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ON POINTWISE 1-TYPE GAUSS MAP OF SURFACES IN $\mathbb{E}^3_1$ CONCERNING CHENG-YAU OPERATOR

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Abstract. In this paper, we study surfaces in 3-dimensional Minkowski space in terms of certain type of their Gauss map. We give several results on these surfaces whose Gauss map G satisfies $\Box G = f(G + C)$ for a smooth function f and a constant vector C, where $\Box$ denotes the Cheng-Yau operator. In particular, we obtain classification theorems on the rotational surfaces in $\mathbb{E}^3_1$ with space-like axis of rotation in terms of type of their Gauss map concerning the Cheng-Yau operator.

1. Introduction

The notion of finite type mappings defined on submanifolds of semi-Euclidean spaces was introduced by B.-Y. Chen at the end of 1970's ([7]). Let $\mathbb{E}^n_s$ denote the semi-Euclidean space of dimension n with index s. Consider a p-dimensional submanifold M of $\mathbb{E}^n_s$. A mapping ϕ from M into the semi-Euclidean space $\mathbb{E}^n_s$ is said to be of k-type if it can be expressed as a sum of eigenvectors of the Laplace operator Δ corresponding to k distinct eigenvalues ([9]). A lot of results on the theory of finite type mappings were summed up in the report of B.-Y. Chen ([6]). Even now, there are some open problems in this area which are still being worked ([5]).

On the other hand, if the mapping ϕ in this definition is the Gauss map of M, then M is said to have k-type Gauss map. In particular, M has 1-type Gauss map if and only if its Gauss map $G : M \to \mathbb{E}^P_s$ satisfies

$$\Delta G = \lambda(G + C) \tag{1.1}$$

for a constant λ and a constant vector $C \in \mathbb{E}^P_s$, where $P = \binom{n}{p}$. For example, the Gauss map of Clifford torus satisfies (1.1) for a constant λ and $C = 0$

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([24]). Some papers dealt with the submanifolds with finite type Gauss map ([3, 4, 10, 20]).

However, the Laplacian of the Gauss map of certain surfaces and hypersurfaces such as helicoid, catenoid in Euclidean 3-space, conjugate Enneper's surface of the second kind and B-scrolls in 3-dimensional Minkowski space, and some special kinds of rotational surfaces in Euclidean 4-space take the form

$$\Delta G = f(G + C) \tag{1.2}$$

for a non-zero smooth function f and a constant vector C ([14, 16, 22]). These submanifolds whose Gauss map satisfying (1.2) are said to have pointwise 1-type Gauss map. There are several works on the submanifolds of semi-Euclidean spaces with pointwise 1-type Gauss map ([2, 8, 12, 13, 15]).

Now, let M be a hypersurface of $\mathbb{E}^n_s$ and L_k denote the linearized operator of the first variation of the k-th mean curvature arising from normal variations of M for $k = 0, 1, \ldots, n-1$. As $\Delta = -L_0$, the following question arises: Are there any hypersurfaces whose Gauss map satisfies

$$L_k G = f(G + C)$$

for a smooth function f and a constant vector C? Recently, authors studied surfaces of Euclidean 3-space whose Gauss map satisfies the above equation for $k = 1$ in [21].

In this paper, we mainly focus on surfaces in the Minkowski 3-space with $\Box$-pointwise 1-type Gauss map, i.e., a surface whose Gauss map satisfies

$$\Box G = f(G + C) \tag{1.3}$$

(see Definition 3.2), where $\Box = L_1$ is the Cheng-Yau operator ([11]). In Section 2, we give the basic facts, definitions and notations on surfaces of Minkowski-3 space. Then, in Section 3, we give some classification and characterization theorems on surfaces with $\Box$-pointwise 1-type Gauss map. We focus on the surfaces with constant mean curvature, surfaces with constant Gaussian curvature and surfaces which have principal curvatures with constant difference. In the last section, we study rotational surfaces in $\mathbb{E}^3_1$ with space-like axis of rotation, and we obtain a complete classification of these surfaces with the Gauss map satisfying (1.3) for $C \neq 0$.

Surfaces we are dealing with are smooth and connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}^3_1$ denote the Minkowski 3-space with the canonical pseudo-Euclidean metric tensor of index 1 given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system in $\mathbb{E}^3_1$. We put

$$\mathbb{S}^2_1(r^2) = \{x \in \mathbb{E}^3_1 : \langle x, x\rangle = r^{-2}\},$$

$$\mathbb{H}^2(-r^2) = \{x \in \mathbb{E}^3_1 : \langle x, x\rangle = -r^{-2}\},$$

where $\langle\ ,\ \rangle$ is the indefinite inner product of $\mathbb{E}^3_1$. Then, $\mathbb{S}^2_1(r^2)$ and $\mathbb{H}^2(-r^2)$ are the complete pseudo-Riemannian manifolds of constant curvature r^2 and $-r^2$, respectively.

Let M be a non-degenerate surface in $\mathbb{E}^3_1$. We denote the Levi-Civita connections of $\mathbb{E}^3_1$ and M by $\widetilde{\nabla}$ and ∇, respectively, and D stands for the normal connection of M. Then, the Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) \tag{2.1}$$

for the vector fields X, Y and Z tangent to M, where h is the second fundamental form of M and $(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Let N be the unit normal vector associated with the orientation of M. The mapping

$$\begin{array}{rcl} G : M & \to & \mathbb{E}^3_1 \\ p & \mapsto & N(p) \end{array}$$

is called the Gauss map of M. From the above definition we have $\langle G, G\rangle = -\varepsilon$ which implies $G(M) \subset Q^2(\varepsilon)$, where

$$Q^2(\varepsilon) = \begin{cases} \mathbb{H}^2(-1) & \text{if } \varepsilon = 1, \\ \mathbb{S}^2_1(1) & \text{if } \varepsilon = -1 \end{cases}$$

and we put

$$\varepsilon = \begin{cases} 1 & \text{if } M \text{ is Riemannian}, \\ -1 & \text{if } M \text{ is Lorentzian}. \end{cases}$$

