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RINGS OF COPURE PROJECTIVE DIMENSION ONE

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Abstract. In this paper, in terms of the notions of strongly copure projective modules and the copure projective dimension $cpD(R)$ of a ring R were defined in [12], we show that a domain R has $cpD(R) \leq 1$ if and only if R is a Gorenstein Dedekind domain.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are left R-modules unless otherwise stated. For an R-module M, $\mathrm{fd}_R M$ (resp. $\mathrm{id}_R M$) stands for the flat (resp. injective) dimension of M. We also use $w.\mathrm{gl.dim}(R)$ (resp. $l.\mathrm{gl.dim}(R)$) to denote the weak global (resp. left global) dimension of R, and use $\mathcal{F}_n$ to denote the class of R-modules with flat dimension at most a fixed nonnegative integer n. And, for a commutative ring R, we also denote by $\mathrm{Max}(R)$ the set of maximal ideals.

In [12], Fu et al. introduced the concepts of copure projective modules, n-copure projective modules, strongly copure projective modules, and the copure projective dimension. A left R-module M is called n-copure projective if $\mathrm{Ext}^1_R(M, N) = 0$ for any R-module $N \in \mathcal{F}_n$. 0-copure projective modules are said simply to copure projective. M is said to be strongly copure projective if $\mathrm{Ext}^{i+1}_R(M, F) = 0$ for any flat R-module F, and all $i \geq 0$. The copure projective dimension $cpd_R(M)$ of an R-module M is defined to be the smallest integer $n \geq 0$ such that $\mathrm{Ext}^{n+i}_R(M, F) = 0$ for any flat left R-module F and for any $i \geq 0$. Of course, if no such n exists, write $cpd_R(M) = \infty$. Thus $cpd_R(M) \leq m$ is equivalent to M has a strongly copure projective resolution

$$0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

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where each P_i is strongly copure projective. The copure projective dimension of a ring R is defined as

$$cpD(R) = \sup\{\, cpd_R(M) \mid M \text{ is an } R\text{-module}\,\}.$$

In this paper, we characterize some classes of rings in terms of copure projective modules.

2. QF rings, IF rings and semihereditary rings

It was shown in [12, Remark 4.2 & Proposition 3.12] that a ring R is a QF ring if and only if $cpD(R) = 0$, and if and only if every right R-module is copure projective. Recall that an R-module D is h-divisible if it is an epic image of an injective R-module. As in [4], we call a ring R a right IF ring if every injective right R-module is flat. Now, we characterize QF rings and IF rings in terms of h-divisible modules and copure projective modules.

Proposition 2.1. *The following statements are equivalent for a ring R:*

(1) *R is a right* IF *ring.*

(2) *R is a left coherent ring, and every finitely presented left R-module is copure projective.*

(3) *R is a left coherent ring, and every finitely presented left R-module is n-copure projective.*

(4) *R is a left coherent ring, and every finitely presented left R-module is strongly copure projective.*

Proof. (1)⇒(4). If R is a right IF ring, by [4, Theorem 2], R is a left coherent. Now, let M be a finitely presented R-module. By [16, Theorem 3.10], R is FP-injective, and so every finitely generated free R-module is FP-injective. Thus for any flat R-module N, then $N = \varinjlim F_i$ by [20, Theorem 5.40], where each F_i is finitely generated free R-module. By [22, Theorem 3.2], we have

$$\mathrm{Ext}^n_R(M, N) = \mathrm{Ext}^n_R(M, \varinjlim F_i) \cong \varinjlim \mathrm{Ext}^n_R(M, F_i) = 0,$$

where $n \geq 1$. Hence M is strongly copure projective.

(4)⇒(2). Trivially.

(2)⇒(1). Let M be a finitely presented R-module. By (2), M is copure projective, then $\mathrm{Ext}^1_R(M, R) = 0$. Thus R is FP-injective. Since R is a left coherent ring, R is a right IF ring by [16, Theorem 3.10].

(1)⇔(3). By [13, Theorem 2.13]. □

Proposition 2.2. *The following statements are equivalent for a ring R:*

(1) *R is a* QF *ring.*

(2) *Every h-divisible module is copure projective.*

(3) *R is a left Neotherian ring and every finitely presented left R-module is copure projective.*

Proof. (1)⇒(2) and (1)⇒(3) are trivial.

(2)⇒(1). For any flat R-module N, pick an exact sequence $0 \to N \to E \to E/N \to 0$ with E injective. By (2), E/N is copure projective. Thus $\mathrm{Ext}^1_R(E/N, N) = 0$, and $0 \to N \to E \to E/N \to 0$ is split. Then N is injective, and $cpD(R) = 0$.

(3)⇒(1). It follows directly from Proposition 2.1. □

Now, we study the localization of commutative QF rings. As in [8], Enochs and Jenda introduce the concepts of copure flat modules and strongly copure flat modules. For a left R-module M, M is called copure flat if $\mathrm{Tor}^R_1(E, M) = 0$ for any injective right R-module E, and M is called strongly copure flat if $\mathrm{Tor}^R_i(E, M) = 0$ for any injective right R-module E and for all $i \geq 1$. In the paper [8] the author defined the copure flat dimension $cfd_R M$ of an R-module M to be the largest integer $n \geq 0$ such that $\mathrm{Tor}^R_n(E, M) \neq 0$ for some injective right R-module E. Of course, if no such n exists, write $cfd_R(M) = \infty$. Thus $cfd_R M = 0$ if and only if M is strongly copure flat. As in [11, Lemma 3.2], it was shown that for a left R-module M, $cfd_R M \leq m$ if and only if $\mathrm{Tor}^R_{m+i}(E, M) = 0$ for any injective right R-module E.

