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EXISTENCE OF THREE POSITIVE SOLUTIONS OF A CLASS OF BVPS FOR SINGULAR SECOND ORDER DIFFERENTIAL SYSTEMS ON THE WHOLE LINE

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Abstract. This paper is concerned with a kind of boundary value problem for singular second order differential systems with Laplacian operators. Using a multiple fixed point theorem, sufficient conditions to guarantee the existence of at least three positive solutions of this kind of boundary value problem are established. An example is presented to illustrate the main results.

1. Introduction

In recent years, existence of solutions of multi-point boundary-value problems (BVPs for short) for second order differential equations or higher order differential equations on finite interval has been studied by different authors, see papers [4–9, 13–25, 28–33]. Different methods are used in these papers such as the Guo-Krasnoselskii fixed point theorem [7], the fixed-point theorem due to Avery and Peterson [3], the Leggett-Williams fixed point theorem [2, 3], the upper and lower solution methods with monotone iterative techniques [30, 31], the five functional fixed point theorem [1], and the Mawhin coincidence degree theory [7], see for examples Ma in [8, 26, 27] and the text book [7]. In many applications, BVPs consist of differential equations coupled with nonhomogeneous BCs, see [21, 22]. Papers [10–12] may be first group of papers concerned with BVPs with two parameter multi-point non-homogeneous BCs for second order ordinary differential equations. In these papers, the existence of lower and upper solutions with certain relations are supposed.

In [4,14,16,17], by using the upper and lower solution method and fixed point index theory, the authors studied the existence, non-existence and multiplicity

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of positive solutions of the following BVP of second order differential systems

$$\begin{cases} [\phi_p(u'(t))]' + \lambda h_1(t) f(u(t), v(t)) = 0, & t \in (0,1), \\ [\phi_p(v'(t))]' + \mu h_2(t) g(u(t), v(t)) = 0, & t \in (0,1), \\ u(0) = a \geq 0,\ v(0) = b \geq 0,\ u(1) = 0,\ v(1) = 0, \end{cases}$$

where $\phi_p(x) = |x|^{p-2}x$ with $p > 1$, λ, μ are non-negative real parameters, $h_1, h_2 \in C((0,1),(0,+\infty))$, $f, g \in C([0,\infty) \times [0,+\infty),[0,+\infty))$, $h_i (i = 1,2)$ are supposed to be singular at $t = 0$ in [16], while $h_i \in C([0,1],[0,+\infty))$ and $\phi_2(x) = x$ in [4].

Motivated by [4,14,16], the purpose of this paper is to investigate the following BVP for singular second order differential systems on the whole line with Laplacian operators

$$\begin{cases} (p(t)\phi(x'(t)))' + f(t, y(t), y'(t)) = 0, & t \in \mathbb{R}, \\ (q(t)\psi(y'(t)))' + g(t, x(t), x'(t)) = 0, & t \in \mathbb{R}, \\ \lim_{t \to -\infty} x(t) = 0, \\ \lim_{t \to +\infty} x(t) = 0, \\ \lim_{t \to -\infty} y(t) = 0, \\ \lim_{t \to +\infty} y(t) = 0, \end{cases} \tag{1.1}$$

where

• f is a p-Caratheodory function and g a q-Caratheodory function (see the definitions in Section 2) and f, g are nonnegative functions;

• both p and q are continuous on $\mathbb{R}$, nonnegative and may be zero at finitely many points;

• $\phi(x) = |x|^{m-2}x$ with $m > 1$ and $\psi(x) = |x|^{n-2}x$ with $n > 1$ are Laplacian operators and their inverse operators are denoted by ϕ^{-1} and ψ^{-1} respectively.

A pair of functions $x, y : \mathbb{R} \to \mathbb{R}$ is called a solution of BVP(1.1) if $x, y \in C^0(\mathbb{R})$, $x', y' \in C^0(\mathbb{R})$, $[p\phi(x')]', [q\phi(y')]' \in L^1(\mathbb{R})$ and all equations in (1.1) are satisfied, where $C^0(\mathbb{R})$ is the set of all functions continuous on R and $L^1(\mathbb{R})$ the set of all functions $\sigma : R \to R$ such that $\int_{-\infty}^{+\infty} |\sigma(s)| ds < +\infty$.