Let $\mathcal{B}$ be a set given by $\mathcal{B} = \{e_1, e_2, e_3\}$ with $e_3 = G$. We will say $\mathcal{B}$ is an orthonormal frame field defined on M if $\langle e_1, e_1\rangle = \varepsilon$, $\langle e_2, e_2\rangle = 1$ and $\langle e_1, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$. In this case, we have $\nabla_{e_1} e_1 = \omega_1 e_2$, $\nabla_{e_1} e_2 = -\varepsilon\omega_1 e_1$, $\nabla_{e_2} e_1 = \omega_2 e_2$ and $\nabla_{e_2} e_2 = -\varepsilon\omega_2 e_1$ for some smooth functions ω_1 and ω_2. On the other hand, we say $\mathcal{B}$ is a pseudo-orthonormal frame field if $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$ and $\langle e_1, e_2\rangle = -1$. In this case, we have $\nabla_{e_1} e_1 = \omega_1 e_1$, $\nabla_{e_1} e_2 = -\omega_1 e_2$, $\nabla_{e_2} e_1 = \omega_2 e_1$ and $\nabla_{e_2} e_2 = -\omega_2 e_2$ for some smooth functions ω_1 and ω_2.

The shape operator (or Weingarten map) $S : TM \to TM$ of M is defined as $S(X) = -\widetilde{\nabla}_X N$ for a tangent vector field X of M, where TM is the tangent bundle of M. The functions $\mathcal{Q}$, H and K defined by $\mathcal{Q}(\lambda) = \det(S - \lambda I) = \lambda^2 - 2H\lambda + K$ are called the characteristic polynomial of S, the mean curvature of M and the Gaussian curvature of M, respectively. Then, the (complex valued) functions λ_1 and λ_2 satisfying $\mathcal{Q}(\lambda_i) = 0$, $i = 1, 2$, are called the principal curvatures of M. A surface M in $\mathbb{E}^3_1$ is said to be isoparametric if its principal curvatures are constant functions. Furthermore, M is said to be flat if its Gaussian curvature K vanishes identically, and we will say M is minimal if $H \equiv 0$.

We will denote by $C^\infty(M)$ the space of smooth functions defined on M into $\mathbb{R}$, and $\chi(M)$ will stand for the space of tangent vector fields defined on M. Let

$f \in C^\infty(M)$ and $X \in \chi(M)$. If $\mathcal{B} = \{e_1, e_2, e_3\}$ is an orthonormal frame field defined on M, then the gradient of f is defined as $\nabla f = \varepsilon e_1(f)e_1 + e_2(f)e_2$ and the divergence of X is given by $\mathrm{div}X = \varepsilon\langle\nabla_{e_1}X, e_1\rangle + \langle\nabla_{e_2}X, e_2\rangle$. On the other hand, if $\mathcal{B}$ is a pseudo-orthonormal frame field defined on M, then $\nabla f = -e_2(f)e_1 - e_1(f)e_2$ and $\mathrm{div}X = -\langle\nabla_{e_1}X, e_2\rangle - \langle\nabla_{e_2}X, e_1\rangle$.

3. Surfaces with □-pointwise 1-type Gauss map

Let M be a surface in $\mathbb{E}^3_1$ and $L_k : C^\infty(M) \to C^\infty(M)$ an operator given by $L_k(f) = tr(P_k \circ \nabla^2 f)$, $k = 1, 2$, where $P_0 = I$, $P_1 = 2HI - S$ and I is the identity operator acting on the tangent bundle of M. Note that we have $L_0 = -\Delta$ and $L_1 = \Box$, where Δ is the Laplace operator of M and $\Box$ is the Cheng-Yau operator introduced in [11].

Let $f \in C^\infty(M)$. As a matter of fact, it turns out to be

$$L_k f = \mathrm{div}\,(P_k\,(\nabla f)) \tag{3.1}$$

(see [1]).

By regarding the shape operator S of M and its characteristic polynomial $\mathcal{Q}$, it is well-known that, without loss of generality, M can be assumed to satisfy one of the three cases below (see, for instance, [18]), locally:

Case I. S is diagonalizable. In this case, there exists an orthonormal frame field $\mathcal{B} = \{e_1, e_2, e_3\}$ defined on M such that $Se_i = k_i e_i$, $i = 1, 2$, for some functions k_1, k_2. The mean curvature and Gaussian curvature of M become $H = \frac{1}{2}(k_1 + k_2)$ and $K = k_1k_2$. Moreover, we obtain

$$e_2(k_1) = \varepsilon\omega_1(k_1 - k_2), \tag{3.2}$$

$$e_1(k_2) = \omega_2(k_1 - k_2) \tag{3.3}$$

from the Codazzi equation (2.1). By a direct calculation with these equations and (3.1), we obtain

$$\Box = \varepsilon k_2(e_1e_1 - \nabla_{e_1}e_1) + k_1(e_2e_2 - \nabla_{e_2}e_2). \tag{3.4}$$

Case II. S is not diagonalizable and $\mathcal{Q}(\lambda) = (\lambda - k)^2$ for some functions k. In this case, we have $\varepsilon = -1$ and there exists a pseudo-orthonormal frame field $\mathcal{B} = \{e_1, e_2, e_3\}$ defined on M such that $Se_1 = ke_1 + e_2$, $Se_2 = ke_2$ and the mean curvature and Gaussian curvature of M becomes $H = k$ and $K = k^2$. Moreover, one can obtain

$$e_1(k) = -2\omega_2, \tag{3.5}$$

$$e_2(k) = 0 \tag{3.6}$$

from the Codazzi equation (2.1). By considering these equations and (3.1), we obtain

$$\Box = -k(e_1e_2 - \nabla_{e_1}e_2 + e_2e_1 - \nabla_{e_2}e_1) + e_2e_2 - \nabla_{e_2}e_2. \tag{3.7}$$