Theorem 2.3. *Let R be a commutative Noetherian ring. Then*

$$cpD(R) = \sup\{cpD(R_{\mathfrak{m}}) \mid \ \mathfrak{m} \in \mathrm{Max}(R)\}.$$

Proof. Let M be a finitely presented R-module. By [12, Proposition 3.7], $cpd_R M \geq cfd_R M$. Now, set $k = cfd_R M$. Then there exists an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where $P_0, P_1, \ldots, P_{k-1}$ is finitely generated projective and P_k is strongly flat. Since R is Noetherian, P_k is finitely presented. For any flat R-module F, F^+ is injective by [9, Theorem 3.2.10]. Then $\mathrm{Ext}^i_R(P_k, F)^+ \cong \mathrm{Tor}^R_i(P_k, F^+) = 0$ by [14, Lemma 1.2.11]. It follows that P_k is strongly copure projective. Hence $cpd_R M = cfd_R M$.

Now, set $n = cfd_R M$. By [11, Lemma 3.2], there exists an injective R-module E such that $\mathrm{Tor}^R_m(E, M) \neq 0$. For some $\mathfrak{m} \in \mathrm{Max}(R)$, $\mathrm{Tor}^{R_{\mathfrak{m}}}_m(E_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \mathrm{Tor}^R_m(E, M)_{\mathfrak{m}} \neq 0$. Since $E_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module by [9, Theorem 3.2.16], $\sup\{cfd_{R_{\mathfrak{m}}} M_{\mathfrak{m}}\} \geq cfd_R M$.

Without loss of generality we can assume $k := cpD(R) < \infty$. Let M be an $R_{\mathfrak{m}}$-module. Then we have an exact sequence $0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is strongly copure projective over R. Thus the sequence $0 \to (P_k)_{\mathfrak{m}} \to (P_{k-1})_{\mathfrak{m}} \to \cdots \to (P_1)_{\mathfrak{m}} \to (P_0)_{\mathfrak{m}} \to M \to 0$ is exact. Let N be a flat $R_{\mathfrak{m}}$-module. Then we have that $k = \mathrm{fd}_R N < \infty$ by hypothesis. Let $0 \to N \to E \to C \to 0$ be an exact sequence with E injective over $R_{\mathfrak{m}}$. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R_{\mathfrak{m}}}((P_i)_{\mathfrak{m}}, E) & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{m}}}((P_i)_{\mathfrak{m}}, C) & \longrightarrow & \mathrm{Ext}^1_{R_{\mathfrak{m}}}((P_i)_{\mathfrak{m}}, N) & \longrightarrow & 0 \\
\cong \downarrow & & \downarrow \cong & & \downarrow \theta & & \\
\mathrm{Hom}_R(P_i, E) & \longrightarrow & \mathrm{Hom}_R(P_i, C) & \xrightarrow{\mu} & \mathrm{Ext}^1_R(P_i, N) & \longrightarrow & X
\end{array}$$

where X is a cokernel of μ. By the Adiont Isomorphic Theorem we have the two vertical arrows on the left are isomorphic. Hence θ is a monomorphism. Let us consider the exact sequence $0 \to K \to F \to N \to 0$ with F flat and $\mathrm{fd}_R K = k-1 < \infty$. Then $\mathrm{Ext}^1_R(P_i, N) \cong \mathrm{Ext}^2_R(P_i, K)$. Hence $\mathrm{Ext}^1_R(P_i, N) = 0$ by induction on k. Then $\mathrm{Ext}^1_{R_\mathfrak{m}}((P_i)_\mathfrak{m}, N) = 0$.

Let $0 \to A \to F \to P_i \to 0$ be an exact sequence, where F is a free R-module. Thus A is also strongly copure projective over R. Since $\mathrm{Tor}^R_1(R_\mathfrak{m}, P_i) = 0$, the sequence $0 = \mathrm{Tor}^R_1(R_\mathfrak{m}, P_i) \to A_\mathfrak{m} \to F_\mathfrak{m} \to (P_i)_\mathfrak{m} \to 0$ is exact. Then $\mathrm{Ext}^{k+1}_{R_\mathfrak{m}}((P_i)_\mathfrak{m}, N) \cong \mathrm{Ext}^k_{R_\mathfrak{m}}(A_\mathfrak{m}, N)$ and $\mathrm{Ext}^{k+1}_R(P_i, N) \cong \mathrm{Ext}^k_R(A, N)$. By dimension shifting we have $\mathrm{Ext}^k_{R_\mathfrak{m}}((P_i)_\mathfrak{m}, N) = 0$ for all $k \geq 1$. Hence $(P_i)_\mathfrak{m}$ is a strongly copure projective $R_\mathfrak{m}$-modules. Thus we get $cpd(R_\mathfrak{m}) \leq k$. Hence $cpD(R) = \sup\{cpD(R_\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)\}$. □

By Theorem 2.3 and [12, Remark 4.2(2)], we have the following proposition.

Proposition 2.4. *Let R be a commutative Noetherian ring. Then R is a* QF *ring if and only if $R_\mathfrak{m}$ is a* QF *ring for every maximal ideal $\mathfrak{m}$ of R.*

A ring R is said to be left semihereditary if its finitely generated left ideals are projective. It was shown in [7, Theorem 4.5] that a left coherent ring R is semihereditary if and only if every finitely presented copure flat right R-module M is flat. Now, we have:

Proposition 2.5. *The following statements are equivalent for a coherent ring R:*

(1) *R is left semihereditary.*

(2) *Every copure projective right R-module M is flat.*

(3) *Every finitely presented copure projective right R-module M is projective.*

Proof. (1)⇒(2). Let M be a copure projective right R-module. By [12, Proposition(1) 3.7], M is copure flat. Hence M is flat by [7, Theorem 4.5].

(2)⇒(3). Let M be a finitely presented copure projective right R-module. By (2), M is flat. Since M is finitely presented, M is projective.