Sufficient conditions for the existence of at least three solutions of BVP(1.1) are established by using the five functional fixed point theorem [2]. The Green functions are not used in the proofs of the main results. Different from [4, 14, 16, 17], we don't use the assumptions of the existence of lower and upper solutions and the boundary value problems are defined on the whole lines which are non-compact spaces and furthermore, both $p(t)$ and $q(t)$ may equal to zero at finitely many points on $\mathbb{R}$.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results and their proofs are presented in Section 3, and an example is given in Section 4.

2. Preliminary results

For the reader's convenience, some background definitions in Banach spaces and an important fixed point theorem are presented.

As usual, let E be a real Banach space. The nonempty convex closed subset P of E is called a cone in E if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0$, and $x \in P$ and $-x \in P$ imply $x = 0$. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave (or convex) functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq \text{ (or } \leq \text{) } t\psi(x) + (1-t)\psi(y) \text{ for all } x, y \in P, \ t \in [0,1].$$

An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Let $c_1, c_2, c_3, c_4, c_5 > 0$ be positive constants, α_1, α_2 be two nonnegative continuous concave functionals on the cone P, $\beta_1, \beta_2, \beta_3$ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$P_{c_5} = \{x \in P : ||x|| < c_5\},$$
$$P(\beta_1, \alpha_1; c_2, c_5) = \{x \in P : \alpha_1(x) \geq c_2, \ \beta_1(x) \leq c_5\},$$
$$P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) = \{x \in P : \alpha_1(x) \geq c_2, \ \beta_3(x) \leq c_4, \ \beta_1(x) \leq c_5\},$$
$$Q(\beta_1, \beta_2; , c_1, c_5) = \{x \in P : \beta_2(x) \leq c_1, \ \beta_1(x) \leq c_5\},$$
$$Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) = \{x \in P : \alpha_2(x) \geq c_3, \ \beta_2(x) \leq c_1, \ \beta_1(x) \leq c_5\}.$$

Lemma 2.1 ([2]). *Let E be a real Banach space, P be a cone in E, α_1, α_2 be two nonnegative continuous concave functionals on the cone P, β_1, β_2, β be three nonnegative continuous convex functionals on the cone P. Suppose*

(A1) *$T : E \to E$ is a completely continuous operator*;

(A2) *there exists a constant $M > 0$ such that*

$$\alpha_1(x) \leq \beta_2(x), \ ||x|| \leq M\beta_1(x) \textit{ for all } x \in P;$$

(A3) *there exist positive numbers c_1, c_2, c_3, c_4, c_5 with $c_1 < c_2$ such that*

(i) $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;

(ii) $\{y \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) | \alpha_1(x) > c_2\} \neq \emptyset$ *and*

$$\alpha_1(Tx) > c_2 \textit{ for every } x \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5);$$

(iii) $\{y \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) | \beta_2(x) < c_1\} \neq \emptyset$ *and*

$$\beta_2(Tx) < c_1 \textit{ for every } x \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5);$$

(iv) $\alpha_1(Ty) > c_2$ *for* $y \in P(\beta_1, \alpha_1; c_2, c_5)$ *with* $\beta_3(Ty) > c_4$;

(v) $\beta_2(Tx) < c_1$ *for each* $x \in Q(\beta_1, \beta_2; c_1, c_5)$ *with* $\alpha_2(Tx) < c_3$.

Then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta_2(y_1) < c_1, \ \alpha_1(y_2) > c_2, \ \beta_2(y_3) > c_1, \ \alpha_1(y_3) < c_2.$$

Definition 2.1. A function $F : \mathbb{R}^3 \to \mathbb{R}$ is called a p-Carathéodory function if

(i) $t \to F\left(t, x, \frac{1}{\phi^{-1}(p(t))} y\right)$ is measurable on $\mathbb{R}$ for each $x, y \in \mathbb{R}$;

(ii) $(x,y) \to F\left(t, x, \frac{1}{\phi^{-1}(p(t))}y\right)$ is continuous on $\mathbb{R}^2$ for almost all $t \in \mathbb{R}$;

(iii) for each $r > 0$ there exists a function $\phi_r \in L^1(\mathbb{R})$ such that

$$\left|F\left(t, x, \frac{1}{\phi^{-1}(p(t))}y\right)\right| \leq \phi_r(t),\ t \in \mathbb{R}, |x|, |y| \leq r.$$

Definition 2.2. A function $G : \mathbb{R}^3 \to \mathbb{R}$ is called a q-Carathéodory function if