Case III. S is not diagonalizable and $\mathcal{Q}(\lambda) = (\lambda - k)^2 + b^2$ for a smooth function k and a nowhere vanishing function b. In this case, we have $\varepsilon = -1$ and there exists an orthonormal frame field $\mathcal{B} = \{e_1, e_2, e_3\}$ defined on M such that $Se_1 = ke_1 - be_2$, $Se_2 = be_1 + ke_2$ and the mean curvature and Gaussian curvature of M become $H = k$ and $K = k^2 + b^2$. Moreover, we obtain

$$e_1(k) + e_2(b) = -2b\omega_1, \tag{3.8}$$

$$e_2(k) - e_1(b) = 2b\omega_2 \tag{3.9}$$

from the Codazzi equation (2.1). By a direct calculation considering these equations and (3.1) we obtain

$$\begin{aligned}\Box = {} & k(-e_1e_1 + \nabla_{e_1}e_1 + e_2e_2 - \nabla_{e_2}e_2) - b(e_1e_2 - \nabla_{e_1}e_2) \\ & - b(e_2e_1 - \nabla_{e_2}e_1).\end{aligned} \tag{3.10}$$

The Cheng-Yau operator acting on the vector fields in $\chi(M)$ is defined as

$$\begin{aligned}\Box : \chi(M) \quad & \rightarrow \quad \chi(M) \\ X \quad & \mapsto \quad \Box X = \sum_{i=1}^{3} \delta_i \Box(\langle X, E_i\rangle)E_i,\end{aligned} \tag{3.11}$$

where $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$ and $\delta_i = \langle E_i, E_i\rangle$. Next, we want to give the following lemma:

Lemma 3.1. *Let M be a surface in $\mathbb{E}^3_1$. Then, its Gauss map G satisfies*

$$\Box G = -\nabla K + 2\varepsilon KHG, \tag{3.12}$$

where H and K are the mean curvature and Gaussian curvature of M and

$$\varepsilon = \begin{cases} 1 & \text{if } M \text{ is Riemannian,} \\ -1 & \text{if } M \text{ is Lorentzian.} \end{cases}$$

Proof. Let M be a surface in $\mathbb{E}^3_1$ and S its shape operator.

Case I. S is diagonalizable. In this case, from (3.4) and (3.11) we have

$$\Box G = \varepsilon k_2(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1}G - \widetilde{\nabla}_{\nabla_{e_1}e_1}G) + k_1(\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2}G - \widetilde{\nabla}_{\nabla_{e_2}e_2}G),$$

where $\{e_1, e_2, G = e_3\}$ is the appropriate orthonormal frame field. By a direct calculation, we get

$$\begin{aligned}\Box G = & -\Big(\varepsilon e_1(k_1)k_2 - \varepsilon\omega_2k_1k_2 + \varepsilon\omega_2k_1^2\Big)e_1 - \Big(k_1e_2(k_2) + \varepsilon\omega_1k_1k_2 - \varepsilon k_2^2\omega_2\Big)e_2 \\ & + \varepsilon\Big(k_1k_2(k_1 + k_2)\Big)G.\end{aligned}$$

Combining the Codazzi equation (3.2)-(3.3) with this equation, we obtain (3.12).

Case II. S is not diagonalizable and its characteristic polynomial is $\mathcal{Q}(\lambda) = (\lambda - k)^2$ for a smooth function k. In this case, from (3.7) and (3.11) we have

$$\begin{aligned}\Box G = & -k\left(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_2}G - \widetilde{\nabla}_{\nabla_{e_1}e_2}G + \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_1}G - \widetilde{\nabla}_{\nabla_{e_2}e_1}G\right) \\ & + \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2}G - \widetilde{\nabla}_{\nabla_{e_2}e_2}G,\end{aligned}$$

where $\{e_1, e_2, G = e_3\}$ is the appropriate pseudo-orthonormal frame field. By a direct calculation, we obtain

$$\Box G = -4k\omega_2 e_2 - 2k^3 G.$$

By using the above equation and the Codazzi equations (3.5)-(3.6) we get (3.12) as $\varepsilon = -1$ for this case.

Case III. S is not diagonalizable and its characteristic polynomial is $\mathcal{Q}(\lambda) = (\lambda - k)^2 + b^2$ for some smooth functions k and b with $b(p) \neq 0$ for all $p \in M$. In this case, from (3.10) and (3.11) we have

$$\begin{aligned}\Box G = k\Big(&-\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1}G + \widetilde{\nabla}_{\nabla_{e_1}e_1}G + \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2}G - \widetilde{\nabla}_{\nabla_{e_2}e_2}G\Big)\\ &- b\Big(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_2}G - \widetilde{\nabla}_{\nabla_{e_1}e_2}G + \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_1}G - \widetilde{\nabla}_{\nabla_{e_2}e_1}G\Big),\end{aligned}$$

where $\{e_1, e_2, G = e_3\}$ is the appropriate orthonormal frame field. After a long calculation, we obtain

$$\begin{aligned}\Box G = &\Big(k(e_1(k) - e_2(b)) + b(e_1(b) + e_2(k)) - 2b^2\omega_2 - 2bk\omega_1\Big)e_1\\ &+ \Big(-k(e_2(k) + e_1(b)) - b(e_2(b) - e_1(k)) + 2b^2\omega_1 - 2bk\omega_2\Big)e_2\\ &+ \Big(-2k^3 - 2kb^2\Big)G.\end{aligned}$$

From this equation and the Codazzi equations (3.8)-(3.9) we get (3.12) as $\varepsilon = -1$ for this case. $\Box$

Next we will give the definition concerning $\Box$-pointwise 1-type Gauss map.

Definition 3.2 ([21])**.** A surface M in the Minkowski 3-space $\mathbb{E}^3_1$ is said to have $\Box$-pointwise 1-type Gauss map if its Gauss map satisfies (1.3) for a non-zero smooth function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3_1$. More precisely, a $\Box$-pointwise 1-type Gauss map is said to be of the first kind if (1.3) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. Moreover, if (1.3) is satisfied for a constant function f, then we say M has $\Box$-(global) 1-type Gauss map. Furthermore, M is said to have proper $\Box$-pointwise 1-type Gauss map if its Gauss map satisfies (1.3) for a non-constant function f.

Remark 3.3. When the Gauss map G is $\Box$-harmonic, i.e., $\Box G = 0$, we regard G to be of $\Box$-pointwise 1-type of the second kind.