(3)⇒(1). Let M be a finitely presented copure flat right R-module. For any flat right R-module F, F^+ is injective by [9, Theorem 3.2.10]. Then $\mathrm{Ext}^1_R(M, F)^+ \cong \mathrm{Tor}^R_1(M, F^+) = 0$ by [14, Lemma 1.2.11]. Hence M is copure projective. By (3), M is projective. Thus, it follows from [7, Theorem 4.5] that R is left semihereditary. □

3. CPH rings and Gorenstein Dedekind domains

For any homological dimension, ones often discuss the rings of this dimension at most one. So it is natural to ask the structure of rings that every submodule of a strongly copure projective module is strongly copure projective, that is, of rings with $cpD(R) \leq 1$. Now, let us call provisionally such a ring R strongly CPH (Copure-Projective-Hereditary) ring if every submodule of a strongly copure projective module is strongly copure projective. In [12] it is proved that if R is right coherent and left Noetherian, then R is a strongly CPH ring if and only if $cpD(R)\leq 1$. 1n 2013, Xiong et al. [23] introduced the notion of CPH rings. A ring R is called a CPH ring if every submodule of a copure projective module is copure projective. In 2015 Gao [13] also introduced the notion of relative hereditary rings for which every submodule of a n-copure projective module is n-copure projective. Naturally, we have the following implications:

$$\text{strongly CPH rings} \Longrightarrow \text{relative hereditary rings} \Longrightarrow \text{CPH rings.}$$

Next we show that these implications can be reversed, that is, a ring R is strongly CPH ring if and only if R is a CPH ring.

Theorem 3.1. *The following statements are equivalent for a ring R:*
(1) $l.cpD(R) \leq 1$, *that is, R is a strongly* CPH *ring.*
(2) *R is a left* CPH *ring.*
(3) *R is a left relative hereditary ring.*
(4) $\mathrm{id}_R F \leq 1$ *for any R-modules F with* $\mathrm{fd}_R F < \infty$.

Proof. (1)⇒(2). Let M be a copure projective left R-module and let N be a submodule of M. Then $0 \to N \to M \to M/N \to 0$ is exact. For any flat left R-module F, $\mathrm{id}_R F \leq 1$ by [12, Theorem 4.11]. Consider the exact sequence $0 = \mathrm{Ext}^1_R(M,F) \to \mathrm{Ext}^1_R(N,F) \to \mathrm{Ext}^2_R(M,F) = 0$. Then we get $\mathrm{Ext}^1_R(N,F) = 0$. Hence N is copure projective.

(2)⇒(1). Let F be a flat left R-module. For any left R-module X, there exists an exact sequence $0 \to K \to P \to X \to 0$ with P projective and K copure projective by (2). Since $0 = \mathrm{Ext}^1_R(K,F) \to \mathrm{Ext}^2_R(X,F) \to \mathrm{Ext}^2_R(P,F) = 0$ is exact, we get $\mathrm{Ext}^2_R(X,F) = 0$. Hence $\mathrm{id}_R F \leq 1$ and $l.cpD(R) \leq 1$.

(1)⇒(3). Let M be a n-copure projective left R-module and let N be a submodule of M. Then $0 \to N \to M \to M/N \to 0$ is exact. For any left R-module F with $\mathrm{fd}_R F \leq n$, then there exists an exact sequence $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to F \to 0$ with each F_i flat. Let I be a left ideal of R. By [12, Corollary 4.12], $cpd_R(R/I) \leq 1$, and hence $\mathrm{Ext}^2_R(R/I, F_i) = 0$. Thus $\mathrm{id}_R F_i \leq 1$. Note $K_0 = F$, $K_1 = \ker(F_0 \to F)$, and $K_i = \ker(F_{i-1} \to F_{i-2})$ for $i \geq 2$. Consider the exact sequence $0 \to F_n \to F_{n-1} \to K_{n-1} \to 0$ and let X be a R-module. Then we can obtain $\mathrm{id}_R K_{n-1} \leq 1$ from the sequence $0 = \mathrm{Ext}^2_R(X, F_{n-1}) \to \mathrm{Ext}^2_R(X, K_{n-1}) \to \mathrm{Ext}^3_R(X, F_n) = 0$. By the same way, we can get $\mathrm{id}_R F \leq 1$. By proof of (1)⇒(2), we get $\mathrm{Ext}^1_R(N,F) = 0$. Hence N is n-copure projective.

(3)⇒(4). Let N be an R-module with $k = \mathrm{fd}_R N < \infty$. Without loss of generality we can assume $k := 2$. Let I be any ideal of R. By (3), I is 2-copure projective. Since $0 = \mathrm{Ext}^1_R(I,N) \to \mathrm{Ext}^2_R(R/I,N) \to \mathrm{Ext}^2_R(R,N) = 0$ is exact, we have $\mathrm{Ext}^2_R(R/I,N) = 0$. Hence $\mathrm{id}_R F \leq 1$.

(4)⇒(1). By [12, Theorem 4.11]. □

Let us call an R-module M ∞-copure projective if $\mathrm{Ext}^1_R(M,N) = 0$ for any R-module N with $\mathrm{fd}_R N < \infty$; and a ring R ∞-CPH ring if the submodule of a

∞-copure projective R-module is also ∞-copure projective. By Theorem 3.1, we have

strongly CPH rings = ∞-CPH rings = relative hereditary rings = CPH rings.

Example 3.2. Let R be a regular local ring with Krull dimension 2. Then $T = R/aR$ is a CPH ring, where a is an element of R that is neither a non-zero-divisor nor a unit.

In [2], Bass introduced the finitistic projective dimension of a ring R as

$$l.\mathrm{FPD}(R) = \sup\{\, \mathrm{pd}_R M \mid M \text{ is a left } R\text{-module with } \mathrm{pd}_R M < \infty \,\}.$$

Theorem 3.3 ([12, Proposition 4.3])**.** *Let R be a left* CPH *ring. Then*

$$l.\mathrm{FPD}(R) \leq 1.$$

Example 3.4. Now we give an example of a ring R with $l.\mathrm{FPD}(R) \leq 1$ which is not CPH. Let L be a field and F an extension field of L with $[F : L] = \infty$. Construct $R = L + xF[x]$. Then R is an almost perfect domain by [21]. Hence $\mathrm{FPD}(R) = 1$ by [1, Proposition 3.2]. Because R is not Noetherian, R is not CPH.