(i) $t \to G\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)$ is measurable on $\mathbb{R}$ for each $x, y \in \mathbb{R}$;

(ii) $(x,y) \to G\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)$ is continuous for almost all $t \in \mathbb{R}$;

(iii) for each $r > 0$ there exists a function $\phi_r \in L^1(\mathbb{R})$ such that

$$\left|G\left(t, x, \frac{1}{\psi^{-1}(q(t))}y\right)\right| \leq \phi_r(t),\ t \in \mathbb{R}, |x|, |y| \leq r.$$

Suppose that

(B1) $f : \mathbb{R} \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is a p-Carathéodory function with $f(t, 0, 0) \not\equiv 0$ on each sub-interval on $\mathbb{R}$;

$g : \mathbb{R}\times[0, +\infty)\times\mathbb{R} \to [0, +\infty)$ is a q-Carathéodory function with $g(t, 0, 0) \not\equiv 0$ on each sub-interval on $\mathbb{R}$;

(B2) $p, q : \mathbb{R} \to [0, +\infty)$ are continuous with $\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds < +\infty$ and $\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{1}{q(s)}\right) ds < +\infty$;

(B3) $\sigma, \varrho : \mathbb{R} \to [0, +\infty)$ are continuous functions and $\sigma, \varrho \in L^1(\mathbb{R})$ with $\varrho(t), \sigma(t) \not\equiv 0$ on each subinterval on $\mathbb{R}$.

Choose

$$X = \left\{ x \in C^0(\mathbb{R}) \begin{array}{l} \phi^{-1}(p)x' \in C^0(\mathbb{R}), \\ \text{the limits } \lim\limits_{t\to\pm\infty} \phi^{-1}(p(t))x'(t), \\ \text{and } \lim\limits_{t\to\pm\infty} x(t) \text{ are finite} \end{array} \right\}.$$

Define its norm by

$$||x|| = ||x||_X = \max\left\{\sup_{t\in\mathbb{R}} |x(t)|,\ \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|x'(t)|\right\}.$$

Choose

$$Y = \left\{ y \in C^0(\mathbb{R}) \begin{array}{l} \psi^{-1}(q)y' \in C^0(\mathbb{R}), \\ \text{the limits } \lim\limits_{t\to\pm\infty} \psi^{-1}(q(t))y'(t), \\ \text{and } \lim\limits_{t\to\pm\infty} y(t) \text{ are finite} \end{array} \right\}.$$

Define its norm by

$$||y|| = ||y||_Y = \max\left\{\sup_{t\in\mathbb{R}} |y(t)|,\ \sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|y'(t)|\right\}.$$

Lemma 2.2. *Suppose that* (B2) *holds. Then both* X *and* Y *are Banach spaces. So* $E = X \times Y$ *with the norm*

$$||(x, y)|| = \max\{||x||,\ ||y||\}$$

for $(x, y) \in E$ *is a Banach space.*

Proof. It is easy to see that X is a normed linear space. Let $\{x_n\}$ be a Cauchy sequence in X. Then

$$x_n, \phi^{-1}(p)x_n' \in C^0(\mathbb{R}),$$

$$\lim_{t\to\pm\infty} x_n(t),\ \lim_{t\to\pm\infty} \phi^{-1}(p(t))x_n'(t) \text{ exist},$$

$$||x_n - x_m|| = \max\left\{\sup_{t\in\mathbb{R}} |x_n(t) - x_m(t)|,\ m, n \to +\infty,\right.$$
$$\left.\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|x_n'(t) - x_m'(t)|\right\} \to 0, m, n \to +\infty.$$

It follows that

$$\sup_{t\in\mathbb{R}} |x_n(t) - x_m(t)| \to 0,\ \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|x_n'(t) - x_m'(t)| \to 0, m, n \to +\infty.$$

Then there exist two functions $x_0, y_0 : \mathbb{R} \to \mathbb{R}$ such that $x_n(t) \to x_0(t)$ and $\phi^{-1}(p(t))x_n'(t) \to y_0(t)$ as $n \to +\infty$. We need to prove that $y_0(t) = \phi^{-1}(p(t))x_0'(t)$ and $x_0 \in X$.

Step 1. Prove that $x_0 \in C^0(\mathbb{R})$.