Theorem 3.4. *Let M be a surface in $\mathbb{E}^3_1$. Then, it has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Proof. The proof directly follows from Lemma 3.1. $\Box$

Remark 3.5. A surface with diagonalizable shape operator in $\mathbb{E}^3_1$ is isoparametric if and only if it is congruent to one of the following surfaces:

(i) An Euclidean plane $\mathbb{E}^2$,
(ii) A Lorentzian plane $\mathbb{E}^2_1$,

(iii) A cylinder $\mathbb{S}^1(r) \times E^1_1$, where $\mathbb{S}^1(r)$ is a circle in $\mathbb{E}^2$ with radius r,
(iv) A cylinder $\mathbb{S}^1_1(r) \times \mathbb{E}^1$, where $\mathbb{S}^1_1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\beta(s) = (r\sinh s, r\cosh s)$,
(v) A cylinder $\mathbb{H}^1(r) \times \mathbb{E}^1$, where $\mathbb{H}^1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\tilde{\beta}(s) = (r\cosh s, r\sinh s)$,
(vi) The de-Sitter space $\mathbb{S}^2_1(r^2)$,
(vii) The hyperbolic space $\mathbb{H}^2(-r^2)$.

Considering Remark 3.3 and Theorem 3.4.

Corollary 3.6. *A surface with diagonalizable shape operator and $\Box$-pointwise 1-type Gauss map of the first kind in $\mathbb{E}^3_1$ if and only if it is congruent to one of the following surfaces*:

(i) *A cylinder $\mathbb{S}^1(r) \times \mathbb{E}^1_1$, where $\mathbb{S}^1(r)$ is a circle in $\mathbb{E}^2$ with radius r,*
(ii) *A cylinder $\mathbb{S}^1_1(r) \times \mathbb{E}^1$, where $\mathbb{S}^1_1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\beta(s) = (r\sinh s, r\cosh s)$,*
(iii) *A cylinder $\mathbb{H}^1(r) \times \mathbb{E}^1$, where $\mathbb{H}^1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\tilde{\beta}(s) = (r\cosh s, r\sinh s)$,*
(iv) *The de-Sitter space $\mathbb{S}^2_1(r^2)$,*
(v) *The hyperbolic space $\mathbb{H}^2(-r^2)$.*

3.1. Lorentzian surfaces with non-diagonalizable shape operator

In this section, we give some characterization and classification theorems for the surfaces with non-diagonalizable shape operator and $\Box$-pointwise 1-type Gauss map.

Example 3.7 ([19]). Let $\beta(s)$ be a null curve in $\mathbb{E}^3_1$ with a Cartan frame $\{A, B, C\}$ such that $\langle A, A\rangle = \langle B, B\rangle = 0$, $\langle A, B\rangle = -1$, $\langle A, C\rangle = \langle B, C\rangle = 0$ and $\langle C, C\rangle = 1$ with $\beta' = A$, $A' = k_1(s)C$ and $B' = k_2C$ for a constant k_2 and a smooth function k_1 which is vanishing only on a subset $\mathcal{U}$ with $\mathrm{int}U = \emptyset$. Then, the surface M given by

$$f(s,t) = \beta(s) + tB(s) \tag{3.13}$$

is said to be a B-scroll. Note that in [23], M. Magid have proved that a surface in $\mathbb{E}^3_1$ with non-diagonalizable shape operator is isoparametric if and only if it is a B-scroll.

The Gauss map of M is

$$G = -k_2tB - C$$

and the matrix representation of the shape operator S of M with respect to the pseudo-orthonormal frame field $\{\partial_s + (t^2k_2/2)\partial_t, \partial_t, G\}$ is

$$S = \begin{pmatrix} k_2 & 0 \\ k_1(s) & k_2 \end{pmatrix}.$$

Thus, the mean curvature and Gaussian curvature of M are given by $H = k_2$ and $K = k_2^2$, which imply $\Box G = -2k_2^3 G$. Hence, M has $\Box$-(global) 1-type Gauss map of the first kind when $k_2 \neq 0$.

Note that a B-scroll given by (3.13) is minimal if and only if $k_2 = 0$. In that case, we have $B' = 0$. Thus, up to congruency, we may assume $B = (1, 1, 0)$ which implies $A(\hat{s}) = (\hat{s}, \hat{s} - 1, (2\hat{s} - 1)^{1/2})$ and $\beta(\hat{s}) = \big(\hat{s}^2/2, \hat{s}^2/2 - \hat{s}, \big((2\hat{s} - 1)^{3/2}\big)/3\big) + c_0$ for a constant vector c_0 and up to translations, one can assume $c_0 = 0$. Thus the position vector of the B-scroll becomes

$$f(\hat{s}, t) = \left(\frac{\hat{s}^2}{2} + t, \frac{\hat{s}^2}{2} - \hat{s} + t, \frac{(2\hat{s} - 1)^{3/2}}{3}\right). \tag{3.14}$$

Remark 3.8. In [23], M. Magid proved that the shape operator of an isoparametric hypersurface of a Minkowski space cannot have complex eigenvalues. Thus, a minimal isoparametric surface in $\mathbb{E}^3_1$ is either an open part of a non-degenerate plane or congruent to the surface given in (3.14).

Theorem 3.9. *Let M be a surface in $\mathbb{E}^3_1$ with non-diagonalizable shape operator whose characteristic polynomial is of the form of $\mathcal{Q}(\lambda) = (\lambda - k)^2$ for a function k. Then, the followings are logically equivalent*:

(i) *M has $\Box$-pointwise 1-type Gauss map.*
(ii) *M has (Δ-)pointwise 1-type Gauss map.*
(iii) *M has $\Box$-(global) 1-type Gauss map of the first kind or $\Box G = 0$.*
(iv) *M has constant Gaussian curvature, i.e., k is constant.*
(v) *M is a B-scroll given by* (3.13) *in Example 3.7.*

Proof. Let $\{e_1, e_2, e_3\}$ be the pseudo-orthonormal frame field such that the corresponding matrix representation of the shape operator S of M is of the form of

$$S = \begin{pmatrix} k & 0 \\ 1 & k \end{pmatrix}.$$

(i)⇔(ii) The Laplace operator Δ acting on $\chi(M)$ can be written as

$$\Delta = \widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_2} + \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_1} - \widetilde{\nabla}_{\nabla_{e_1}e_2} - \widetilde{\nabla}_{\nabla_{e_2}e_1}.$$

From this equation and (3.7) we have $\Box = k\Delta - \widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2} + \widetilde{\nabla}_{\nabla_{e_2}e_2}$. In addition, we have $-\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2}G + \widetilde{\nabla}_{\nabla_{e_2}e_2}G = 0$. Therefore, we have

$$\Box G = -k\Delta G.$$

(iii)⇒ (i) is obvious and (iii)⇔(iv) directly follows from Lemma 3.1 and Codazzi equations (3.5)-(3.6).