Now we are in a position to discuss the relations between the class of left CPH rings and the class of left hereditary rings. By Theorem 3.1, a CPH ring is not in general a (left) hereditary ring.

Theorem 3.5. *Let R be a left* CPH *ring. Then either $l.\mathrm{gl.dim}(R) \leq 1$ or $w.\mathrm{gl.dim}(R) = \infty$.*

Proof. Assume that $w.\mathrm{gl.dim}(R) < \infty$. Let F be a projective R-module and M a submodule of F. By hypothesis, M is copure projective with $k := \mathrm{fd}_R M < \infty$. We will show $k = 0$. Assume $k > 0$. Let $0 \to P_k \to P_{k-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence, where $P_0, P_1, \ldots, P_{k-1}$ are projective. Thus P_k is flat. Since R is a CPH ring, all syzygies in this long exact sequence are copure projective. Clearly, $\mathrm{fd}_R P_k \leq 1$. Consider the exact sequence $0 \to F_{k1} \to F_{k0} \to P_k \to 0$ with F_{k0} free and F_{k1} flat. Hence the exact sequence is split and P_k is projective, whence $\mathrm{fd}_R M \leq k - 1$, a contradiction. Thus $k = 0$, and hence M is projective, which implies that R is hereditary. □

Corollary 3.6. *Let R be a* CPH *ring. Then R is hereditary if and only if every copure projective left R-module is projective.*

Proof. Assume that R is hereditary. Let C be a copure projective module. Pick an exact sequence $0 \to A \to P \to C \to 0$, where P is projective. By Theorem 3.1, A is strongly copure projective. By [12, Proposition 3.5], C is strongly copure projective. By hypothesis, $\mathrm{fd}_R C \leq 1$. By [12, Proposition 3.4], C is projective. Conversely, assume that every copure projective left R-module is projective, let $0 \to N \to P \to P/N \to 0$ be an exact sequence with N being a submodule of a projective module P. By hypothesis, N is projective. Hence R is hereditary. □

Example 3.7. We give another example of a CPH ring without being hereditary. In fact, construct $R = \mathbb{Q}[x,y]/(x^2+2y^2)$. Since x^2+2y^2 is an irreducible polynomial, we have that R is a CPH domain. Noting that R is not integrally closed, we have $\mathrm{gl.dim}(R) = \infty$.

Let x be an indeterminate over R. Now, we raise a question that if R is a QF ring, then whether $R[x]$ is a CPH ring.

Theorem 3.8. *The following statements are equivalent for a ring R:*
(1) *R is a* QF *ring.*
(2) *For any indeterminate x over R, $R[x]$ is a* CPH *ring.*

Proof. (1)⇒(2). Equivalently, we prove that $cpD(R[x]) = cpD(R)+1$. First, for an R-module M, we prove that $cpd_{R[x]}(M[x]) = cpd_R(M)$. Let m be a nonnegative integer. If $cpd_R(M) \le m$, then there is an exact sequence $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where every P_i is a strongly copure projective R-module. Thus $0 \to P_m[x] \to P_{m-1}[x] \to \cdots \to P_1[x] \to P_0[x] \to M[x] \to 0$ is exact. Let N be a flat $R[x]$-module. Then we have that N is a flat R-module. Let $0 \to N \to E \to C \to 0$ be an exact sequence where E is an injective $R[x]$-module. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_{R[x]}(P_i[x],E) & \longrightarrow & \mathrm{Hom}_{R[x]}(P_i[x],C) & \longrightarrow & \mathrm{Ext}^1_{R[x]}(P_i[x],N) & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\cong & & \downarrow\theta & & \\
\mathrm{Hom}_R(P_i,E) & \longrightarrow & \mathrm{Hom}_R(P_i,C) & \xrightarrow{\alpha} & \mathrm{Ext}^1_R(P_i,N) & \longrightarrow & \mathrm{cok}\alpha
\end{array}$$

By the Adiont Isomorphic Theorem we have the two vertical arrows on the left are isomorphic. Hence θ is a monomorphism. Consider the exact sequence $0 \to K \to F \to N \to 0$ with F flat and $\mathrm{fd}_R K = k-1 < \infty$. Then $\mathrm{Ext}^1_R(P_i,N) \cong \mathrm{Ext}^2_R(P_i,K)$. Hence $\mathrm{Ext}^1_R(P_i,N) = 0$ by induction on k. Then $\mathrm{Ext}^1_{R[x]}(P_i[x],N) = 0$. Let $k \ge 0$. Let $0 \to A \to F \to P_i \to 0$ be an exact sequence, where F is a free R-module. Thus A is also strongly copure projective over R. Then the sequence $0 = \mathrm{Tor}^R_1(R[x],P_i) \to A[x] \to F[x] \to P_i[x] \to 0$ is exact, and $\mathrm{Ext}^{k+1}_{R[x]}(P_i[x],N) \cong \mathrm{Ext}^k_{R[x]}(A[x],N)$, $\mathrm{Ext}^{k+1}_R(P_i,N) \cong \mathrm{Ext}^k_R(A,N)$. By dimension shifting, $\mathrm{Ext}^k_{R[x]}(P_i[x],N) = 0$ for all $k \ge 1$. Hence $P_i[x]$ is a strongly copure projective $R[x]$-module. Hence $cpd_{R[x]}(M[x]) \le m$. On the other hand, let $cpd_{R[x]}(M[x]) \le m$. Then there is an exact sequence

$$0 \to F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M[x] \to 0,$$

where $F_0, F_1, \ldots, F_m$ are strongly copure projective $R[x]$-modules. Since x is certainly a non-zero-divisor of $M[x]$, we have