For every $\epsilon > 0$, since $\sup_{t\in\mathbb{R}} |x_n(t) - x_m(t)| \to 0$ as $m, n \to +\infty$, then there exists N_1 such that $|x_n(t) - x_m(t)| < \epsilon$ for all $m, n > N_1$ and $t \in \mathbb{R}$. So $|x_n(t) - x_0(t)| \le \epsilon$ for all $n > N_1$ and $t \in R$. Then $x_n(t) \to x_0(t)$ as $n \to +\infty$ uniformly on $\mathbb{R}$.

For each $t_0 \in \mathbb{R}$, $x_n(t_0) \to x_0(t_0)$ as $n \to +\infty$ implies that there exists N_2 such that $|x_n(t_0) - x_0(t_0)| < \epsilon$ for all $n > N_2$. Since x_n is continuous at $t = t_0$, Thus for $n > \max\{N_1, N_2\}$, there exists $\delta > 0$ such that $|x_n(t) - x_n(t_0)| < \epsilon$ for all $|t - t_0| < \delta$. We have

$$|x_0(t) - x_0(t_0)| \le |x_0(t) - x_n(t)| + |x_n(t) - x_n(t_0)| + |x_n(t_0) - x_0(t_0)| \le 3\epsilon$$

for all $|t - t_0| < \delta$. It follows that x_0 is continuous at t_0. Thus $x_0 \in C^0(\mathbb{R})$.

Step 2. Prove that the limits $\lim_{t\to\pm\infty} x_0(t)$ exist.

For every $\epsilon > 0$, since $\sup_{t\in\mathbb{R}} |x_n(t) - x_m(t)| \to 0$ as $m, n \to +\infty$, then there exists N_1 such that $|x_n(t) - x_m(t)| < \epsilon$ for all $m, n > N_1$ and $t \in \mathbb{R}$. Let $t \to \pm\infty$, we get that $\left|\lim_{t\to\pm\infty} x_n(t) - \lim_{t\to\pm\infty} x_m(t)\right| < \epsilon$ for all $m, n > N_1$. Then the limit $\lim_{n\to+\infty} \lim_{t\to\pm\infty} x_n(t)$ exists.

Since $x_n(t) \to x_0(t)$ as $n \to +\infty$ uniformly on $\mathbb{R}$, we have that

$$\lim_{t\to\pm\infty} x_0(t) = \lim_{t\to\pm\infty} \lim_{n\to+\infty} x_n(t) = \lim_{n\to+\infty} \lim_{t\to\pm\infty} x_n(t).$$

Step 3. Prove that $y_0 \in C^0(\mathbb{R})$ and the limits $\lim\limits_{t\to\pm\infty} y_0(t)$ exist.

It is similar to those proofs in Steps 1 and 2.

Step 4. Prove that $y_0(t) = \phi^{-1}(p(t))x_0'(t)$ and $x_0 \in X$.

We have that

$$\begin{aligned}
&\left| x_n(t) - x_n(t_0) - \int_{t_0}^{t} \frac{y_0(s)}{\phi^{-1}(p(s))} ds \right| \\
\le &\left| \int_{t_0}^{t} \left| x_n'(s) - \frac{y_0(s)}{\phi^{-1}(p(s))} \right| ds \right| \\
= &\left| \int_{t_0}^{t} \frac{1}{\phi^{-1}(p(s))} \left| \phi^{-1}(p(s))x_n'(s) - y_0(s) \right| ds \right| \\
\le &\int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in R} \left| \phi^{-1}(p(t))x_n'(t) - y_0(t) \right| \\
\to &\ 0 \text{ as } n \to +\infty.
\end{aligned}$$

So

$$\lim_{n\to+\infty} (x_n(t) - x_n(t_0)) = \int_{t_0}^{t} \frac{y_0(s)}{\phi^{-1}(p(s))} ds.$$

Then

$$x_0(t) - x_0(t) = \int_{t_0}^{t} \frac{y_0(s)}{\phi^{-1}(p(s))} ds.$$

It follows that $y_0(t) = \phi^{-1}(p(t))x_0'(t)$. It follows that $x_0 \in X$. This show us that X is a Banach space.