(i)⇒ (iii) Suppose M has pointwise 1-type Gauss map. Then, there exist a smooth function f and a constant vector $C = C_1e_1 + C_2e_2 + C_3G$ such that $\Box G = f(G + C)$, where $\{e_1, e_2, G\}$ is the pseudo-orhonormal frame field defined on Case II. Note that we have $C_i = -\langle C, e_{3-i}\rangle$, $i = 1, 2$, $C_3 = \langle C, G\rangle$.

Suppose that $C \neq 0$. From (3.5)-(3.6) and (3.12) one can obtain $f(G+C_1e_1+C_2e_2+C_3G) = -4k\omega_2 e_2 - 2k^3G$. On the open subset $\mathcal{O}_1 = \{p \in M : f(p) \neq 0\}$, $C_1 = -\langle C, e_2\rangle = 0$. As $e_1(C_1) = -\langle C, \widetilde{\nabla}_{e_1} e_2\rangle$, we have

$$kC_3 = 0. \tag{3.15}$$

Now, consider the open subset $\mathcal{O}_2 = \{p \in \mathcal{O}_1 : k(p) \neq 0\}$. If the open set $\mathcal{O}_2$ is not empty, then we have $C_3 = 0$ on $\mathcal{O}_2$, On $\mathcal{O}_2$, we have $e_1(C_3) = -\langle C, \widetilde{\nabla}_{e_1} G\rangle = \langle C, ke_1 + e_2\rangle = kC_2 = 0$ and thus $C_2 = 0$, a contradiction. Thus, $\mathcal{O}_2$ is empty and $k = 0$ on $\mathcal{O}_1$. Hence, $\Box G = 0$ on $\mathcal{O}_1$. Therefore, $\Box G = 0$ on M.

If $C = 0$, then $\Box G = -2k^3G$ and $\nabla K = \nabla(k^2) = 0$. It follow that M has $\Box$-(global) 1-type Gauss map of the first kind.

(iv)⇔(v) was proved in [23]. □

We will use the following proposition later:

Proposition 3.10. *Let M be a minimal surface in $\mathbb{E}^3_1$. Then, M has $\Box$-pointwise 1-type Gauss map if and only if it is either an open part of a non-degenerate plane or congruent to the surface given by* (3.14).

Proof. If the mean curvature of M vanishes identically, then (3.12) implies

$$\Box G = -\nabla K. \tag{3.16}$$

In order to prove the necessary condition, we assume that M has $\Box$-pointwise 1-type Gauss map. Then, there exist a smooth function f and a constant vector $C = \varepsilon C_1 e_1 + c_2 e_2 - \varepsilon C_3 G$ such that (1.3) is satisfied. From (1.3) and (3.16) we have $-\nabla K = f(G + C)$ which implies $C_3 = \varepsilon$ on M. Thus, we have $\langle G, C\rangle = -\langle G, G\rangle = \varepsilon$ and we get

$$\langle C, \widetilde{\nabla}_{e_i} G\rangle = 0, \quad i = 1, 2, \tag{3.17}$$

where e_1 and e_2 form an appropriated orthonormal (or pseudo-orthonormal) frame field.

Now, we consider the open subset $\mathcal{O} = \{p \in M : K(p) \neq 0\}$ of M. Note that if we prove that $K \equiv 0$ on M, then M will be either an open part of a non-degenerate plane or congruent to the surface given in (3.14) because of Remark 3.8. Thus, we will prove that $\mathcal{O}$ is empty.

On $\mathcal{O}$, the shape operator S of M is of the form of either $S = \text{diag } (k, -k)$, or

$$S = \text{adiag}(k, -k) = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

for a function k. For both cases, (3.17) imply $kC_1 = kC_2 = 0$ which implies $C_1 = C_2 = 0$. Therefore, we have $C = -\varepsilon G$, i.e., the Gauss map of $\mathcal{O}$ is constant. However, this implies $\mathcal{O}$ is an open part of a plane which is a contradiction. Thus, we have $\mathcal{O} = \emptyset$, i.e., $K \equiv 0$ on M.

Conversely, if M is an open part of a non-degenerate plane, then we have $\Box G = 0$ from (3.12). Hence, (1.3) is satisfied for an arbitrary non-zero smooth function f and $C = -G$. Furthermore, the surface given by (3.14) has also $\Box$-pointwise 1-type Gauss map because of Theorem 3.9. $\Box$

Theorem 3.11. *Let M be a surface in $\mathbb{E}^3_1$ with constant mean curvature and non-diagonalizable shape operator whose characteristic polynomial has complex roots. Then, M has $\Box$-pointwise 1-type Gauss map if and only if it has proper $\Box$-pointwise 1-type Gauss map of the second kind.*

Proof. If the characteristic polynomial of the shape operator S of M has complex roots, then it satisfies $Se_1 = ke_1 - be_2$ and $Se_2 = be_1 + ke_2$ for an appropriate orthogonal frame field $\mathcal{B} = \{e_1, e_2, e_3\}$ and a smooth function k and a nowhere vanishing smooth function b. Since M has constant mean curvature and we have

$$H = k, \tag{3.18}$$

we see that k is constant. Thus, from (3.12) and Codazzi equations (3.8)-(3.9) we obtain

$$\Box G = -4b^2\omega_2 e_1 + 4b^2\omega_1 e_2 - 2k(k^2 + b^2)G. \tag{3.19}$$

Note that if M has $\Box$-pointwise 1-type Gauss map of the first kind, then (3.12) implies its Gaussian curvature $K = k^2 + b^2$ is constant. Thus, M is an isoparametric surface. However, this is not possible as the characteristic polynomial of shape operator M has complex roots (see, [23, Theorem 4.10]).