$$0 \to F_m/xF_m \to F_{m-1}/xF_{m-1} \to \cdots \to F_1/xF_1 \to F_0/xF_0 \to M \to 0$$

is exact. Notice that $R \cong R[x]/xR[x]$. Let N be a flat R-module. Let $0 \to N \to E \to C \to 0$ be an exact sequence where E is an injective R-module. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Hom}_R(F_i/xF_i,E) & \longrightarrow & \mathrm{Hom}_R(F_i/xF_i,C) & \longrightarrow & \mathrm{Ext}^1_R(F_i/xF_i,N) & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\cong & & \downarrow\alpha & & \\
\mathrm{Hom}_R(F_i,E) & \longrightarrow & \mathrm{Hom}_R(F_i,C) & \xrightarrow{\xi} & \mathrm{Ext}^1_R(F_i,N) & \longrightarrow & \mathrm{cok}\xi
\end{array}$$

Hence α is a monomorphism. Consider the exact sequence $0 \to K \to F \to N \to 0$ with K, F flat. Then $\mathrm{Ext}^1_{R[x]}(F_i, N) \cong \mathrm{Ext}^2_{R[x]}(F_i, K)$. Hence $\mathrm{Ext}^1_{R[x]}(F_i, N) = 0$. Then

$$\mathrm{Ext}^1_R(F_i/xF_i, N) = 0.$$

Let $k \geq 0$. Let $0 \to A \to F \to F_i \to 0$ be an exact sequence, where F is a free $R[x]$-module. Thus A is also strongly copure projective over $R[x]$. Since x is a non-zero-divisor of F_i, the sequence $0 \to A/aA \to F/aF \to F_i/xF_i \to 0$ is exact. Then $\mathrm{Ext}^{k+1}_R(F_i/xF_i, N) \cong \mathrm{Ext}^k_R(A/xA, N)$ and $\mathrm{Ext}^{k+1}_R(F_i, N) \cong \mathrm{Ext}^k_R(A, N)$. We have $\mathrm{Ext}^k_R(F_i/xF_i, N) = 0$ for all $k \geq 1$ by dimension shifting. Hence F_i/xF_i is a strongly copure projective R-modules. Thus we get $cpd_{R[x]}(M[x]) = cpd_R(M)$.

Now, we can assume $m := cpD(R) < \infty$. There is an R-module $M \neq 0$ with $cpd_R(M) = m$. Let N be any flat $R[x]$-module. Certainly, x is a non-zero-divisor of N. By Rees Theorem, $\mathrm{Ext}^{m+2}_{R[x]}(M, N) \cong \mathrm{Ext}^{m+1}_R(M, N/xN) = 0$. Thus $cpd_{R[x]}(M) \leq m + 1$. Since $cpd_R(M) = m$, there is a flat R-module N with $\mathrm{Ext}^m_R(M, N) \neq 0$. Let $0 \to A \to F/xF \to N \to 0$ be exact, where F is a free R-module. Thus A is also a flat R-module. Hence the exact sequence $\mathrm{Ext}^m_R(M, F/xF) \to \mathrm{Ext}^m_R(M, N) \to \mathrm{Ext}^{m+1}_R(M, A) = 0$, which implies $\mathrm{Ext}^m_{R[x]}(M, F/xF) \neq 0$. By using Rees Theorem again we get $\mathrm{Ext}^{m+1}_{R[x]}(M, F) \neq 0$. Therefore, $cpd_{R[x]}(M) \geq m + 1$. Hence $cpd_{R[x]}(M) = m + 1$. Therefore, $cpD(R[x]) \geq m + 1$.

Let A be an $R[x]$-module. Consider the canonical exact $R[x]$-sequence $0 \longrightarrow A[x] \longrightarrow A[x] \longrightarrow A \longrightarrow 0$. By [12, Proposition 4.10(3)], we have $cpd_{R[x]}(A) \leq cpd_{R[x]}(A[x]) + 1 = cpd_R(A) + 1 \leq m + 1$. Consequently, $cpD(R[x]) \leq m + 1$. Thus we are done.

(2)⇒(1). Set $m = cpD(R)$. There is an R-module $M \neq 0$ with $cpd_R M = m$, and a flat R-module N with $\mathrm{Ext}^m_R(M, N) \neq 0$. Let $0 \to A \to F/xF \to N \to 0$ be exact, where F is a free $R[x]$-module. Thus A is also a flat R-module. Hence we have the exact sequence

$$\mathrm{Ext}^m_R(M, F/xF) \to \mathrm{Ext}^m_R(M, N) \to \mathrm{Ext}^{m+1}_R(M, A) = 0,$$

which implies $\mathrm{Ext}^m_R(M, F/xF) \neq 0$. By using Rees Theorem again we get $\mathrm{Ext}^{m+1}_{R[x]}(M, F) \neq 0$. Therefore, $1 \geq cpd_{R[x]} M \geq m + 1$ by Theorem 3.1. Hence $m = 0$. Therefore, $cpD(R) = 0$. Hence by [12, Remark 4.2], R is a QF ring. □

Now, we study the CPH domains.

Theorem 3.9. *Let R be a* CPH *domain. Then R is a Noetherian ring of Krull dimension* $\dim(R) \leq 1$.

