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SELF-HOMOTOPY EQUIVALENCES RELATED TO COHOMOTOPY GROUPS

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Abstract. Given a topological space X and a non-negative integer k, we study the self-homotopy equivalences of X that do not change maps from X to n-sphere S^n homotopically by the composition for all $n \geq k$. We denote by $\mathcal{E}_k^\sharp(X)$ the set of all homotopy classes of such self-homotopy equivalences. This set is a dual concept of $\mathcal{E}_\sharp^k(X)$, which has been studied by several authors. We prove that if X is a finite CW complex, there are at most a finite number of distinguishing homotopy classes $\mathcal{E}_k^\sharp(X)$, whereas $\mathcal{E}_\sharp^k(X)$ may not be finite. Moreover, we obtain concrete computations of $\mathcal{E}_k^\sharp(X)$ to show that the cardinal of $\mathcal{E}_k^\sharp(X)$ is finite when X is either a Moore space or co-Moore space by using the self-closeness numbers.

1. Introduction

Throughout this paper, all topological spaces are based and have the based homotopy type of a CW-complex, and all maps and homotopies preserve base points. For the spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. No distinction is made between the notation of a map $X \to Y$ and that of its homotopy class in $[X, Y]$. Let S^n be the n-sphere. Then, $[S^n, Y]$ is known as the n-th homotopy group of space Y, denoted by $\pi_n(Y)$ and $[X, S^n]$ is referred to as the n-th cohomotopy group of X, denoted by $\pi^n(X)$.

Given X, we denote by $\mathcal{E}(X)$ the set of all homotopy classes of self-homotopy equivalences of X. Then, $\mathcal{E}(X)$ is a subset of $[X, X]$ and has a group structure given by the composition of homotopy classes. $\mathcal{E}(X)$ has been studied extensively by various authors, including Arkowitz [2], Maruyama [3], Lee [7], Rutter [8], Sawashita [9], and Sieradski [10]. Moreover, several subgroups of $\mathcal{E}(X)$ have also been studied, notably the subgroup $\mathcal{E}_\sharp^k(X)$, which consists of the elements of $\mathcal{E}(X)$ that induce the identity homomorphism on homotopy

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groups $\pi_i(X)$ for $i = 0, 1, 2, \ldots, k$. In [3], Arkowitz and Maruyama introduced and determined these subgroups for Moore spaces and co-Moore spaces using the homological method. In [4], the second and third authors used homotopy techniques to calculate these subgroups for the wedge products of Moore spaces.

Given a topological space X and a non-negative integer k, consider the self-map $f : X \to X$ such that $g \circ f$ is homotopic to g for each $g : X \to S^n$ and for each $n \geq k$. We denote by $[X, X]^\sharp_k$ the set of all homotopy classes of such self-maps of X, that is,

$$[X, X]^\sharp_k = \{f \in [X, X] \,|\, g \circ f \sim g \text{ for each } g : X \to S^n, \text{ for all } n \geq k\}.$$

This set has a monoid structure by composition.

We define

$$\mathcal{E}^\sharp_k(X) = \mathcal{E}(X) \cap [X, X]^\sharp_k.$$

Then, it is easy to prove that $\mathcal{E}^\sharp_k(X)$ is a subgroup of $\mathcal{E}(X)$ and has a lower bound, whereas $\mathcal{E}^k_\sharp(X)$ has an upper bound.

In this paper, we compute these subgroups of Moore spaces and co-Moore spaces, by first showing that if X is a finite CW complex, then there are at most a finite number of distinguishing subgroups $\mathcal{E}^\sharp_k(X)$.

When G is an abelian group, we let $M(G, n)$ denote the Moore space; i.e., the space in which G is a single non-vanishing homology group at the n-level. We note that if $n \geq 3$, then $M(G, n)$ is characterized by

$$\widetilde{H}_i(M(G, n)) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Furthermore, we let $C(G, n)$ denote the co-Moore space of type (G, n) defined by

$$\widetilde{H}^i(C(G, n)) \cong \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Next, we compute $\mathcal{E}^\sharp_k(X)$ for $X = M(\mathbb{Z}_p, n)$ or $X = C(\mathbb{Z}_p, n)$ to obtain the following tables:

	p odd	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}^\sharp_{n+1}(M(\mathbb{Z}_p, n))$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathcal{E}^\sharp_{n}(M(\mathbb{Z}_p, n))$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathcal{E}^\sharp_{n-1}(M(\mathbb{Z}_p, n))$	1	1	1

	p odd	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}^\sharp_{n}(C(\mathbb{Z}_p, n))$	1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathcal{E}^\sharp_{n-1}(C(\mathbb{Z}_p, n))$	1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$\mathcal{E}^\sharp_{n-2}(C(\mathbb{Z}_p, n))$	1	1	1

Henceforth, when a group G is generated by a set $\{a_1, \ldots, a_n\}$, then we denote the group by $G\{a_1, \ldots, a_2\}$ or $G = \langle a_1, \ldots, a_n \rangle$. Moreover, when $f : X \to Y$ is a map, $f_{\sharp k} : \pi_k(X) \to \pi_k(Y)$ and $f^{\sharp k} : \pi^k(Y) \to \pi^k(X)$ denote the induced homomorphisms in k-th homotopy group and k-th cohomotopy group, respectively.

2. Preliminaries

In this section, we review and summarize selected definitions and results provided in [1, 3, 5, 12], knowledge of which would be useful when reading this paper.

First, we summarize the concepts and results introduced in [5]. For any non-negative integer n, $\mathcal{A}_\sharp^n(X)$ consists of the homotopy classes of the self-map of X that induce an automorphism from $\pi_i(X)$ to $\pi_i(X)$ for $i = 0, 1, \dots, n$. $\mathcal{A}_\sharp^k(X)$ is a submonoid of $[X, X]$ and always contains $\mathcal{E}(X)$. If $n = \infty$, we briefly denote $\mathcal{A}_\sharp^\infty(X)$ as $\mathcal{A}_\sharp(X)$. If $k < n$, then $\mathcal{A}_\sharp^n(X) \subseteq \mathcal{A}_\sharp^k(X)$; thus, we have the following chain by inclusion:

$$\mathcal{E}(X) \subseteq \mathcal{A}_\sharp(X) \subseteq \cdots \subseteq \mathcal{A}_\sharp^1(X) \subseteq \mathcal{A}_\sharp^0(X) = [X, X].$$

Definition 2.1. Let X be a CW complex. The self closeness number of X is the minimum number n such that $\mathcal{A}_\sharp^n(X) = \mathcal{E}(X)$ and is denoted here by $N\mathcal{E}(X)$. That is,

$$N\mathcal{E}(X) = \min\{n \mid \mathcal{A}_\sharp^n(X) = \mathcal{E}(X) \ \text{ for } n \geq 0\}.$$

By [5, Theorem 1], the self-closeness number is a homotopy invariant. Moreover, if X is an n-connected space with dimension m and $\mathcal{E}(X) \neq [X, X]$, then we have $n < N\mathcal{E}(X) \leq m$ by [5, Lemma 4 and Theorem 2].