Now, we assume that M has $\Box$-(global) 1-type Gauss map of the second kind. Then, there exist a constant λ and a constant vector $C = -C_1e_1 + C_2e_2 + C_3G$ such that (1.3) is satisfied for the constant function $f = \lambda$. From (1.3) and (3.19) we obtain

$$\lambda C_1 = 4b^2\omega_2, \tag{3.20}$$

$$\lambda C_2 = 4b^2\omega_1, \tag{3.21}$$

$$\lambda(C_3 + 1) = -2k(k^2 + b^2). \tag{3.22}$$

Moreover, Proposition 3.10 implies that $k \neq 0$ because of (3.18).

As $C_3 = \langle C, G\rangle$, by using (3.20)-(3.21) we obtain

$$\lambda e_1(C_3) = 4b^3\omega_1 - 4kb^2\omega_2, \tag{3.23}$$

$$\lambda e_2(C_3) = -4kb^2\omega_1 - 4b^3\omega_2. \tag{3.24}$$

On the other hand, (3.22) and Codazzi equations (3.8)-(3.9) imply

$$\lambda e_1(C_3) = 8kb^2\omega_2, \tag{3.25}$$

$$\lambda e_2(C_3) = 8kb^2\omega_1. \tag{3.26}$$

Thus, (3.23)-(3.26) imply

$$\begin{pmatrix} 3\omega_1 & \omega_2 \\ 3\omega_2 & -\omega_1 \end{pmatrix} \begin{pmatrix} k \\ b \end{pmatrix} = 0. \tag{3.27}$$

As $k \neq 0$ and b is nowhere vanishing, this equation implies $\omega_1 = \omega_2 = 0$ on M. Codazzi equations (3.8)-(3.9) imply that b is constant. Therefore, we have M is isoparametric which is a contradiction ([23, Theorem 4.10]).

Hence, we have if M has $\Box$-pointwise 1-type Gauss map, then it must have proper $\Box$-pointwise 1-type Gauss map of the second kind. The converse is obvious. $\Box$

3.2. *CDPC*-surfaces

In this subsection, we work on surfaces with constant difference of the principal curvatures in the Minkowski 3-space $\mathbb{E}^3_1$, which are abbreviated by $CDPC$-surfaces. Note that the complete classification of helicoidal $CDPC$-surfaces was given in [17].

We give the following theorem:

Theorem 3.12. *Let M be an $CDPC$-surface in $\mathbb{E}^3_1$ with non-diagonalizable shape operator. Then, M has $\Box$-pointwise 1-type Gauss map if and only if it has proper $\Box$-pointwise 1-type Gauss map of the second kind.*

Proof. Because of the hypothesis, the characteristic polynomial of the shape operator S of M is of the form of $\mathcal{Q}(\lambda) = (\lambda - k)^2 + b_0^2$ for a function k and a non-zero constant b_0. Therefore, the gradient of the Gaussian curvature K of M becomes $\nabla K = -e_1(k^2)e_1 + e_2(k^2)e_2 = 4kb_0(\omega_1 e_1 + \omega_2 e_2)$ for an appropriate orthonormal frame $\mathcal{B} = \{e_1, e_2, e_3\}$ because of Codazzi equations (3.8)-(3.9). Thus, (3.12) implies

$$\Box G = -4kb_0\omega_1 e_1 - 4kb_0\omega_2 e_2 - 2k(k^2 + b_0^2)G. \tag{3.28}$$

Note that if M has $\Box$-pointwise 1-type Gauss map of the first kind, then Theorem 3.4 implies that $K = k^2 + b_0^2$ is constant which implies that $H = k$ is also a constant. It is not possible because of the same reason we mentioned on the proof of Theorem 3.11.

Now, we assume that M has $\Box$-(global) 1-type Gauss map of the second kind. Then, there exist a constant vector $C = -C_1 e_1 + C_2 e_2 + C_3 e_3$ and a constant λ such that $\lambda(G + C) = -4kb\omega_1 e_1 - 4kb\omega_2 e_2 - 2k(k^2 + b_2)G$. Therefore, we have

$$\lambda C_1 = -4kb_0\omega_1, \tag{3.29}$$

$$\lambda C_2 = -4kb_0\omega_2, \tag{3.30}$$

$$\lambda(C_3 + 1) = -2k(k^2 + b_0^2). \tag{3.31}$$

By a similar calculation in the proof of Theorem 3.11, we obtain

$$\begin{pmatrix} 2k^2 + b_0^2 & kb_0 \\ -kb_0 & 2k^2 + b_0^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = 0. \tag{3.32}$$

Thus, at a point p of M we have either $(k^2 + b_0^2)^2 + k^2 b_0^2 = 0$ or $\omega_1 = \omega_2 = 0$. However, as $b_0 \neq 0$, we have $\omega_1 = \omega_2 = 0$ on M. Thus, Codazzi equations (3.8)-(3.9) imply that k is a constant, which yields a contradiction.

Hence, we have if M has $\Box$-pointwise 1-type Gauss map, then it must be proper $\Box$-pointwise 1-type Gauss map of the second kind. The converse is obvious. $\Box$

Next, we give the following theorem concerning $CDPC$-surfaces in $\mathbb{E}^3_1$ with diagonalizable shape operator.