Proof. Let $I \neq 0$ be any finitely generated ideal of R. For any $0 \neq a \in I$. Set $T = R/aR$ and let M be a T-module with $k := cpd_T(M)$. Let $P = F/aF$ is a free T-module, where F is a free R-module. Then by Rees Theorem, $\mathrm{Ext}^k_T(M,P) \cong \mathrm{Ext}^{k+1}_R(M,F) = 0$ for all $k \geq 1$. Therefore, $\mathrm{Ext}^k_T(M,P) = 0$ for all projective T-modules P and for $k \geq 1$. Now let N be a flat T-module. By Theorem 3.3, $\mathrm{FPD}(R) \leq 1$. Let M be a T-module with $k := \mathrm{pd}_T(M) < \infty$. Then, by [19, Theorem 9.32], $\mathrm{pd}_R(M) = k+1 \leq 1$. Thus $k = 0$, whence $\mathrm{FPD}(T) = 0$. By Jensen Lemma [17], $s := \mathrm{pd}_T(N) < \infty$, whence $s = 0$. Thus N is a projective T-module. Particularly, $\mathrm{Ext}^k_T(M,N) = 0$ for all $k \geq 0$. Therefore, $cpd_T(M) = 0$, whence $cpD(T) = 0$. By [12, Remark 4.2], T is a QF ring. Then T is an IF ring. By [4, Theorem 2], T is coherent, and so $I/(u)$ is finitely presented over $R/(u)$. Then there exist a finitely generated free R-module F and a finitely generated $R/(u)$-module A such that $0 \to A \to F/IF \to I/(u) \to 0$ is an $R/(u)$-exact sequence. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & IF & = & IF & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & N & \to & F & \to & I/(u) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & F/IF & \to & I/(u) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Notice that IF is a finitely generated R-module, so is N. Hence $I/(u)$ is finitely presented over R. Since $0 \to (u) \to I \to I/(u) \to 0$ is exact, I is finitely presented. Consequently, R is coherent. Let P be a nonzero prime ideal of R. Pick $0 \neq a \in P$. Set $m = cpD(T = R/aR)$. By the proof of Theorem 3.8, we get $cpD(T) = 0$. Hence by [12, Remark 4.2], T is a QF ring. Since a QF ring is Artinian, $P/(a)$ is finitely generated. Consequently, P is finitely generated, and hence R is Noetherian. Now $\dim(R) = \mathrm{FPD}(R) \leq 1$ is obtained by [18] and Theorem 3.3. □

Example 3.10. CPH rings are not necessarily Noetherian. For example, let R be an umbrella ring with $\mathrm{gl.dim}(R) \leq 2$ and let P be the maximum non-finitely generated prime ideal of R. Let $a \in P$ but $a \neq 0$. Then $R/(a)$ is a coherent CPH ring but not Noetherian.

Example 3.11. A coherent domain is not necessarily CPH. Let $(R, \mathfrak{m})$ be a regular local ring of Krull dimension 2. Then R is a coherent domain, but not a CPH ring by Theorem 3.3 and [12, Corollary 4.4].

Let us say that an R-module M is torsion-free if, $ax = 0$ for $x \in M$, and for a non-zero-divisor $a \in C(R)$, where $C(R)$ is the center of R, we have $x = 0$. Note that projective modules are torsion-free. We pose the following question: whether copure projective modules whether are also torsion-free.

Theorem 3.12. *Let R be a commutative coherent* CPH *ring. Then every copure projective R-module M is torsion-free. Moreover, if R is a* CPH *domain, then every finitely generated torsion-free R-module is finitely presented copure projective.*

Proof. Let M be a copure projective R-module. For any $a \in R$ which is neither a non-zero-divisor nor a unit, $\mathrm{fd}_R R/aR \leq 1$ and the sequence $0 \to aR \to R \to R/aR \to 0$ is exact. Let I be an ideal of R. By hypothesis, R is a CPH ring, and so $cpD(R) \leq 1$ by Theorem 3.1. Then $cpd_R(R/I) \leq 1$, and hence $\mathrm{Ext}^2_R(R/I, R) = 0$ and $\mathrm{Ext}^2_R(R/I, aR) = 0$. Thus $\mathrm{id}_R R \leq 1$ and $\mathrm{id}_R aR \leq 1$. Now, let X be an R-module. Then we can obtain $\mathrm{id}_R R/aR \leq 1$ from the sequence $0 = \mathrm{Ext}^2_R(X, R) \to \mathrm{Ext}^2_R(X, R/aR) \to \mathrm{Ext}^3_R(X, aR) = 0$. So there is an exact sequence $0 \to R/aR \to E \to C \to 0$ with E, C injective. By [5, Theorem 1], E^+, C^+ are flat. For any ideal I of R, $\mathrm{Tor}^R_2(R/I, E)^+ \cong \mathrm{Ext}^2_R(R/I, E^+) = 0$ since $cpd_R(R/I) \leq 1$. Hence $\mathrm{fd}_R E \leq 1$ by $\mathrm{Tor}^R_2(R/I, E) = 0$. By the same way, $\mathrm{fd}_R C \leq 1$. Then $0 \to \mathrm{Tor}^R_1(R/aR, M) \xrightarrow{f} \mathrm{Tor}^R_1(E, M) \to \mathrm{cok}(f) \to 0$ is exact. Hence $0 \to \mathrm{cok}(f)^+ \to \mathrm{Tor}^R_1(E, M)^+ \to \mathrm{Tor}^R_1(R/aR, M)^+ \to 0$ is also exact. By hypothesis, M is copure projective and so $\mathrm{Tor}^R_1(E, M)^+ \cong \mathrm{Ext}^1_R(M, E^+) = 0$. Hence $M^a = \{ m \in M \mid am = 0 \} \cong \mathrm{Tor}^R_1(R/aR, M) = 0$. Thus M is torsion-free, as desired.

Now, assume R is a CPH domain. Let M be a finitely generated torsionfree module. Then M can be imbedded into a finitely generated free module. Hence M is finitely presented copure projective. □

Example 3.13. A copure projective R-module is not necessarily torsion-free. In fact, let L be a field and set $R = L[x, y]$. Set $M = R/(x, y)$. Then for any flat R-module N, we have $\mathrm{Ext}^1_R(M, N) = 0$, but $\mathrm{Ext}^2_R(M, R) \cong \mathrm{Hom}_R(M, M) \neq 0$. Hence M is copure projective but is not torsion-free.