Proposition 2.1 ([3]). *If X is $(k-1)$-connected, Y is $(l-1)$-connected, and further, if $k, l \geq 2$, and $\dim P \leq k + l - 1$, then the projections $X \vee Y \to X$ and $X \vee Y \to Y$ induce a bijection.*

$$[P, X \vee Y] \to [P, X] \oplus [P, Y].$$

By [12], the generators of some homotopy groups of spheres can be summarized as follows.

	$i < 0$	$i = 0$	$i = 1$	$i = 2$	$i = 3$	$i = 4, 5$
$[S^{n+i}, S^n]$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0
Generator		ι_n	η_n	η_n^2	ν_n	0

Here, we note that for the Moore space $M(\mathbb{Z}_p, n) = S^n \cup_p e^{n+1}$, there exists a mapping cone sequence

$$S^n \xrightarrow{p} S^n \xrightarrow{i} S^n \cup_p e^{n+1} \xrightarrow{\pi} S^{n+1} \xrightarrow{p} S^{n+1},$$

where p is a map of degree p, i is an inclusion and π is a quotient map. In [1], Araki and Toda computed the homotopy groups, and cohomotopy groups of $M(\mathbb{Z}_p, n)$, and set of homotopy classes of self-maps on $M(\mathbb{Z}_p, n)$. The results can be summarized as follows.

1. Homotopy group $\pi_k(M(\mathbb{Z}_p, n))$ for $k = n,\ n+1$:

$$\pi_n(M(\mathbb{Z}_p, n)) \cong \mathbb{Z}_p\{i_\sharp(\iota_n)\}$$

and

$$\pi_{n+1}(M(\mathbb{Z}_p,n)) \cong \begin{cases} 0 & \text{if } p = \text{odd}, \\ \mathbb{Z}_2\{i_\sharp(\eta_n)\} & \text{if } p = \text{even}. \end{cases}$$

2. Cohomotopy groups $\pi^{n+i-3}(M(\mathbb{Z}_p,n))$:

TABLE 1

	$i \leq 0$	$i=1$	$i=2$	$i=3$	$i=4$
$p \equiv 1 \pmod 2$	0	$\mathbb{Z}_p$	0	0	$\mathbb{Z}_{(p,24)}$
$p \equiv 0 \pmod 4$	0	$\mathbb{Z}_p$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{(p,24)}$
$p \equiv 2 \pmod 4$	0	$\mathbb{Z}_p$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{(p,24)}$
Generators	-	$\iota_n \circ q$	$\eta_n \circ q$	$\bar{\eta}_n, \eta_n^2 \circ q$	$\eta_n \circ \bar{\eta}, \nu_n \circ q$

3. The set of homotopy classes $[M(\mathbb{Z}_p,n), M(\mathbb{Z}_p,n)]$:

TABLE 2

	$p \equiv 1 \pmod 2$	$p \equiv 2 \pmod 4$	$p \equiv 0 \pmod 4$
$[M(\mathbb{Z}_p,n), M(\mathbb{Z}_p,n)]$	$\mathbb{Z}_p$	$\mathbb{Z}_{2p}$	$\mathbb{Z}_p \oplus \mathbb{Z}_2$
Generators	1_X	1_X	$1_X, i \circ \eta_n \circ q$

3. Self-homotopy equivalences that induce the identity on co-homotopy groups

In this section, we study the properties of the sets $\mathcal{E}_k^\sharp(X)$. We recall

$$\mathcal{E}_k^\sharp(X) = \mathcal{E}(X) \cap [X,X]_k^\sharp,$$

where

$$[X,X]_k^\sharp = \{f \in [X,X] \,|\, g \circ f \sim g \text{ for each } g : X \to S^n, \text{ for all } n \geq k\}.$$

Equivalently,

$$\mathcal{E}_k^\sharp(X) = \{f \in \mathcal{E}(X) \,|\, f^{\sharp n} = id_{\pi^n(X)} \text{ on } \pi^n(X) \text{ for } n \geq k\}.$$

This definition indicates that $\mathcal{E}_m^\sharp(X) \subseteq \mathcal{E}_n^\sharp(X)$ for $n \geq m$. Hence, we obtain a chain of subsets as follows:

$$\mathcal{E}(X) \supseteq \cdots \supseteq \mathcal{E}_n^\sharp(X) \supseteq \mathcal{E}_{n-1}^\sharp(X) \supseteq \cdots \supseteq \mathcal{E}_1^\sharp(X). \tag{3.1}$$

Proposition 3.1. *$\mathcal{E}_k^\sharp(X)$ is a subgroup of $\mathcal{E}(X)$.*

Proof. Let $f,\ g \in \mathcal{E}_k^\sharp(X)$ and $\bar{g}$ be the homotopy inverse map of g. Because $g \circ \bar{g} = id_X$ and $\bar{g} \circ g = id_X$,

$$id_{\pi^k(X)} = (g \circ \bar{g})^{\sharp k} = \bar{g}^{\sharp k} \circ g^{\sharp k} = \bar{g}^{\sharp k}.$$

Thus,

$$(f \circ \bar{g})^{\sharp k} = \bar{g}^{\sharp k} \circ f^{\sharp k} = \bar{g}^{\sharp k} = id_{\pi^k(X)}.$$

Hence, $f \circ \bar{g} \in \mathcal{E}_k^\sharp(X)$. Consequently, $\mathcal{E}_k^\sharp(X)$ is a subgroup of $\mathcal{E}(X)$. □

Lemma 3.2. *If X is a finite CW complex, then there exists a positive integer N such that $[X, S^N] = 0$.*

Proof. Let $\dim(X) = m < \infty$. We choose N such that $N > m$. If $f \in [X, S^N]$, then $f(X) \subseteq (S^N)_m$ by the cellular approximation theorem, where $(S^N)_m$ is m-skeleton of S^N. Because $S^N = e^0 \cup e^N$, $(S^N)_m = e^0$. Thus $f = 0$ and consequently, $[X, S^N] = 0$. □

Theorem 3.3. *If X is a finite CW complex, then there are at most a finite number of distinguishing subgroups $\mathcal{E}_k^\sharp(X)$.*

Proof. Let $m < \infty$ be the dimension of X. By definition of $\mathcal{E}_k^\sharp(X)$, we see that

$$\mathcal{E}(X) \supseteq \cdots \supseteq \mathcal{E}_n^\sharp(X) \supseteq \mathcal{E}_{n-1}^\sharp(X) \supseteq \cdots \supseteq \mathcal{E}_1^\sharp(X).$$

By Lemma 3.2, $[X, S^{m+i}] = 0$ for $i = 1, 2, \ldots$. Hence $\mathcal{E}(X) = \mathcal{E}_{m+1}^\sharp(X)$. Consequently, we have the following finite chain of subsets:

$$\mathcal{E}(X) = \mathcal{E}_{m+1}^\sharp(X) \supseteq \mathcal{E}_m^\sharp(X) \supseteq \cdots \supseteq \mathcal{E}_n^\sharp(X) \supseteq \cdots \supseteq \mathcal{E}_1^\sharp(X).$$ □

Next, we consider abelian groups G_1 and G_2 and Moore spaces $M_1 = M(G_1, n_1)$ and $M_2 = M(G_2, n_2)$. Let $X = M_1 \vee M_2$. We denote by $i_j : M_j \to X$ the inclusion and by $q_j : X \to M_j$ the projection, where $j = 1, 2$. If $f : X \to X$, then we define $f_{jk} : M_k \to M_j$ by $f_{jk} = q_j \circ f \circ i_k$ for $j, k = 1, 2$.