Similarly we can prove that Y is a Banach space. So $E = X \times Y$ is a Banach space. □

Consider the following BVP

$$\begin{cases} [p(t)\phi(u'(t))]' + \sigma(t) = 0, \quad t \in \mathbb{R}, \\ \lim\limits_{t\to\pm\infty} u(t) = 0. \end{cases} \tag{2.1}$$

Lemma 2.3. *Suppose that both* (B2) *and* (B3) *hold. Then*

(i) $u \in X$ *is a solution of* BVP(2.1) *implies that* u *is concave with respect to* τ, *where* τ *is defined by*

$$\tau = \tau(t) = \left(\int_{-\infty}^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) ds \right) \left(\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds \right)^{-1}; \tag{2.2}$$

(ii) $u \in X$ *is a solution of* BVP(2.1) *implies that* u *is positive on* $\mathbb{R}$;

(iii) *Let* $k > 0$. $u \in X$ *is a solution of* BVP(2.1) *implies that* u *satisfies that*

$$\min_{t\in[-k,k]} u(t) \ge \mu \max_{t\in\mathbb{R}} u(t), \tag{2.3}$$

where μ is defined by

$$\mu = \left(\int_{-\infty}^{-k} \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right)\left(2\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right)^{-1}; \tag{2.4}$$

(iv) *$u \in X$ is a solution of* BVP(2.1) *if and only if there exists a unique numbers A_σ such that*

$$u(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) \phi^{-1}\left(A_\sigma - \int_{-\infty}^{s} \sigma(w)dw\right) ds, \quad t \in R, \tag{2.5}$$

where

$$A_\sigma \in \left[0, \int_{-\infty}^{+\infty} \sigma(w)dw\right] \tag{2.6}$$

such that

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) \phi^{-1}\left(A_\sigma - \int_{-\infty}^{s} \sigma(w)dw\right) ds = 0.$$

Proof. Suppose that u is a solution of (2.1). Firstly we prove that u is concave with respect to τ defined by (2.2). It is easy to see from (2.2) that $\tau \in C^0(\mathbb{R}, (0,1))$ and

$$\phi\left(\frac{du}{d\tau}\right) = p(t)\phi\left(\frac{du}{dt}\right)\phi\left(\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right). \tag{2.7}$$

Since $[p(t)\phi(u'(t))]' = -\sigma(t) \leq 0$, we get $p(t)\phi(u'(t))$ is decreasing (non-increasing) on $\mathbb{R}$. Then (2.7) implies that $\phi\left(\frac{du}{d\tau}\right)$ decreases as t increases. By definition of τ, we know that τ increases if and only if t increases. Hence $\phi\left(\frac{du}{d\tau}\right)$ decreases as τ increases. So u is concave with respect to τ on $(0,1)$.

Secondly, we prove that u is positive on $\mathbb{R}$. In fact, since $[p(t)\phi(u'(t))]' \leq 0$, we see that $p(t)\phi(u'(t))$ is non-increasing on $\mathbb{R}$. Then $\phi^{-1}(p(t))u'(t)$ is non-increasing on $\mathbb{R}$. We prove that $\lim_{t\to-\infty} u'(t) \geq 0$ and $\lim_{t\to+\infty} u'(t) \leq 0$ from $\lim_{t\to\pm\infty} u(t) = 0$ and p is eventually positive. Then there exists $t_0 \in \mathbb{R}$ such that $u'(t_0) = 0$. So

$$\phi(u'(t)) = \frac{1}{p(t)} \begin{cases} -\displaystyle\int_{t_0}^{t} \sigma(w)dw, & t \geq t_0, \\ \displaystyle\int_{t}^{t_0} \sigma(w)dw, & t \leq t_0. \end{cases}$$

Then

$$u(t) = \begin{cases} \displaystyle\int_{t}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\int_{t_0}^{s} \sigma(w)dw\right) ds, & t \geq t_0, \\ \displaystyle\int_{-\infty}^{t} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{t_0} \sigma(w)dw\right) ds, & t \leq t_0. \end{cases}$$

It follows that $u(t) \geq 0$ for all $t \in \mathbb{R}$. We note that $\sigma(t) \not\equiv 0$ on each subinterval on $\mathbb{R}$. It is easy to show that $u(t) > 0$ on $\mathbb{R}$.