An R-module M is said to be reflexive if $M \cong M^{**}$, where $M^* = \mathrm{Hom}_R(M, R)$. Let R be a domain and K the quotient field of R. For a submodule A of K, we denote $A^{-1} = \{ x \in K \mid xA \subseteq R \}$ and $A_v = (A^{-1})^{-1}$. An ideal I of R is said to be a v-ideal if $I = I_v$. For an R-module N, we set $\mathrm{ann}(N) = \{ r \in R \mid rN = 0 \}$. Now, for a Noetherian domain R, we study when R is a CPH ring.

Theorem 3.14. *The following statements are equivalent for a Noetherian domain R*:

(1) *R is a* CPH *domain.*
(2) *Every ideal I of R is a* v*-ideal.*
(3) *Every prime ideal of R is copure projective.*
(4) *Every finitely generated copure projective module is reflexive.*

(5) *For every* $0 \neq u \in R$ *that is not a unit,* $R/(u)$ *is a* QF *ring.*

Proof. (1)⇒(4). Let M be a finitely generated copure projective R-module. There is an exact sequence $0 \to A \to P \to M \to 0$ where P is finitely generated projective. Then $0 \to M^* \to P^* \to A^* \to 0$ is exact and P^* is finitely generated projective. Hence A^* is finitely generated torsion-free. Consider the exact sequence $0 \to A^* \to F \to F/A^* \to 0$ with F being a finitely generated free R-module. Then we get $\mathrm{Ext}^1_R(A^*, R) \cong \mathrm{Ext}^2_R(F/A^*, R) = 0$ by Theorem 3.1 and [12, Theorem 4.11]. Hence $0 \to A^{**} \to P^{**} \to M^{**} \to 0$ is exact. Notice that P is a reflexive submodule of a finitely generated torsion-free R-module. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho} & & \| & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & A^{**} & \longrightarrow & P & \longrightarrow & P/A^{**} & \longrightarrow & 0 \end{array}$$

Then $0 \to \ker f \cong \mathrm{cok}\rho \to M \xrightarrow{f} P/A^{**} \to 0$ is exact. Because $\mathrm{rank}(A) = \mathrm{rank}(A^{**})$, we have $\mathrm{rank}(M) = \mathrm{rank}(P/A^{**})$. By Theorem 3.12, M is finitely generated torsion-free. Hence $\ker f = 0$ since $\mathrm{rank}(\ker f) = 0$ and $\ker f$ is torsion-free. That is $A \cong A^{**}$. We infer that M is reflexive by the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\ 0 & \longrightarrow & A^{**} & \longrightarrow & P^{**} & \longrightarrow & M^{**} & \longrightarrow & 0 \end{array}$$

(4)⇒(2) and (1)⇒(3) are both obvious.

(2)⇒(5). Set $T = R/uR$. It is clear that T is Noetherian. For any ideal $J = I/(u)$ of T, where $I \supseteq (u)$ is an ideal of R, by hypothesis, $I_v = I$. Then $\mathrm{ann}_T(\mathrm{ann}_T(J)) = I_v/(u) = J$. Hence T is a QF ring.

(5)⇒(1). Let $I \neq 0$ be an ideal of R. Set $M = R/I$. Pick $0 \neq u \in I$ and note $\overline{R} = R/uR$. Then $uM = 0$ and M is $\overline{R}$-module. By hypothesis, M is a copure projective $\overline{R}$-module. Let N be a flat R-module. Certainly, u is a non-zero-divisor of N. By Rees Theorem $\mathrm{Ext}^2_R(M, N) \cong \mathrm{Ext}^1_{\overline{R}}(M, N/uN) = 0$. Thus $cpd_R(M) \leq 1$. By [12, Corollary 4.12], $cpD(R) \leq 1$. Hence R is a CPH ring by Theorem 3.1.

(3)⇒(1). Let F be a flat R-module. Consider the exact sequence $0 \to F \to E \to E/F \to 0$ in which E is injective. Then for any prime ideal $\mathfrak{p}$ of R, we get $\mathrm{Ext}^1_R(R/\mathfrak{p}, E/F) \cong \mathrm{Ext}^2_R(R/\mathfrak{p}, F) \cong \mathrm{Ext}^1_R(\mathfrak{p}, F) = 0$ by (3). Hence E/F is injective. Hence $\mathrm{id}_R F \leq 1$. By Theorem 3.1 and [12, Theorem 4.11], R is a CPH domain. □

An R-module M is said to be Gorenstein projective (G-projective for short) if there is an exact sequence of projective modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module. Recall that a ring R is called Gorenstein hereditary if all submodules of a projective R-module are Gorenstein projective. Also, a Gorenstein hereditary domain is called a Gorenstein Dedekind domain. Now, we will prove that Gorenstein Dedekind domains are exactly CPH domains.

Theorem 3.15. *Let R be a domain. Then the following statements are equivalent*:

(1) *R is a Gorenstein Dedekind domain.*

(2) *R is a* CPH *domain.*

(3) *R is a Noetherian ring and for any maximal ideal $\mathfrak{m}$ of R, $R_{\mathfrak{m}}$ is a* CPH *ring.*

Proof. (1)⇒(2). For every $0 \neq u \in R$ that is not a unit, by [15, Corollary 2.7], $R/(u)$ is a QF ring. Hence R is a CPH domain by Theorem 3.14.

(2)⇒(1). By Theorem 3.9 and Theorem 3.1, R is a Noetherian ring with $cpD(R) \leq 1$. Hence $\mathrm{id}_R({}_RR) \leq 1$ and R is a Gorenstein Dedekind domain.

(2)⇔(3). By Theorem 2.3. □

It was shown in [15, Corollary 1.3] that a domain R is Gorenstein Dedekind domain if and only if every ideal of R is G-projective. We are in the position of characterizing Gorenstein Dedekind domains in terms of copure projective (n-copure projective, strongly copure projective) modules.

