(dx, y|\mathscr{F}).$$

Further, we will study that

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n+1]} \le y|\mathscr{F})$$
$$= P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} \le y, Y_{[n-k+1,n+1]} \le y|\mathscr{F})$$
$$+ P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} > y, Y_{[n-k+1,n+1]} \le y|\mathscr{F}).$$

Again, by symmetry and $\mathscr{F}$ independence, we have

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} \le y, Y_{[n-k+1,n+1]} \le y|\mathscr{F})$$

$$= \frac{n!}{(n-k-1)!(k-1)!} \int_R \int_y^{r_G} \int_x^{r_H} \int_{l_G}^{y} P(X_3 \leq x, \ldots, X_{n-k+1} \leq x,$$
$$X_{n-k+2} > x_1, \ldots, X_n > x_1, E_{n+1} | \mathscr{F}) F(dx_1, dv_1 | \mathscr{F}) F(dx, dv | \mathscr{F}),$$

where

$$E_{n+1} = \Big\{ (X, Y) \in \Pi_{(x, r_H] \times [l_G, y]} \bigcup \Pi_{(x_1, r_H] \times (y, r_G]} \Big\} \text{ and}$$
$$\Pi_{(x, r_H] \times [l_G, y]} \bigcup \Pi_{(x_1, r_H] \times (y, r_G]}$$

are half-open rectangles with vertices a, b, c, d. In the same way

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} > y, Y_{[n-k+1,n+1]} \leq y | \mathscr{F})$$
$$= \frac{n!}{(n-k-1)!(k-1)!} \int_R \int_y^{r_G} \int_x^{r_H} \int_{l_G}^{y} P(X_3 \leq x, \ldots, X_{n-k+1} \leq x,$$
$$X_{n-k+2} > x_1, \ldots, X_n > x_1, \widetilde{E}_{n+1} | \mathscr{F}) F(dx_1, dv_1 | \mathscr{F}) F(dx, dv | \mathscr{F}),$$

where $\widetilde{E}_{n+1} = \big\{ (X_{n+1}, Y_{n+1}) \in \Pi_{(x, x_1] \times [l_G, y]} \big\}$. It can be shown that

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} \leq y, Y_{[n-k+1,n+1]} \leq y | \mathscr{F})$$
$$= \frac{n!}{(n-k-1)!(k-1)!} \int_R \int_y^{r_G} [H(x|\mathscr{F})]^{n-k-1} \int_x^{r_H} \int_{l_G}^{y} (1 - H(x_1|\mathscr{F}))^{k-1}$$
$$\times [1 - H(x_1|\mathscr{F}) + F(x_1, y|\mathscr{F}) - F(x, y|\mathscr{F})] F(dx_1, dv_1|\mathscr{F}) F(dx, dv|\mathscr{F})$$

and

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} > y, Y_{[n-k+1,n+1]} \leq y | \mathscr{F})$$
$$= \frac{n!}{(n-k-1)!(k-1)!} \int_R \int_y^{r_G} [H(x|\mathscr{F})]^{n-k-1} \int_x^{r_H} \int_{l_G}^{y} (1 - H(x_1|\mathscr{F}))^{k-1}$$
$$\times [F(x_1, y|\mathscr{F}) - F(x, y|\mathscr{F})] F(dx_1, dv_1|\mathscr{F}) F(dx, dv|\mathscr{F}).$$

Evaluating $P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} \leq y, Y_{[n-k+1,n+1]} \leq y|\mathscr{F})$, we get

$$\int_x^{r_H} \int_{l_G}^{y} (1 - H(x_1|\mathscr{F}))^{k-1} [1 - H(x_1|\mathscr{F}) + F(x_1, y|\mathscr{F}) - F(x, y|\mathscr{F})]$$
$$\times F(dx_1, dv_1|\mathscr{F})$$
$$\leq (1 - H(x_1|\mathscr{F}))^k \int_x^{r_H} \int_{l_G}^{y} F(dx_1, dv_1|\mathscr{F})$$
$$\leq (1 - H(x_1|\mathscr{F}))^k [G(y|\mathscr{F}) - F(x, y|\mathscr{F})].$$