Thirdly, we prove (iii). Let the inverse function of $\tau = \tau(t)$ be $t = t(\tau)$. It follows from above discussion that $\sup\limits_{t\in\mathbb{R}} u(t) = u(t_0)$. One sees from the concavity of u in τ that

$$\min_{t\in[-k,k]} u(t) = \min_{t\in[-k,k]} u(t(\tau(t))) = \min_{t\in[-k,k]} u(t(\tau)) = \min\{u(-k),\ u(k)\}.$$

If $\min\{u(-k),\ u(k)\} = u(-k) = u(t(\tau(-k)))$, then we have $t_0 \geq -k$. Then for $t \in [-k,k]$, one has

$$\begin{aligned} u(t) &\geq u(t(\tau(-k))) \\ &= u\left(t\left(\frac{1-\tau(-k)+\tau(t_0)}{1+\tau(t_0)}\frac{\tau(-k)}{1-\tau(-k)+\tau(t_0)} + \frac{\tau(-k)}{1+\tau(t_0)}\tau(t_0)\right)\right). \end{aligned}$$

Noting that $1 > \tau(-k)$ and $u(t)$ is concave with respect to τ. Then, for $t \in [-k,k]$,

$$\begin{aligned} u(t) &\geq \frac{1-\tau(-k)+\tau(t_0)}{1+\tau(t_0)} u\left(t\left(\frac{\tau(-k)}{1-\tau(-k)+\tau(t_0)}\right)\right) + \frac{\tau(-k)}{1+\tau(t_0)} u\Big(\ t\left(\tau(t_0)\right)\Big) \\ &\geq \frac{\int_{-\infty}^{-k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{1+\int_{-\infty}^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds} u\Big(\ t_0\Big) \geq \frac{\int_{-\infty}^{-k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)ds} u(t_0) \geq \mu \sup_{t\in\mathbb{R}} u(t). \end{aligned}$$

Similarly, if $\min\{u(-k),\ u(k)\} = u(t(\tau(k)))$, we have $t_0 \leq k$. For $t \in [-k,k]$, one has

$$\begin{aligned} u(t) &\geq u(t(\tau(k))) \\ &= u\left(t\left(\frac{1+\tau(t_0)-\tau(k)}{1+\tau(t_0)}\frac{\tau(k)}{1+\tau(t_0)-\tau(k)} + \frac{\tau(k)}{1+\tau(t_0)}\tau(t_0)\right)\right) \\ &\geq \frac{1+\tau(t_0)-\tau(k)}{1+\tau(t_0)} u\left(t\left(\frac{\tau(k)}{1+\tau(t_0)-\tau(k)}\right)\right) + \frac{\tau(k)}{1+\tau(\tau_0)} u\left(t\left(\tau(t_0)\right)\right) \\ &\geq \frac{\int_{-\infty}^{k}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{2\int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)ds} u(t_0) \geq \mu \sup_{t\in\mathbb{R}} u(t). \end{aligned}$$

Hence (2.3) holds and μ is defined by (2.4).

Finally, we prove (iv). Firstly we prove that there exists a unique $A_\sigma \in \left[0, \int_{-\infty}^{+\infty}\sigma(w)dw\right]$ such that

$$\int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_\sigma - \int_{-\infty}^{s}\sigma(w)dw\right)ds = 0. \tag{2.8}$$

Let

$$G(c) = \int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(c - \int_{-\infty}^{s}\sigma(u)du\right)ds.$$

It is easy to see that $G(c)$ is increasing on $\mathbb{R}$. One gets easily that $G(0) \leq 0$ and $G\left(\int_{-\infty}^{+\infty} \sigma(w)dw\right) \geq 0$. Hence we get that there exists a unique constant $A_\sigma \in \left[0, \int_{-\infty}^{+\infty} \sigma(w)dw\right]$ such that (2.8) holds. So (2.6) holds.

If $u \in X$ is a solution of (2.1), we get from $\sigma \in L^1(\mathbb{R})$ that

$$p(t)\phi(u'(t)) = \lim_{t\to-\infty} p(t)\phi(u'(t)) - \int_{-\infty}^{t} \sigma(w)dw, \quad t \in \mathbb{R}.$$

So

$$u(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) \phi^{-1}\left(\lim_{t\to-\infty} p(t)\phi(u'(t)) - \int_{-\infty}^{s} \sigma(w)dw\right) ds.$$

Choose $A_\sigma = \lim_{t\to-\infty} p(t)\phi(u'(t))$. Then (2.5) and (2.6) hold.