By Proposition 2.1, let $X = M_1 \vee M_2$ then

$$[X, X] = [M_1, M_1] \oplus [M_1, M_2] \oplus [M_2, M_1] \oplus [M_2, M_2].$$

By [3, Proposition 2.6], the function θ which assigns to each $f \in [X, X]$, the 2×2 matrix

$$\theta(f) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{jk} \in [M_k, M_j]$, is a bijection. In addition,
(1) $\theta(f + g) = \theta(f) + \theta(g)$, so θ is an isomorphism $[X, X] \to \bigoplus_{j,k=1,2}[M_k, M_j]$.
(2) $\theta(fg) = \theta(f)\theta(g)$, where fg denotes composition in $[X, X]$ and $\theta(f)\theta(g)$ denotes matrix multiplication.

Further, [3] also introduced the forms of the homomorphism induced by f on homotopy, homology, and cohomology groups, respectively.

Now, we determine the form of the homomorphism induced by f on cohomotopy groups.

Proposition 3.4. *For any $f \in [X, X]$, we have*

$$f^{\sharp k}(\gamma_1, \gamma_2) = (f_{11}^{\sharp k}(\gamma_1) + f_{21}^{\sharp k}(\gamma_2), f_{12}^{\sharp k}(\gamma_1) + f_{22}^{\sharp k}(\gamma_2)),$$

where $\gamma_1 \in \pi^k(M_1)$ and $\gamma_2 \in \pi^k(M_2)$.

Proof. For any $f \in [X, X]$, we identify that

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Thus, f induces the homomorphism $f^{\sharp k}$ on cohomotopy groups as follows:

$$f^{\sharp k} = \begin{pmatrix} f_{11}^{\sharp k} & f_{12}^{\sharp k} \\ f_{21}^{\sharp k} & f_{22}^{\sharp k} \end{pmatrix}.$$

Because $\pi^k(X) = \pi^k(M_1) \oplus \pi^k(M_2)$, we are able to identify $\gamma \in \pi^k(X)$ as $\gamma = (\gamma_1, \gamma_2)$, for some $\gamma_i \in \pi^k(M_i)$. Then

$$\begin{aligned} f^{\sharp k}(\gamma) = \gamma f &= (\gamma_1, \gamma_2) \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \\ &= (\gamma_1 f_{11} + \gamma_2 f_{21}, \gamma_1 f_{12} + \gamma_2 f_{22}) \\ &= (f_{11}^{\sharp k}(\gamma_1) + f_{21}^{\sharp k}(\gamma_2), f_{12}^{\sharp k}(\gamma_1) + f_{22}^{\sharp k}(\gamma_2)). \end{aligned}$$

□

Proposition 3.5. *If $f \in \mathcal{E}_k^\sharp(X)$, then*

$$f^{\sharp k} = \begin{pmatrix} 1_{\pi^k(M_1)} & 0 \\ 0 & 1_{\pi^k(M_2)} \end{pmatrix}. \tag{3.2}$$

Proof. Because f induces the identity on $\pi^k(X)$, $f^{\sharp k} = id_{\pi^k(X)} = 1_X^{\sharp k}$, where $1_X \in [X, X]$ is the identity map. As $1_X = \begin{pmatrix} 1_{M_1} & 0 \\ 0 & 1_{M_2} \end{pmatrix}$, we have

$$f^{\sharp k} = 1_X^{\sharp k} = \begin{pmatrix} 1_{\pi^k(M_1)} & 0 \\ 0 & 1_{\pi^k(M_2)} \end{pmatrix}.$$

□

Here, we review the group of self homotopy equivalences of Moore space. Let p be a positive integer. In [11], Sieradski proved the following result by using the universal coefficient theorem for homotopy:

$$\mathcal{E}(M(\mathbb{Z}_p, n)) \cong \begin{cases} \mathbb{Z}_p \rtimes \mathbb{Z}_p^* & n = 2 \\ \mathbb{Z}_{(2,p)} \rtimes \mathbb{Z}_p^* & n \geq 3, \end{cases}$$

where $\mathbb{Z}_p^*$ is the automorphism group of $\mathbb{Z}_p$.

Our computations require us to determine the definite forms of elements in $\mathcal{E}(M(\mathbb{Z}_p, n))$ and we use the concept of the self-closeness number introduced in [5] for this purpose. Because the Moore space of type (G, n) has the self closeness number n by [5, Corollary 3], $\mathcal{A}_\sharp^n(M(\mathbb{Z}_p, n)) = \mathcal{E}(M(\mathbb{Z}_p, n))$ by [5, Definition 2.1 or Theorem 4], where $\mathcal{A}_\sharp^n(M(\mathbb{Z}_p, n))$ is the set of homotopy classes of self-maps of $M(\mathbb{Z}_p, n)$ that induce an automorphism of $\pi_i(X)$ for $i = 0, 1, \ldots, n$. To determine the definite forms of elements in $\mathcal{E}(M(\mathbb{Z}_p, n))$, we compute $\mathcal{A}_\sharp^n(M(\mathbb{Z}_p, n))$ rather than $\mathcal{E}(M(\mathbb{Z}_p, n))$.

Consider the mapping cone sequence

$$S^n \xrightarrow{p} S^n \xrightarrow{i} S^n \cup_p e^{n+1} \xrightarrow{\pi} S^{n+1} \xrightarrow{p} S^{n+1},$$

where p is a map of degree p, i is the inclusion and π is the quotient map.

Theorem 3.6. *Let $X = M(\mathbb{Z}_p, n)$ be a Moore space. Then we have*

$$\mathcal{A}^n_\sharp(X) = \begin{cases} \{\, k\cdot 1_X \,|\, (k,p)=1\} & p\equiv 1 \pmod 2,\\ \{\, \ell\cdot i\circ\eta_n\circ\pi + k\cdot 1_X \,|\, (k,p)=1\} & p\equiv 0 \pmod 4,\\ \{\, k\cdot 1_X, (k+p)\cdot 1_X \,|\, (k,p)=1\} & p\equiv 2 \pmod 4.\end{cases}$$

Proof. We first note that $\pi_n(X)\cong \mathbb{Z}_p\{i_\sharp(\iota_n)\}$.