Evaluating $P(Y_{[n-k,n]} > y, Y_{[n-k+1,n]} > y, Y_{[n-k+1,n+1]} \leq y|\mathscr{F})$, we get

$$\int_x^{r_H} \int_{l_G}^{y} (1 - H(x_1|\mathscr{F}))^{k-1} [F(x_1, y|\mathscr{F}) - F(x, y|\mathscr{F})] F(dx_1, dv_1|\mathscr{F})$$
$$\leq (1 - H(x_1|\mathscr{F}))^k \int_x^{r_H} \int_{l_G}^{y} [G(y) - F(x, y)] F(dx_1, dv_1|\mathscr{F})$$

$$\leq (1 - H(x_1|\mathscr{F}))^k [G(y|\mathscr{F}) - F(x, y|\mathscr{F})].$$

Then

$$P(Y_{[n-k,n]} > y, Y_{[n-k+1,n+1]} \leq y|\mathscr{F}) \leq \frac{2n!}{(n-k-1)!(k-1)!}$$
$$\int_R \int_y^{r_G} [H(x|\mathscr{F})]^{n-k-1} [G(y|\mathscr{F}) - F(x, y|\mathscr{F})] F(dx, dy|\mathscr{F})$$
$$= \frac{2n!}{(n-k-1)!(k-1)!} \int_R [H(x|\mathscr{F})]^{n-k-1} (1 - H(x_1|\mathscr{F}))^k$$
$$[G(y|\mathscr{F}) - F(x, y|\mathscr{F})][h(x|\mathscr{F}) - F(dx, y|\mathscr{F})].$$

Because $\frac{1}{(1-x)^{2+k}} = \sum_{n=0}^{\infty} \frac{(n+k+1)!}{(k+1)!n!} x^n = \sum_{n=k+1}^{\infty} \frac{n!}{(n-k-1)!(k+1)!} x^{n-k-1}$, we obtain the following inequality

$$\sum_{n=k+1}^{\infty} P(Y_{[n-k,n]} > y, Y_{[n-k+1,n+1]} \leq y|\mathscr{F}) \leq 2k(k+1)$$
$$\times \int_R \frac{G(y|\mathscr{F}) - F(x, y|\mathscr{F})}{(1 - H(x|\mathscr{F}))^2} [h(x|\mathscr{F}) - F(dx, y|\mathscr{F})] < \infty.$$

Namely $\sum_{n=1}^{\infty} P(A_n^c A_{n+1}|\mathscr{F}) < \infty$, by Lemma 2.2, we have $P(\limsup A_n) = 0$, thus the conclusion $Y_{[n-k,n]} \xrightarrow{\text{a.s.}} r_G$ as $n \to \infty$ is true. □

Theorem 5.3. *Let for all* $\epsilon > 0, \beta(c-\epsilon|\mathscr{F}) = 0, \beta(c+\epsilon|\mathscr{F}) = 1$, *the inequalities*

$$\int_R \frac{G(c+\epsilon|\mathscr{F}) - F(x, c+\epsilon|\mathscr{F})}{(1 - H(x|\mathscr{F}))^2} [h(x|\mathscr{F}) - F(dx, c+\epsilon|\mathscr{F})] < \infty \tag{5.3}$$

and

$$\int_R \frac{G(c-\epsilon|\mathscr{F}) - F(x, c-\epsilon|\mathscr{F})}{(1 - H(x|\mathscr{F}))^2} [h(x|\mathscr{F}) - F(dx, c-\epsilon|\mathscr{F})] < \infty \tag{5.4}$$

hold true, then $Y_{[n-k,n]} \xrightarrow{a.s.} c$ *as* $n \to \infty$.