Corollary 3.16. *The following statements are equivalent for a domain R*:

(1) *R is a Gorenstein Dedekind domain.*

(2) *Every ideal I of R is n-copure projective, where $0 \leq n < \infty$.*

(3) *Every ideal I of R is strongly projective.*

(4) *Every submodule of free* (*or projective, or m-copure projective, or strongly projective*) *modules is n-copure projective* (*or strongly projective*), *where $0 \leq n, m < \infty$.*

(5) $\mathrm{id}_R F \leq 1$ *for any R-modules F with $\mathrm{fd}_R F < \infty$.*

Proof. By Theorem 3.15, Theorem 3.1 and [12, Theorem 4.11]. □

We say that a module M has Gorenstein projective dimension at most a positive integer n and we write $\mathrm{Gpd}_R M \leq n$, if there is an exact sequence of modules $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ where each P_i is Gorenstein projective. The Gorenstein global dimension G-gl.dim(R) of R is defined as $G\text{-gl.dim}(R) = \sup\{\mathrm{Gpd}_R M \mid M \text{ is any } R\text{-module}\}$.

Let R be a ring. By [3, Proposition 1.3] and [12, Remark 4.2], R is a QF ring if and only if G-gl.dim$(R) = 0$; if and only if $cpD(R) = 0$. By Theorem 3.15 and Theorem 3.1, a Noetherian domain R is Gorenstein Dedekind if and only if G-gl.dim$(R) \leq 1$; if and only if $cpD(R) \leq 1$. Now, for a ring R with G-gl.dim$(R) < \infty$, we conclude this article with the following proposition.

Proposition 3.17. *Let R be a ring with G-gl.dim$(R) < \infty$. G-gl.dim$(R) \leq cpD(R)$. Moreover, if R is a Noetherian ring, then G-gl.dim$(R) = cpD(R)$.*

Proof. Set $cpD(R) = n$. Let P be a projective module. By [12, Theorem 4.11], $\mathrm{id}_R P \leq n$. Then by [3, Theorem 1.2], G-gl.dim$(R) \leq n$. Hence G-gl.dim$(R) \leq cpD(R)$.

Now, let R be a Noetherian ring. Set G-gl.dim$(R) = n$. Let F be any flat R-module and let I be an ideal of R. By [20, Theorem 5.40], $F = \varinjlim F_i$, where each F_i is a finitely generated free R-module. By [3, Theorem 1.2], $\mathrm{Ext}_R^{n+1}(R/I, F_i) = 0$. Then by [22, Theorem 3.2], we have $\mathrm{Ext}_R^{n+1}(R/I, F) = \mathrm{Ext}_R^{n+1}(R/I, \varinjlim F_i) \cong \varinjlim \mathrm{Ext}_R^{n+1}(R/I, F_i) = 0$. Hence $\mathrm{id}_R F \leq n$. By [12, Theorem 4.11], $cpD(R) \leq n$. We get G-gl.dim$(R) = cpD(R)$. □

References

[1] J. Abuhlail and M. Jarrar, *Tilting modules over almost perfect domains*, J. Pure Appl. Algebra **215** (2011), no. 8, 2024–2033.

[2] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[3] D. Bennis, *A short survey on Gorenstein global dimension*, Actes des rencontres du C.I.R.M. **2** (2010), no. 2, 115–117.

[4] R. R. Colby, *Rings which have flat injective modules*, J. Algebra **35** (1975), 239–252.

[5] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 175–177.

[6] N. Q. Ding and J. L. Chen, *The flat dimensions of injective modules*, Manuscripta Mathematica **78** (1993), no. 2, 165–177.

[7] ———, *On copure flat modules and flat resolvents*, Comm. Algebra **24** (1996), no. 3, 1071–1081.

[8] E. E. Enochs and O. M. G. Jenda, *Copure injective resolutions, flat resolvents and dimensions*, Comment. Math. Univ. Carolin. **34** (1993), no. 2, 203–211.

[9] ———, *Relative Homological Algebra*, Walter de Gruyter, Berlin-New York 2000.

[10] D. J. Fieldhouse, *Character modules, dimension and purity*, Glasgow Math. J. **13** (1972), 144–146.

[11] X. H. Fu and N. Q. Ding, *On strongly copure flat modules and copure flat dimensions*, Comm. Algebra **38** (2010), no. 12, 4531–4544.

[12] X. H. Fu and H. Y. Zhu, and N. Q. Ding, *On copure projective modules and copure projective dimensions*, Comm. Algebra **40** (2012), no. 1, 343–359.

[13] Z. H. Gao, *n-copure projective modules*, Mathematical Notes **97** (2015), no. 1, 50–56.

[14] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, Berlin-New York: Walter de Gruyter, 2006.

[15] K. Hu and F. G. Wang, *Some results on Gorenstein Dedekind domains and their factor rings*, Comm. Algebra **41** (2013), no. 1, 284–293.

[16] S. Jain, *Flat and FP-injective*, Proc. AMS. **41** (1979), 437–442.

[17] C. U. Jensen, *On the vanishing of* $\varprojlim^{(i)}$, J. Algebra **15** (1970), 155–166.

[18] M. Raynaud and L. Gruson, *Gritéres de plattitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.

[19] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York 1979.

[20] ———, *An Introduction to Homological Algebra*, 2nd ed. New York: Springer Science+Business Media, LLC, 2009.

[21] L. Salce, *Almost perfect domains and their modules*, In Commutative algebra: Noetherian and non-Noetherian perspectives, pp. 363–386, New York, Springer, 2011.

[22] B. Stënstrom, *Coherent rings and FP-injective modules*, J. London Math. Soc. **2** (1970), 323–329.

[23] T. Xiong, F. G. Wang, and K. Hu, *Copure Projective Modules and CPH-rings*, J. Sichuan Normal University (Natural Science) **36** (2013), no. 2, 198–201.

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