Theorem 3.13. *Let M be an $CDPC$-surface in $\mathbb{E}^3_1$ with diagonalizable shape operator. Then, M has $\Box$-pointwise 1-type Gauss map of the first kind if and only if it is isoparametric.*

Proof. Proof directly follows from Theorem 3.4. $\Box$

Theorem 3.14. *Let M be an $CDPC$-surface in $\mathbb{E}^3_1$ with diagonalizable shape operator. Then, M has $\Box$-(global) 1-type Gauss map of the second kind if and only if it is congruent to a non-degenerate plane, a cylinder $\mathbb{S}^1(r) \times \mathbb{E}^1_1$, $\mathbb{S}^1_1(r) \times \mathbb{E}^1$ or $\mathbb{H}^1(r) \times \mathbb{E}^1$.*

Proof. If M is a $CDPC$-surface, then its principal curvatures can be put as k_1 and $k_2 = k_1 - c$ for a constant c and the gradient of its Gaussian curvature becomes $\nabla K = \nabla(k_1k_2) = \varepsilon c(k_1 + k_2)(\omega_2 e_1 + \omega_1 e_2)$ for an appropriated orthonormal frame field $\{e_1, e_2, G\}$. Thus, from (3.12) we have

$$\Box G = \varepsilon(k_1 + k_2)(-c\omega_2 e_1 - c\omega_1 e_2 + k_1k_2G). \tag{3.33}$$

Now, we assume that M has $\Box$-(global) 1-type Gauss map of the second kind, i.e., there exist a constant λ and a constant vector $C = \varepsilon C_1 e_1 + C_2 e_2 - \varepsilon C_3 G$ satisfying $\Box G = \lambda(G + C)$. From this equation and (3.33) we have

$$\lambda C_1 = -c\omega_2(k_1 + k_2), \tag{3.34}$$

$$\lambda C_2 = -\varepsilon c\omega_1(k_1 + k_2), \tag{3.35}$$

$$\lambda(1 - \varepsilon C_3) = \varepsilon k_1k_2(k_1 + k_2). \tag{3.36}$$

By using these equations and Codazzi equations (3.2)-(3.3), we obtain

$$\omega_1 c(k_1^2 + 2k_2^2 + 5k_1k_2) = 0, \tag{3.37}$$

$$\omega_2 c(2k_1^2 + k_2^2 + 5k_1k_2) = 0. \tag{3.38}$$

Note that if $c = 0$, then M is an open part of a non-degenerate plane, $\mathbb{S}^2_1(k_1^{-2})$ or $\mathbb{H}^2(-k_1^{-2})$. Since $\mathbb{S}^2_1(k_1^{-2})$ or $\mathbb{H}^2(-k_1^{-2})$ have $\Box$-(global) 1-type Gauss map of the first kind, M is an open part of a non-degenerate plane.

Now, suppose that $c \neq 0$. If $\omega_1 = \omega_2 = 0$ on an open subset $\mathcal{O}_1$ of M, then (3.2)-(3.3) implies that k_1 and k_2 are constants. Then $\mathcal{O}_1$ is an open part of an isoparemetric surface and continuity implies that $\mathcal{O}_1 = M$. On the other hand, if $k_1^2 + 2k_2^2 + 5k_1k_2 = 0$ or $2k_1^2 + k_2^2 + 5k_1k_2 = 0$ on an open subset $\mathcal{O}_2$ of M, then again we have k_1 and k_2 are constants on $\mathcal{O}_2$. Thus, $\mathcal{O}_2$ is an open part of an isoparemetric surface because of the hypothesis, and we have $\mathcal{O}_2 = M$. Hence, the proof is completed by considering Remark 3.5.

Conversely, if M is an open part of surfaces given in the theorem, then its Gauss map satisfies $\Box G = 0$. Obviously, M has $\Box$-(global) 1-type Gauss map of the second kind in this case. □

3.3. Surfaces with constant mean curvature

In [21], we focused on surfaces of Euclidean 3-space with constant mean curvature. We proved that such a surface has $\Box$-(global) 1-type Gauss map if and only if it is isoparametric. With the same techniques, the following proposition can be proved when the ambient space is Minkowski 3-space:

Proposition 3.15. *Let M be a surface in $\mathbb{E}^3_1$ with diagonalizable shape operator and constant mean curvature. Then, M has $\Box$-(global) 1-type Gauss map if and only if it is isoparametric.*

By combining Theorem 3.9, Theorem 3.11 and Proposition 3.15 we state:

Theorem 3.16. *Let M be a surface in $\mathbb{E}^3_1$ with constant mean curvature. Then, M has $\Box$-(global) 1-type Gauss map if and only if it is isoparametric.*

4. Rotational surfaces with $\Box$-pointwise 1-type Gauss map

In this section, we focus on the rotational surfaces in Minkowski 3-space $\mathbb{E}^3_1$ with space-like axis of rotation. We give a classification of such surfaces with $\Box$-pointwise 1-type Gauss map of the second kind. Note that if M is a flat surface, then its Gauss map satisfies $\Box G = 0$. In this case, M obviously has $\Box$-pointwise 1-type Gauss map of second kind. Thus, we will focus on non-flat rotational surfaces.

Without loss of generality, we assume the axis of rotation is the line $L = \{(0, 0, t) : t \in \mathbb{R}\}$. If the profile curve $\beta_1 = (x, 0, z)$ is contained in the x_1x_3-plane, then the position vector f_1 of the rotational surface becomes

$$f_1(s,t) = (x(s)\cosh t, x(s)\sinh t, z(s))\,. \tag{4.1}$$

On the other hand, if the profile curve $\beta_2 = (0, y, z)$ is contained in the x_2x_3-plane, then the position vector f_2 of the rotational surface becomes

$$f_2(s,t) = (y(s)\sinh t, y(s)\cosh t, z(s))\,. \tag{4.2}$$

4.1. Rotational surfaces given by (4.1)

Let M be a rotational surface in $\mathbb{E}^3_1$ given by (4.1). Without loss of generality, we may assume $x > 0$ and $z'^2 - x'^2 = \varepsilon$, where

$$\varepsilon = \begin{cases} 1 & \text{if } M \text{ is Riemannian} \\ -1 & \text{if } M \text{ is Lorentzian.} \end{cases}$$

The Gauss map of M becomes

$$G = (z'\cosh t, z'\sinh t, x') \tag{4.3}$$

and the Gaussian curvature of M is given by

$$K = \frac{x''}{x}. \tag{4.4}$$

By a direct calculation and using (3.12), we obtain

$$\Box G = -\varepsilon\left(\frac{x''}{x}\right)'\partial_s - \varepsilon\frac{x''}{x}\left(\theta' + \frac{z'}{x}\right)G, \tag{4.5}$$

where θ is a function satisfying $z' = \cosh\theta$ and $x' = \sinh\theta$ if $\varepsilon = 1$ or $x' = \cosh\theta$ and $z' = \sinh\theta$ if $\varepsilon = -1$.