Suppose that p is odd. Then $[X,X]=\mathbb{Z}_p\{1_X\}$. Moreover, we have

$$1_{X\sharp}(i_\sharp(\iota_n)) = 1_X\circ i\circ\iota_n = i_\sharp(\iota_n).$$

Thus, $(k\cdot 1_X)_\sharp(i_\sharp(\iota_n)) = k\cdot(i_\sharp(\iota_n))$. It follows that

$$\mathcal{A}^n_\sharp(X) = \{k\cdot 1_X \mid (k,p)=1\} = \mathbb{Z}_p^*.$$

Suppose that $p\equiv 0 \pmod 4$. In this case,

$$[X,X] = \mathbb{Z}_2\oplus\mathbb{Z}_p\{i\circ\eta_n\circ\pi, 1_X\}.$$

Because $1_{X\sharp}(i_\sharp(\iota_n)) = i_\sharp(\iota_n)$ and $(i\circ\eta_n\circ\pi)_\sharp(i_\sharp(\iota_n)) = i\circ\eta_n\circ\pi\circ i\circ\iota_n = 0$, we have

$$(\ell\cdot(i\circ\eta_n\circ\pi)+k\cdot 1_X)_\sharp(i_\sharp(\iota_n)) = (k\cdot 1_X)_\sharp(i_\sharp(\iota_n)) = k\cdot(i_\sharp(\iota_n))$$

for $\ell\in\mathbb{Z}_2$ and $k\in\mathbb{Z}_p$. Therefore

$$\mathcal{A}^n_\sharp(X) = \{\ell\cdot(i\circ\eta_n\circ\pi)+k\cdot 1_X \mid (k,p)=1\}.$$

Suppose that $p\equiv 2 \pmod 4$. In this case, we have $[X,X]=\mathbb{Z}_{2p}\{1_X\}$. As $k\cdot 1_{X\sharp}(i_\sharp(\iota_n)) = k\cdot i_\sharp(\iota_n)$ for $0<k\le p$ and $(p+k)\cdot 1_{X\sharp}(i_\sharp(\iota_n)) = k\cdot i_\sharp(\iota_n)$ for $0<k\le p$, we have

$$\mathcal{A}^n_\sharp(X) = \{k\cdot 1_X \mid (k,p)=1, 1<k\le p\}\cup\{(p+k)\cdot 1_X \mid (k,p)=1, 0<k<p\}.$$

□

4. Computation of $\mathcal{E}^\sharp_k(M(\mathbb{Z}_p,n))$

In this section, we compute $\mathcal{E}^\sharp_k(M(\mathbb{Z}_p,n))$ and determine their generators for $k=n+1, n$, and $n-1$. Throughout this section, we let $X=M(\mathbb{Z}_p,n)$.

Theorem 4.1. *For $\mathcal{E}^\sharp_{n+1}(X)$, we have the following table*:

	p odd	$p\equiv 0 \pmod 4$	$p\equiv 2 \pmod 4$
$\mathcal{E}^\sharp_{n+1}(X)$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
generators	1_X	$\ell i\circ\eta_n\circ q\oplus 1_X$	$(\ell+1)1_X$

Proof. Because $\pi^k(X)=0$ for $k>n+1$ by Theorem 3.3, it is sufficient to consider the $(n+1)$-th cohomotopy group of X. From Table 1, $\pi^{n+1}(X) = \mathbb{Z}_p\{\iota_{n+1}\circ q\}$.

Case 1. Let p be odd.

By Theorem 3.6, for each $f \in \mathcal{E}(X)$, $f = k1_X$ for some k such that $0 \leq k \leq p-1$ and $(k,p) = 1$. Thus, we have

$$f^{\sharp n+1}(\iota_{n+1} \circ \pi) = \iota_{n+1} \circ \pi \circ (k1_X) = k(\iota_{n+1} \circ \pi \circ 1_X) = k(\iota_{n+1} \circ \pi).$$

Therefore, to ensure that $f^{\sharp n+1} = 1_{\pi_{n+1}}$ holds, k is require to be 1. Hence $\mathcal{E}^{\sharp}_{n+1}(X) \cong 1\{1_X\}$.

Case 2. Let $p \equiv 0 \pmod 2$.

By Theorem 3.6, for each $f \in \mathcal{E}(X)$, $f = \ell i \circ \eta_n \circ \pi \oplus k1_X$, for some $\ell = 0,\ 1$, where k is an integer such that $0 \leq k \leq p-1$ and $(k,p) = 1$. Thus, we have

$$\begin{aligned} f^{\sharp n+1}(\iota_{n+1} \circ \pi) &= \iota_{n+1} \circ \pi(\ell i \circ \eta_n \circ \pi \oplus k1_X) \\ &= \ell \iota_{n+1} \circ \pi \circ i \circ \eta_n \circ \pi \oplus k\iota_{n+1} \circ \pi \circ 1_X \\ &= k\iota_{n+1} \circ \pi \end{aligned}$$

because $q \circ i$ is homotopic to the constant map. Thus k is require to be 1 and $\ell = 0$ or 1 to ensure that $f^{\sharp n+1} = 1_{\pi_{n+1}}$ holds. Hence

$$\mathcal{E}^{\sharp}_{n+1}(X) \cong \mathbb{Z}_2\{\ell i \circ \eta_n \circ q \oplus 1_X |\ \ell = 0, 1\}.$$

Case 3. Let $p \equiv 2 \pmod 4$.

By Theorem 3.6, for each $f \in \mathcal{E}(X)$, $f = (k+\ell)1_X$ for some k and ℓ such that $0 \leq k \leq p-1$, $(k,p) = 1$ and $\ell = 0, p$. Thus, we have

$$f^{\sharp n+1}(\iota_{n+1} \circ \pi) = (k+\ell)(\iota_{n+1} \circ \pi \circ 1_X) = (k+\ell)\iota_{n+1} \circ \pi.$$

Thus k is require to be 1 and $\ell = 0$ or p to ensure that $f^{\sharp n+1} = 1_{\pi_{n+1}}$ holds. Hence,

$$\mathcal{E}^{\sharp}_{n+1}(X) \cong \mathbb{Z}_2\{(\ell+1)1_X \mid \ell = 0, p\}.$$ □

Theorem 4.2. *For $\mathcal{E}^{\sharp}_n(X)$, we have the following table*:

	p odd	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}^{\sharp}_n(X)$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
generators	1_X	$1_X \oplus \ell i \circ \eta_n \circ \pi$	$(\ell+1)1_X$

Proof. We first note that $\mathcal{E}^{\sharp}_n(X) \subseteq \mathcal{E}^{\sharp}_{n+1}(X)$. From Table 1,

$$\pi^n(X) = \begin{cases} 0 & p \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_n \circ \pi\} & p \equiv 0 \pmod 2. \end{cases}$$

Case 1. Let p be odd.

By Theorem 4.1, $\mathcal{E}^{\sharp}_n(X) \subseteq \mathcal{E}^{\sharp}_{n+1}(X) \cong 1$.

Case 2. Let $p \equiv 0 \pmod 4$.

By Theorem 4.1, for each $f \in \mathcal{E}(X)$, we have $f = \ell i \circ \eta_n \circ \pi \oplus 1_X$ for $\ell = 0, 1$. Thus, we have

$$\begin{aligned} f^{\sharp n}(\eta_n \circ \pi) &= \eta_n \circ \pi \circ (\ell i \circ \eta_n \circ \pi \oplus 1_X) \\ &= \ell \eta_n \circ \pi \circ i \circ \eta_n \circ \pi \oplus \eta_n \circ \pi \circ 1_X \\ &= \eta_n \circ \pi \end{aligned}$$

because $\pi\circ i$ is homotopic to the constant map. Thus for any $\ell = 0, 1$, $f^{\sharp n} = 1_{\pi_n}$ holds. Hence,

$$\mathcal{E}_n^\sharp(X) \cong \mathbb{Z}_2\{\,\ell i \circ \eta_n \circ q \oplus 1_X \,|\, \ell = 0, 1\}.$$

Case 3. Let $p \equiv 2 \pmod 4$.