Proof. Suppose event$A_n = \{Y_{[n-k,n]} \leq c - \epsilon\}$, because $\beta(c-\epsilon|\mathscr{F}) = 0$, then $P(A_n|\mathscr{F}) \to 0$. From(5.3), we get $\sum_{n=1}^{\infty} P(A_n^c A_{n+1}|\mathscr{F}) < \infty$. By Lemma 2.2, $P(\limsup A_n) = 0$, thus the conclusion $Y_{[n-k,n]} \xrightarrow{\text{a.s.}} c$ as $n \to \infty$ is true. □

Theorem 5.3 is a slight extension of Theorem 5.2.

5.2. Limit results for $Y_{[n-k,n]}$ when k tends to infinity

In this part, we suppose that the distribution $F(x, y)$ is absolutely continuous and $F_x(x, y)$ is the density-distribution of $F(x, y)$. Let for some $y < r_G$

$$\lim_{x \to r_H^-} \frac{F_x(x, y|\mathscr{F})}{h(x|\mathscr{F})} = \beta(y|\mathscr{F}). \tag{5.5}$$

Observe that the limit in (5.5) equals the limit in (5.1) for the same y. When $F(x, y)$ is absolutely continuous, the limit in (5.1) boils down to the limit in

(5.5). Let $k = k_n$ be such that $k_n \to \infty$ as $n \to \infty$, under which condition we will study the sequence $Y_{[n-k,n]}$ is convergent in probability. Before the discussion, we need the following lemma from Dembinska et al. [8].

Lemma 5.4. *Let k_n depend on n in such way that $k_n \to \infty$, $n - k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Then, for any fixed $0 < d < 1$,*

$$\frac{n!}{(n-k_n)!(k_n-1)!} d^{n-k_n} \to 0 \quad as \quad n \to \infty.$$

Theorem 5.5. *Let $F(x,y)$ be absolutely continues, the limit in (5.5) exists and be equal to 0 for all $y < r_G$. Let $k_n > 0$ be an increasing sequence of positive integers such that $k_n \to \infty, n - k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$. Then*

$$Y_{[n-k,n]} \xrightarrow{P} r_G \text{ as } n \to \infty.$$

Proof. Because the limit in (5.5) exists and be equal to 0, then for any small $\epsilon > 0$, we choose x_ϵ such that for any $x > x_\epsilon, y < r_G$,

$$F_x(x, y|\mathscr{F}) < \epsilon h(x|\mathscr{F}). \tag{5.6}$$

We have

$$P(Y_{[n-k,n]} \le y|\mathscr{F}) = I_1(y|\mathscr{F}) + I_2(y|\mathscr{F}),$$

where

$$I_1(y|\mathscr{F}) = \frac{n!}{(n-k_n-1)!k_n!} \int_{x_\epsilon}^{r_H} (1 - H(x|\mathscr{F}))^{k_n} H^{n-k_n-1}(x|\mathscr{F}) F(dx, y|\mathscr{F}),$$

$$I_2(y|\mathscr{F}) = \frac{n!}{(n-k_n-1)!k_n!} \int_{l_H}^{x_\epsilon} (1 - H(x|\mathscr{F}))^{k_n} H^{n-k_n-1}(x|\mathscr{F}) F(dx, y|\mathscr{F}).$$

It follows from (5.6) that for any $y < r_G$,

$$\begin{aligned} I_1(y|\mathscr{F}) &< \frac{\epsilon n!}{(n-k_n-1)!k_n!} \int_{x_\epsilon}^{r_H} (1 - H(x|\mathscr{F}))^{k_n} H^{n-k_n-1}(x|\mathscr{F}) h(x|\mathscr{F}) dx \\ &< \epsilon \int_{l_H}^{r_H} h(x|\mathscr{F}) dx = \epsilon. \end{aligned}$$

For any $y < r_G$, we have

$$I_2(y|\mathscr{F}) < \frac{n!}{(n-k_n-1)!k_n!} H^{n-k_n-1}(x_\epsilon|\mathscr{F}).$$