Now, we suppose that M has $\Box$-pointwise 1-type Gauss map of the second kind. Then, there exist a smooth function f and a constant vector $C = (c_1, c_2, c_3) \neq 0$ such that (1.3) is satisfied. From (1.3) and (4.5) we have $\langle C, \partial_t\rangle = 0$ which implies $c_1 = c_2 = 0$. Thus, we have $C = \varepsilon(C_1\partial_s - C_3 G) = (0, 0, c)$ for a constant $c \neq 0$ and $C_1 = \langle C, \partial_s\rangle = cz'$, $C_3 = \langle C, G\rangle = cx'$. Therefore, (1.3) and (4.5) imply

$$-\varepsilon\left(\frac{x''}{x}\right)'\partial_s - \varepsilon\frac{x''}{x}\left(\theta' + \frac{z'}{x}\right)G = f\big(G + \varepsilon(cz'\partial_s - cx'G)\big),$$

$$fcz' = -\left(\frac{x''}{x}\right)',$$

$$f(1 - \varepsilon cx') = -\varepsilon\frac{x''}{x}\left(\theta' + \frac{z'}{x}\right).$$

By a further calculation, we see that these equations imply

$$xx'''(\varepsilon cx' - 1) + \varepsilon cxx''^2 + x''(x' + c) = 0.$$

By integrating this equation, we obtain

$$xx''(\varepsilon cx' - 1) - \varepsilon\frac{c}{3}x'^3 + x'^2 + cx' = A, \tag{4.6}$$

where A is a constant.

Therefore, we state:

Lemma 4.1. *Let M be a rotational surface in $\mathbb{E}^3_1$ given by (4.1). Then, it has $\Box$-pointwise 1-type Gauss map of the second kind if and only if the function x satisfies (4.6) for some constants A and $c \neq 0$. In this case, (1.3) is satisfied for $C = (0, 0, c)$.*

Next, we use the transformation

$$x = e^Y, \quad s = \phi e^Y \tag{4.7}$$

in (4.6), and obtain

$$(\varepsilon c - \psi)\dot{\psi} = A\psi^4 + c\psi^3 + \psi^2 - \varepsilon\frac{c}{3}\psi, \tag{4.8}$$

where $\cdot$ denote the ordinary differentiation with respect to Y and $\psi = \phi + \dot{\phi}$. Note that the transformation given by (4.7) implies

$$x' = \frac{1}{\psi}. \tag{4.9}$$

Consider the subset $\mathcal{U}$ of the points of M such that $A\psi^4+c\psi^3+\psi^2-\varepsilon\frac{c}{3}\psi = 0$. Suppose $\mathrm{int}\,\mathcal{U} \neq \emptyset$. On a component $\mathcal{O}$ of $\mathcal{U}$, we have ψ is a constant. Thus, (4.4) and (4.9) imply that $\mathcal{O}$ is flat. Moreover, because of the continuity, we have $\mathcal{O} = M$. Therefore, if we assume that M is not flat, then we also locally suppose that $A\psi^4 + c\psi^3 + \psi^2 - \varepsilon\frac{c}{3}\psi \neq 0$. In this case, we obtain $\psi = \eta^{-1}(Y)$, where η is a function defined by

$$\eta(\psi) = \int_{\psi_0}^{\psi} \frac{\varepsilon c - \xi}{\xi(A\xi^3 + c\xi^2 + \xi - \varepsilon\frac{c}{3})} d\xi \tag{4.10}$$

due to (4.8). Thus, from (4.9) we have $x' = \frac{1}{\eta^{-1}(\ln x)}$ which implies $\frac{dz}{dx} = \sqrt{\varepsilon(\eta^{-1}(\ln x))^2 + 1}$. Hence, we obtain the following theorem:

Theorem 4.2. *Let M be a non-flat rotational surface given by* (4.1). *Then, M has $\Box$-pointwise 1-type Gauss map of the second kind if and only if it is locally congruent to the surface given by*

$$F(x,t) = \left(x\cosh t, x\sinh t, \int_{x_0}^{x} \sqrt{\varepsilon(\eta^{-1}(\ln x))^2 + 1} d\xi\right) \tag{4.11}$$

for some constants x_0, A and $c \neq 0$, where η is the function defined by (4.10).

4.2. Rotational surfaces given by (4.2)

Let M be a rotational surface in $\mathbb{E}^3_1$ given by (4.2). Without loss of generality, we assume $y'^2 + z'^2 = 1$ and $y > 0$. The Gauss map of M is given by

$$G = (z'\sinh t, z'\cosh t, -y'), \tag{4.12}$$

which satisfies

$$\Box G = \left(\frac{y''}{y}\right)' \partial_s - \frac{y''}{y}\left(\theta' + \frac{z'}{y}\right) G, \tag{4.13}$$

where θ is a function satisfying $y' = \cos\theta$ and $z' = \sin\theta$.

By a similar way described in the previous subsection, we obtain:

Lemma 4.3. *Let M be a rotational surface in $\mathbb{E}^3_1$ given by* (4.2). *Then, it has $\Box$-pointwise 1-type Gauss map of the second kind if and only if the function y satisfies*

$$yy''(cy' - 1) - \frac{c}{3}y'^3 + y'^2 - cy' = A \tag{4.14}$$

for some constants A and $c \neq 0$. In this case, equation (1.3) *is satisfied for the constant vector $C = (0, 0, c)$.*

By using the same method we used in the previous subsection, we obtain the following theorem:

Theorem 4.4. *Let M be a non-flat rotational surface given by* (4.2). *Then, M has $\Box$-pointwise 1-type Gauss map of the second kind if and only if it is locally congruent to the surface given by*

$$F(x,t)=\left(y\sinh t, y\cosh t, \int_{y_0}^{y}\sqrt{(\tilde{\eta}^{-1}(\ln y))^2-1}d\xi\right) \tag{4.15}$$

for some constants y_0, A and $c\neq 0$, where $\tilde{\eta}$ is the function defined by

$$\tilde{\eta}(\psi)=\int_{\psi_0}^{\psi}\frac{c-\xi}{\xi(A\xi^3-c\xi^2+\xi-\frac{c}{3})}d\xi. \tag{4.16}$$

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