By Theorem 4.1, for each $f \in \mathcal{E}(X)$, $f = (\ell+1)1_X$, for $\ell = 0, p$. Thus, we have

$$f^{\sharp n}(\eta_n \circ \pi) = (\ell+1)(\eta_n \circ \pi \circ 1_X) = (\ell+1)\eta_n \circ \pi.$$

Therefore, for any $\ell = 0, p$, $f^{\sharp n} = 1_{\pi_n}$ holds. Hence,

$$\mathcal{E}_n^\sharp(X) \cong \mathbb{Z}_2\{\,(1+\ell)1_X \,|\, \ell = 0, p\}.$$

□

Theorem 4.3. *If $n > 3$, then $\mathcal{E}_{n-1}^\sharp(M(\mathbb{Z}_p, n)) = 1\{1_{M(\mathbb{Z}_p,n)}\}$.*

Proof. First we note that $\mathcal{E}_{n-1}^\sharp(X) \subseteq \mathcal{E}_n^\sharp(X)$ and

$$\pi^{n-1}(X) = \begin{cases} 0 & p : \text{odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2\{\bar\eta_{n-1}, \eta_{n-1}^2 \circ q\} & p \equiv 0 \pmod 4 \\ \mathbb{Z}_4\{\eta_{n-1}^2 \circ q\} & p \equiv 2 \pmod 4 \end{cases}$$

by Table 1.

In case that p is odd, it is clear because $\mathcal{E}_{n-1}^\sharp(X) \subseteq \mathcal{E}_n^\sharp(X) \cong 1$ by Theorem 4.2.

Suppose $p \equiv 0 \pmod 4$. By Theorem 4.2, for each $f \in \mathcal{E}(X)$, $f = \ell i \circ \eta_n \circ q \oplus 1_X$ for $\ell = 0, 1$. Thus, we have

$$\begin{aligned} f^{\sharp n-1}(\bar\eta_{n-1} \oplus \eta_{n-1}^2 \circ q) &= (\bar\eta_{n-1} \oplus \eta_{n-1}^2 \circ q) \circ (\ell i \circ \eta_n \circ q \oplus 1_X) \\ &= \bar\eta_{n-1} \circ (\ell i \circ \eta_n \circ q \oplus 1_X) \\ &\quad \oplus \eta_{n-1}^2 \circ q \circ (\ell i \circ \eta_n \circ q \oplus 1_X) \\ &= (\ell \bar\eta_{n-1} \circ i \circ \eta_n \circ q \oplus \bar\eta_{n-1} \circ 1_X) \\ &\quad + (\ell \eta_{n-1}^2 \circ q \circ i \circ \eta_n \circ q \oplus \eta_{n-1}^2 \circ q \circ 1_X) \\ &= (\ell \eta_{n-1} \circ \eta_n \circ q \oplus \bar\eta_{n-1}) + (\eta_{n-1}^2 \circ q \oplus 0) \\ &= (\ell+1)\eta_{n-1}^2 \circ q \oplus \bar\eta_{n-1} \end{aligned}$$

because $q \circ i$ is homotopic to the constant map.

For $f^{\sharp n-1} = 1_{\pi_{n-1}}$ to be valid, $1 + \ell = 1$; thus $\ell = 0$. Hence, $\mathcal{E}_{n-1}^\sharp(X) \cong 1\{1_X\}$.

Suppose $p \equiv 2 \pmod 4$. By Theorem 4.2, for each $f \in \mathcal{E}(X)$, $f = (\ell+1)1_X$ for $\ell = 0, p$. Thus, we have

$$f^{\sharp n-1}(\eta_{n-1}^2) = (\ell+1)\eta_{n-1}^2 \circ q \circ 1_X = (\ell+1)\eta_{n-1}^2 \circ q.$$

If $\ell = p$, then $(\ell+1)\eta_{n-1}^2 \circ q = 3\eta_{n-1}^2 \circ q \neq \eta_{n-1}^2 \circ q$ because $\pi^{n-1}(X) = \mathbb{Z}_4$. Thus, to ensure that $f^{\sharp n-1} = 1_{\pi_{n-1}}$ holds, ℓ is require to be 0. Hence $\mathcal{E}_{n-1}^\sharp(X) \cong 1\{1_X\}$. □

Theorem 4.4. *For $X = M(\mathbb{Z}_p, 3)$, we have the following table*:

	p odd	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}^\sharp_{n-1}(X)$	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$
generators	1_X	$1_X \oplus i \circ \eta_3 \circ q$	$(p+1)1_X$

Proof. Based on the Puppe Sequence, we have

$$\pi^2(X) = \begin{cases} 0 & p \equiv 1 \pmod 2, \\ \mathbb{Z}_2\{\eta_2^2 \circ q\} & p \equiv 0 \pmod 2. \end{cases}$$

<u>Case 1.</u> Let p be odd.

By Theorem 4.2, $\mathcal{E}^\sharp_{n-1}(X) \subseteq \mathcal{E}^\sharp_n(X) \cong 1$.

<u>Case 2.</u> Let $p \equiv 0 \pmod 4$.

By Theorem 3.6, for each $f \in \mathcal{E}(X)$, $f = k1_X \oplus \ell i \circ \eta_n \circ \pi$ for some k and ℓ such that $0 \le k \le p-1$, $(k,p) = 1$ and $\ell = 0, 1$. Thus we have

$$\begin{aligned} f^{\sharp n-1}(\eta_2^2 \circ \pi) &= (\eta_2^2 \circ \pi) \circ (k1_X \oplus \ell i \circ \eta_3 \circ \pi) \\ &= k\eta_2^2 \circ \pi \circ 1_X \oplus \ell \eta_2^2 \circ \pi \circ i \circ \eta_3 \circ \pi \\ &= k\eta_2^2 \circ \pi \end{aligned}$$

because $\pi \circ i$ is homotopic to the constant map. Thus, for $f^{\sharp n-1} = 1_{\pi_{n-1}}$ to hold, k is required to be 1. Hence,

$$\mathcal{E}^\sharp_{n-1}(X) \cong \mathbb{Z}_2\{1_X \oplus \ell i \circ \eta_3 \circ q |\, \ell = 0, 1\}.$$

<u>Case 3.</u> Let $p \equiv 2 \pmod 4$.

By Theorem 3.6, for each $f \in \mathcal{E}(X)$, $f = (k+\ell)1_X$ for some k and l such that $0 \le k \le p-1$, $(k,p) = 1$ and $\ell = 0, p$. Thus, we have

$$f^{\sharp n-1}(\eta_2^2 \circ q) = (k+\ell)\eta_2^2 \circ q \circ 1_X = (k+\ell)\eta_2^2 \circ q.$$

Therefore, to ensure that $f^{\sharp n-1} = 1_{\pi_{n-1}}$ holds, k is required to be 1. Hence,

$$\mathcal{E}^\sharp_{n-1}(X) \cong \mathbb{Z}_2\{\, (\ell+1)1_X |\, \ell = 0, p\}.$$

□

5. Computation of $\mathcal{E}^\sharp_k(C(\mathbb{Z}_p, n))$

In this section, we compute $\mathcal{E}^\sharp_k(C(\mathbb{Z}_p, n))$ and determine their generators for $k = n, n-1$ and $n-2$. Throughout this section, we let $X = C(\mathbb{Z}_p, n)$ and $M_2 = M(\mathbb{Z}_p, n-1)$.