Because $0 < H(x_\epsilon|\mathscr{F}) < 1$, by Lemma 5.4, $I_2(y|\mathscr{F}) \to 0$ for any $y < r_G$. Thus, we obtain for any $y < r_G$, $P(Y_{[n-k,n]} \le y|\mathscr{F}) \to 0$, namely $Y_{[n-k,n]} \xrightarrow{P} r_G$ as $n \to \infty$. □

5.3. Limit results for $C_n Y_{[n-k,n]}$

In this part, we will study the limit of $C_n Y_{[n-k,n]}$, where C_n is a non-stochastic sequence.

Theorem 5.6. *Let $F(x, y)$ be a 0-stable maximum-concomitants conditional distribution*

(1) *Let $C_n \to +\infty$ be such that for any $y > 0$*

$$n[G(y/C_n|\mathscr{F}) - G(-y/C_n|\mathscr{F})] \to 0 \ (n \to \infty),$$

then for any fixed $k \geq 0$,

$$C_n|Y_{[n-k,n]}| \xrightarrow{P} \infty \ \text{as} \ n \to \infty.$$

(2) *Let $C_n \to +\infty$ be such that for any $y > 0$*

$$\sum_{n=1}^{\infty} n[G(y/C_n|\mathscr{F}) - G(-y/C_n|\mathscr{F})] < \infty,$$

then for any fixed $k \geq 0$,

$$C_n|Y_{[n-k,n]}| \xrightarrow{a.s.} \infty \ \text{as} \ n \to \infty.$$

Proof. First, observe that even if l_G and r_G are both finite, the support for the limiting distribution of $C_n Y_{[n-k,n]}$ is unbounded. It follows from (5.2) that

$$\sum_{k=0}^{n-1} P(a < Y_{[n-k,n]} < b|\mathscr{F}) = n[G(b|\mathscr{F}) - G(a|\mathscr{F})],$$

then we have

$$P(a < Y_{[n-k,n]} < b|\mathscr{F}) \leq n[G(b|\mathscr{F}) - G(a|\mathscr{F})],$$

thus

$$\begin{aligned} P(|C_n Y_{[n-k,n]}| \leq y|\mathscr{F}) &= P(-y < C_n Y_{[n-k,n]} < y|\mathscr{F}) \\ &\leq n[G(y/C_n|\mathscr{F}) - G(-y/C_n|\mathscr{F})] \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $y > 0, C_n > 0$. The first statement of Theorem 5.6 readily follows. The second statement can be similarly obtained if we apply Lemma 2.1 in Section 2. □

By slightly modifying the arguments in Theorem 5.6, we get the following limit result.

Theorem 5.7. *Let $F(x, y)$ be a $+\infty$ (or $-\infty$)-stable maximum-concomitants conditional distribution.*

(1) *Let $C_n \to 0$ be such that for any $y > 0$*

$$n[1 - G(y/C_n|\mathscr{F}) + G(-y/C_n|\mathscr{F})] \to 0 \ \text{as} \ n \to \infty,$$

then for any fixed $k \geq 0$,

$$C_n Y_{[n-k,n]} \xrightarrow{P} 0 \ \text{as} \ n \to \infty.$$

(2) *Let $C_n \to +\infty$ be such that for any $y > 0$*

$$\sum_{n=1}^{\infty} n[1 - G(y/C_n|\mathscr{F}) + G(-y/C_n|\mathscr{F})] < \infty,$$

then for any fixed $k \geq 0$,

$$C_n Y_{[n-k,n]} \xrightarrow{a.s.} 0 \text{ as } n \to \infty.$$

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Qianmin Chen
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan, 430074, P. R. China
E-mail address: www.haoojo@qq.com

Jicheng Liu
School of Mathematics and Statistics
Huazhong University of Science and Technology
Wuhan, 430074, P. R. China
E-mail address: jcliu@hust.edu.cn