First of all, we determine the generators of $[X, X]$ and $[X, S^n]$.

It is well known that

$$C(\mathbb{Z}_p, n) = M(\mathbb{Z}, n) \vee M(\mathbb{Z}_p, n-1) = S^n \vee M(\mathbb{Z}_p, n-1)$$

for $n > 3$. Thus we have

$$[X, X] \cong [S^n, X] \oplus [M_2, X]$$

and by Proposition 2.1,

$$[X, X] \cong [X, S^n] \oplus [X, M_2].$$

Consequently,

$$[X, X] \cong [S^n, S^n] \oplus [M_2, S^n] \oplus [S^n, M_2] \oplus [M_2, M_2].$$

In [3], we have

$$\mathcal{E}(X) = \mathcal{E}(S^n) \oplus [M_2, S^n] \oplus [S^n, M_2] \oplus \mathcal{E}(M_2).$$

Consider the mapping cone sequence

$$S^n \xrightarrow{p} S^n \xrightarrow{i} M(\mathbb{Z}_p, n) \xrightarrow{\pi} S^{n+1} \xrightarrow{p} S^{n+1}$$

and let $i_1 : S^n \to X$ and $i_2 : M_2 \to X$ be inclusion maps and $q_1 : X \to S^n$ and $q_2 : X \to M_2$ be projection maps.

Then, from Table 1 and Theorem 3.6, we have the following lemmas.

Lemma 5.1. *If $i_1 : S^n \to X$ and $i_2 : M_2 \to X$ are inclusion maps and $q_1 : X \to S^n$ and $q_2 : X \to M_2$ are projection maps, then we have*:

$p \equiv 1 \pmod 2$	$[X, X]$	$\mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$
	Generators	$i_1 \circ \iota_n \circ q_1,\ i_1 \circ \iota_n \circ \pi \circ q_2,\ i_1 \circ 1_{M_2} \circ q_2$
$p \equiv 0 \pmod 4$	$[X, X]$	$\mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_p$
	Generators	$i_1 \circ \iota_n \circ q_1,\ i_1 \circ \iota_n \circ \pi \circ q_2,$ $i_2 \circ i \circ \eta_{n-1} \circ q_1,\ i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2,\ i_2 \circ 1_{M_2} \circ q_2$
$p \equiv 2 \pmod 4$	$[X, X]$	$\mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$
	Generators	$i_1 \circ \iota_n \circ q_1,\ i_1 \circ \iota_n \circ \pi \circ q_2,\ i_2 \circ i \circ \eta_{n-1} \circ q_1,\ i_2 \circ 1_{M_2} \circ q_2$

As $\mathcal{E}(S^n) \cong \mathbb{Z}_2$, we have the following lemma.

Lemma 5.2. *$\mathcal{E}(X)$ is isomorphic to*

$p \equiv 1 \pmod 2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus (\mathbb{Z}_p^*)$
$p \equiv 0 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \rtimes \mathbb{Z}_p^*)$
$p \equiv 2 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \rtimes \mathbb{Z}_p^*)$

By Proposition 2.1,

$$\pi^r(X) = \pi^r(S^n) \oplus \pi^r(M(\mathbb{Z}_p, n-1)).$$

Thus, we have the following lemma from Table 1,

Lemma 5.3. *For $\pi^r(X)$, we have the following table*:

		$\pi^r(X)$	Generators
$r = n$		$\mathbb{Z} \oplus \mathbb{Z}_p$	$\iota_n \circ q_1,\ \iota_n \circ \pi \circ q_2$
$r = n-1$	$p \equiv 1 \pmod 2$	$\mathbb{Z}_2$	$\eta_{n-1} \circ q_1$
	$p \equiv 0 \pmod 2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\eta_{n-1} \circ q_1,\ \eta_{n-1} \circ \pi \circ q_2$
$r = n-2$	$p \equiv 1 \pmod 2$	$\mathbb{Z}_2$	$\eta_{n-2}^2 \circ q_1$
	$p \equiv 0 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\eta_{n-2}^2 \circ q_1,\ \eta_{n-2}^2 \circ \pi \circ q_2,\ \bar{\eta}_n \circ q_2$
	$p \equiv 2 \pmod 4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4$	$\eta_{n-2}^2 \circ q_1,\ \bar{\eta}_n \circ q_2$

Now, we compute $\mathcal{E}_k^\sharp(X)$ and determine their generators for $k = n, n-1$, and $n-2$.

Theorem 5.4. *For $\mathcal{E}_n^\sharp(X)$, we have the following table*:

	$p \equiv 1 \pmod 2$	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}_n^\sharp(X)$	1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Proof. Because $\pi^k(X) = 0$ for $k > n$, it is sufficient to consider the case that $k = n$. From Lemma 5.3, $\pi^n(X) = \mathbb{Z} \oplus \mathbb{Z}_p\{\iota_n \circ q_1, \iota_n \circ \pi \circ q_2\}$. By Proposition 3.4, each $f \in \mathcal{E}(X)$ can be identified as

$$f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}.$$

Let $\gamma = (\gamma_1, \gamma_2) = (\iota_n \circ q_1, \iota_n \circ \pi \circ q_2)$ be the generator.

Case 1. Let p be odd.

By Lemma 5.2, for each $f \in \mathcal{E}(X)$, we have

$$\begin{aligned} f_{11} &= si_1 \circ \iota_n \circ q_1, \\ f_{12} &= ti_1 \circ \iota_n \circ \pi \circ q_2, \\ f_{21} &= 0, \\ f_{22} &= ki_1 \circ 1_{M_2} \circ q_2 \end{aligned}$$

for some k, s and t such that $0 \le t \le p-1$, $0 \le k \le p-1$ and $(k, p) = 1$, $s = -1, 1$. Thus we have

$$\begin{aligned} f_{11}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ s(i_1 \circ \iota_n \circ q_1) \\ &= s\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ q_1 = s\iota_n \circ q_1, \\ f_{12}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ t(i_1 \circ \iota_n \circ \pi \circ q_2) \\ &= t\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ \pi \circ q_2 = t\iota_n \circ \pi \circ q_2, \\ f_{22}^{\sharp n}(\gamma_2) &= \iota_n \circ \pi \circ q_2 \circ (ki_2 \circ 1_{M_2} \circ q_2) = k\iota_n \circ \pi \circ q_2. \end{aligned}$$

By Proposition 3.5, $s = 1,\ t = 0$ and $k = 1$. Hence

$$\mathcal{E}_n^\sharp(X) \cong 1 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ 0 & i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\}.$$

Case 2. $p \equiv 0 \pmod 4$.

By Lemma 5.2, for each $f \in \mathcal{E}(X)$, we have

$$\begin{aligned} f_{11} &= si_1 \circ \iota_n \circ q_1, \\ f_{12} &= ti_1 \circ \iota_n \circ \pi \circ q_2, \\ f_{21} &= mi_2 \circ i \circ \eta_{n-1} \circ q_1, \\ f_{22} &= \ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus ki_2 \circ 1_{M_2} \circ q_2 \end{aligned}$$

for k, ℓ, m, s and t such that $0 \le t \le p-1, 0 \le k \le p-1$, $m, \ell = 0, 1, s = -1, 1$ and $(k, p) = 1$. Thus we have

$$\begin{aligned} f_{11}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ s(i_1 \circ \iota_n \circ q_1) \\ &= s\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ q_1 = s\iota_n \circ q_1, \\ f_{12}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ t(i_1 \circ \iota_n \circ \pi \circ q_2) \end{aligned}$$

$$
\begin{aligned}
&= t\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ q \circ q_2 = t\iota_n \circ \pi \circ q_2,\\
f_{21}^{\sharp n}(\gamma_2) &= \iota_n \circ \pi \circ q_2 \circ m(i_2 \circ i \circ \eta_{n-1} \circ q_1)\\
&= m\iota_n \circ q \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 = 0,\\
f_{22}^{\sharp n}(\gamma_2) &= \iota_n \circ \pi \circ q_2 \circ (\ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus k i_2 \circ 1_{M_2} \circ q_2)\\
&= \ell\iota_n \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus k\iota_n \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2\\
&= k\iota_n \circ \pi \circ q_2
\end{aligned}
$$

because $\pi \circ i$ is homotopic to the constant map.

By Proposition 3.5, $s = 1, t = 0, k = 1$ and $m, \ell = 0, 1$. Hence

$$
\mathcal{E}_n^{\sharp}(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ m i_2 \circ i \circ \eta_{n-1} \circ q_1 & \ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \middle| \, m, \ell = 0, 1 \right\},
$$

where $\alpha = i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2$.

<u>Case 3.</u> Let $p \equiv 2 \pmod 4$.

By Lemma 5.2, for each $f \in \mathcal{E}(X)$, we have

$$
\begin{aligned}
f_{11} &= s i_1 \circ \iota_n \circ q_1,\\
f_{12} &= t i_1 \circ \iota_n \circ \pi \circ q_2,\\
f_{21} &= m i_2 \circ i\eta_{n-1} \circ q_1,\\
f_{22} &= (k + \ell) i_2 \circ 1_{M_2} \circ q_2
\end{aligned}
$$

for k, ℓ, m, s and t such that $0 \le t \le p-1, 0 \le k \le p-1$, $m = 0, 1$, $s = -1, 1$, $\ell = 0, p$ and $(k, p) = 1$. Thus we have

$$
\begin{aligned}
f_{11}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ s(i_1 \circ \iota_n \circ q_1)\\
&= s\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ q_1 = s\iota_n \circ q_1,\\
f_{12}^{\sharp n}(\gamma_1) &= \iota_n \circ q_1 \circ t(i_1 \circ \iota_n \circ \pi \circ q_2)\\
&= t\iota_n \circ q_1 \circ i_1 \circ \iota_n \circ \pi \circ q_2 = t\iota_n \circ \pi \circ q_2,\\
f_{21}^{\sharp n}(\gamma_2) &= \iota_n \circ q \circ q_2 \circ m(i_2 \circ i\eta_{n-1} \circ q_1)\\
&= m\iota_n \circ q \circ q_2 \circ i_2 \circ i\eta_{n-1} \circ q_1 = 0,\\
f_{22}^{\sharp n}(\gamma_2) &= \iota_n \circ \pi \circ q_2 \circ (k + \ell)(i_2 \circ 1_{M_2} \circ q_2)\\
&= (k + \ell)\iota_n \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 = (k + \ell)\iota_n \circ \pi \circ q_2
\end{aligned}
$$

because $\pi \circ i$ is homotopic to the constant map.

By Proposition 3.5, $s = 1, t = 0$, $k = 1$, $\ell = 0, p$ and $m = 0, 1$. Hence

$$
\mathcal{E}_n^{\sharp}(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ m i_2 \circ i \circ \eta_{n-1} \circ q_1 & (\ell + 1) i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \middle| \; m = 0, 1 \text{ and } \ell = 0, p \right\}. \qquad \square
$$

Theorem 5.5. *For* $\mathcal{E}_{n-1}^{\sharp}(X)$, *we have the following table*:

	$p \equiv 1 \pmod 2$	$p \equiv 0 \pmod 4$	$p \equiv 2 \pmod 4$
$\mathcal{E}^{\sharp}_{n-1}(X)$	1	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Proof. From Lemma 5.3, we have

$$\pi^{n-1}(X) = \begin{cases} \mathbb{Z}_2\{\eta_{n-1} \circ q_1\} & p \equiv 1 \pmod 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2\{\eta_{n-1} \circ q_1, \eta_{n-1} \circ \pi \circ q_2\} & p \equiv 0 \pmod 2. \end{cases}$$

Case 1. Let p be odd.

By Theorem 5.4, we have $\mathcal{E}^{\sharp}_{n-1}(X) \subseteq \mathcal{E}^{\sharp}_{n}(X) \cong 1$.

Case 2. Let $p \equiv 0 \pmod 4$.

By Theorem 5.4, for each $f \in \mathcal{E}(X)$, we have

$$\begin{aligned} f_{11} &= i_1 \circ \iota_n \circ q_1, \\ f_{12} &= 0, \\ f_{21} &= mi_2 \circ i \circ \eta_{n-1} \circ q_1, \\ f_{22} &= \ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus i_2 \circ 1_{M_2} \circ q_2 \end{aligned}$$

for $m, \ell = 0, 1$. By Proposition 3.5, it is sufficient to determine f_{21} and f_{22}; however, we have

$$\begin{aligned} f^{\sharp n-1}_{21}(\gamma_2) &= \eta_{n-1} \circ \pi \circ q_2 \circ m(i_2 \circ i \circ \eta_{n-1} \circ q_1) \\ &= m\eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 = 0, \\ f^{\sharp n-1}_{22}(\gamma_2) &= \eta_{n-1} \circ \pi \circ q_2 \circ (\ell i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus i_2 \circ 1_{M_2} \circ q_2) \\ &= \ell\eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ \pi \circ q_2 \oplus \eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 \\ &= \eta_{n-1} \circ \pi \circ q_2 \end{aligned}$$

because $\pi \circ i$ is homotopic to the constant map.

By Proposition 3.5, $m, \ell = 0, 1$. Hence,

$$\mathcal{E}^{\sharp}_{n-1}(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ mi_2 \circ i \circ \eta_{n-1} \circ q_1 & \ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \middle| \; \ell = 0, 1 \text{ and } m = 0, 1 \right\},$$

where $\alpha = i_2 \circ i \circ \eta_{n-1} \circ q \circ q_2$.

Case 3. Let $p \equiv 2 \pmod 4$

By Theorem 5.4, for each $f \in \mathcal{E}(X)$, we have

$$\begin{aligned} f_{11} &= i_1 \circ \iota_n \circ q_1, \\ f_{12} &= 0, \\ f_{21} &= mi_2 \circ i\eta_{n-1} \circ q_1, \\ f_{22} &= (1 + \ell)i_2 \circ 1_{M_2} \circ q_2 \end{aligned}$$

for $m = 0, 1$, $\ell = 0, p$. By Proposition 3.5, it is sufficient to determine f_{21} and f_{22}; however, we have

$$f^{\sharp n-1}_{21}(\gamma_2) = \eta_{n-1} \circ \pi \circ q_2 \circ m(i_2 \circ i\eta_{n-1} \circ q_1)$$

$$
\begin{aligned}
&= m\eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ i\eta_{n-1} \circ q_1 = 0, \\
f_{22}^{\sharp n-1}(\gamma_2) &= \eta_{n-1} \circ \pi \circ q_2 \circ (1+\ell)(i_2 \circ 1_{M_2} \circ q_2) \\
&= (1+\ell)\eta_{n-1} \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 = (1+\ell)\eta_{n-1} \circ \pi \circ q_2
\end{aligned}
$$

because $q \circ i$ is homotopic to the constant map.

By Proposition 3.5, $\ell = 0, p$ and $m = 0, 1$. Hence

$$
\mathcal{E}^{\sharp}_{n-1}(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ i_2 \circ i \circ \eta_{n-1} \circ q_1 & (\ell+1) i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \middle| \; \ell = 0, p \text{ and } m = 0, 1 \right\}. \qquad \square
$$

Theorem 5.6. $\mathcal{E}^{\sharp}_{n-2}(C(\mathbb{Z}_p, n)) \cong 1$.

Proof. From Theorem 5.3,

$$
\pi^{n-2}(X) = \begin{cases} \mathbb{Z}_2\{\eta^2_{n-2} \circ q_1\} & p \equiv 1 \pmod 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\{\eta^2_{n-2} \circ q_1, \eta^2_{n-2} \circ \pi \circ q_2, \bar{\eta}_n \circ q_2\} & p \equiv 0 \pmod 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4\{\eta^2_{n-2} \circ q_1, \bar{\eta}_n \circ q_2\} & p \equiv 2 \pmod 4. \end{cases}
$$

<u>Case 1.</u> Let p be odd. By Theorem 5.5, $\mathcal{E}^{\sharp}_{n-2}(X) \subseteq \mathcal{E}^{\sharp}_{n-1}(X) \cong 1$.

<u>Case 2.</u> Let $p \equiv 0 \pmod 4$.

Then, the generator of $\pi^{n-2}(X)$ is

$$
\gamma = (\gamma_1, \gamma_2) = (\eta^2_{n-2} \circ q_1, \eta^2_{n-2} \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2).
$$

By Theorem 5.5, for each $f \in \mathcal{E}(X)$, we have

$$
\begin{aligned}
f_{11} &= i_1 \circ \iota_n \circ q_1, \\
f_{12} &= 0, \\
f_{21} &= m i_2 \circ i \circ \eta_{n-1} \circ q_1, \\
f_{22} &= \ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2
\end{aligned}
$$

for $m, \ell = 0, 1$ and $\alpha = i_2 \circ i \circ \eta_{n-1} \circ q \circ q_2$. By Proposition 3.5, it is sufficient to determine f_{21} and f_{22}; however, we have

$$
\begin{aligned}
f_{21}^{\sharp n-2}(\gamma_2) &= \eta^2_{n-2} \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2 \circ m(i_2 \circ i \circ \eta_{n-1} \circ q_1) \\
&= m\eta^2_{n-2} \circ \pi \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 \oplus m\bar{\eta}_n \circ q_2 \circ i_2 \circ i \circ \eta_{n-1} \circ q_1 \\
&= m\bar{\eta}_n \circ i \circ \eta_n \circ q_1 = m\eta_{n-2}\eta_{n-1} \circ q_1 \\
&= m\eta^2_{n-2} \circ q_1, \\
f_{22}^{\sharp n-2}(\gamma_2) &= \eta^2_{n-2} \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2 \circ (\ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2) \\
&= \eta^2_{n-2} \circ \pi \circ q_2 \circ (\ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2) + \bar{\eta}_n \circ q_2 \circ (\ell\alpha \oplus i_2 \circ 1_{M_2} \circ q_2) \\
&= \ell\eta^2_{n-2} \circ \pi \circ q_2 \circ \alpha \oplus \eta^2_{n-2} \circ \pi \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 \\
&\quad + \ell\bar{\eta}_n \circ q_2 \circ \alpha \oplus \bar{\eta}_n \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 \\
&= \eta^2_{n-2} \circ \pi \circ q_2 + \ell\bar{\eta}_n \circ i\eta_{n-1} \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2
\end{aligned}
$$

$$= \eta_{n-2}^2 \circ \pi \circ q_2 + \ell\eta_{n-2}^2 \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2$$
$$= (1+\ell)\eta_{n-2}^2 \circ \pi \circ q_2 \oplus \bar{\eta}_n \circ q_2$$

because $q \circ i$ is homotopic to the constant map and $\bar{\eta}_n \circ i\eta_{n-2} = \eta_{n-2}^2 = \eta_{n-2} \circ \eta_{n-1}$.

By Proposition 3.5, $\ell = 0$ and $m = 0$. Hence,

$$\mathcal{E}_{n-2}^{\sharp}(X) \cong 1 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ 0 & i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\}.$$

Case 3. Let $p \equiv 2 \pmod 4$.

Then the generator of $\pi^{n-2}(X)$ is $\gamma = (\gamma_1, \gamma_2) = (\eta_{n-2}^2 \circ q_1, \bar{\eta}_n \circ q_2)$. By Theorem 5.5, for each $f \in \mathcal{E}(X)$, we have

$$f_{11} = i_1 \circ \iota_n \circ q_1,$$
$$f_{12} = 0,$$
$$f_{21} = mi_2 \circ i\eta_{n-1} \circ q_1,$$
$$f_{22} = (1+\ell)i_2 \circ 1_{M_2} \circ q_2$$

for $\ell = 0, p$ and $m = 0, 1$. By Proposition 3.5, it is sufficient to determine f_{21} and f_{22}; however, we have

$$f_{21}^{\sharp n-2}(\gamma_2) = \bar{\eta}_n \circ q_2 \circ m(i_2 \circ i\eta_{n-1} \circ q_1)$$
$$= m\bar{\eta}_n \circ q_2 \circ i_2 \circ i\eta_{n-1} \circ q_1 = m\eta_{n-2}^2 \circ q_1,$$
$$f_{22}^{\sharp n-2}(\gamma_2) = \bar{\eta}_n \circ q_2 \circ (1+\ell)(i_2 \circ 1_{M_2} \circ q_2)$$
$$= (1+\ell)\bar{\eta}_n \circ q_2 \circ i_2 \circ 1_{M_2} \circ q_2 = (1+\ell)\bar{\eta}_n \circ q_2$$

because $\bar{\eta}_n \circ i\eta_{n-2} = \eta_{n-2} \circ \eta_{n-1} = \eta_{n-2}^2$.

If $\ell = p$, then $(1+p)\bar{\eta}_n \circ q_2 = 3\bar{\eta}_n \circ q_2 \neq \bar{\eta}_n \circ q_2$ because $\bar{\eta}_n \circ q_2$ has order 4. Thus, by Proposition 3.5, $\ell = 0$ and $m = 0$. Hence,

$$\mathcal{E}_{n-2}^{\sharp}(X) \cong 1 \left\{ \begin{pmatrix} i_1 \circ \iota_n \circ q_1 & 0 \\ 0 & i_2 \circ 1_{M_2} \circ q_2 \end{pmatrix} \right\}.$$

□

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