On the other hand, if u satisfies (2.5) and (2.6), it is easy to show that $u \in X$ and u is a solution of BVP(2.1). The proof is complete. □

Similarly we consider the following BVP

$$\begin{cases} [q(t)\psi(v(t))]' + \varrho(t) = 0, \quad t \in \mathbb{R}, \\ \lim_{t\to\pm\infty} v(t) = 0. \end{cases} \tag{2.9}$$

We get the following lemma similarly to Lemma 2.3:

Lemma 2.4. *Suppose that both* (B2) *and* (B3) *hold. Then*

(i) $v \in Y$ *is a solution of* BVP(2.9) *implies that* v *is concave with respect to* ς, *where* ς *is defined by*

$$\varsigma = \varsigma(t) = \frac{\int_{-\infty}^{t} \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{1}{q(s)}\right) ds};$$

(ii) $v \in Y$ *is a solution of* BVP(2.9) *implies that* v *is positive on* $\mathbb{R}$;

(iii) *Let* $k > 0$. $v \in Y$ *is a solution of* BVP(2.9) *implies that* v *satisfies that*

$$\min_{t\in[-k,k]} v(t) \geq \nu \max_{t\in\mathbb{R}} v(t),$$

where ν *is defined by*

$$\nu = \left(\int_{-\infty}^{-k} \psi^{-1}\left(\frac{1}{q(s)}\right) ds\right) \left(2\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{1}{q(s)}\right) ds\right)^{-1};$$

(iv) $v \in Y$ *is a solution of* BVP(2.9) *if and only if there exists a unique numbers* A_ϱ *such that*

$$v(t) = \int_{-\infty}^{t} \psi^{-1}\left(\frac{1}{q(s)}\right) \psi^{-1}\left(A_\varrho - \int_{-\infty}^{s} \varrho(w)dw\right) ds, \quad t \in \mathbb{R},$$

where

$$A_\varrho \in \left[0, \int_{-\infty}^{+\infty} \varrho(w)dw\right]$$

such that

$$\int_{-\infty}^{+\infty}\psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_\varrho-\int_{-\infty}^{s}\varrho(w)dw\right)ds=0.$$

Proof. The proof is similar to that of Lemma 2.3 and is omitted. □

Let $k>0$ be fixed. Define

$$P=\left\{(u,v)\in E:\ \begin{array}{c} u(t),v(t)\geq 0,\ t\in\mathbb{R},\\ \min\limits_{t\in[-k,k]}u(t)\geq\mu\sup\limits_{t\in\mathbb{R}}u(t),\\ \min\limits_{t\in[-k,k]}v(t)\geq\nu\sup\limits_{t\in\mathbb{R}}v(t),\\ \lim\limits_{t\to\pm\infty}u(t)=0,\\ \lim\limits_{t\to\pm\infty}v(t)=0\end{array}\right\}.$$

It is easy to see that P is a cone in the Banach space E.

Define the functionals on $P\to\mathbb{R}$ by

$$\beta_1(u,v)=\max\left\{\sup_{t\in\mathbb{R}}\phi^{-1}(p(t))|u'(t)|,\ \sup_{t\in\mathbb{R}}\psi^{-1}(q(t))|v'(t)|\right\},\ (u,v)\in P,$$

$$\beta_2(u,v)=\max\left\{\sup_{t\in\mathbb{R}}|u(t)|,\ \sup_{t\in\mathbb{R}}|v(t)|\right\},\ (u,v)\in P,$$

$$\beta_3(u,v)=\max\left\{\max_{t\in\mathbb{R}}|u(t)|,\ \sup_{t\in\mathbb{R}}|v(t)|\right\},\ (u,v)\in P,$$

$$\alpha_1(u,v)=\min\left\{\min_{t\in[-k,k]}|u(t)|,\ \min_{t\in[-k,k]}|v(t)|\right\},\ (u,v)\in P,$$

$$\alpha_2(u,v)=\min\left\{\min_{t\in[-k,k]}|u(t)|,\ \min_{t\in[-k,k]}|v(t)|\right\},\ (u,v)\in P.$$

Define the nonlinear operator $T:P\to E$ by $T(u,v)=(T_1v,T_2u)$ with

$$(T_1v)(t)=\int_{-\infty}^{t}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v-\int_{-\infty}^{s}f(w,v(w)+h_2,v'(w))dw\right)ds,$$

$$(T_2u)(t)=\int_{-\infty}^{t}\psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u-\int_{-\infty}^{s}g(w,u(w)+h_1,u'(w))dw\right)ds,$$

where A_v, A_u satisfy

$$\int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{1}{p(s)}\right)\phi^{-1}\left(A_v-\int_{-\infty}^{s}f(w,v(w),v'(w))dw\right)ds=0,$$

$$\int_{-\infty}^{+\infty}\psi^{-1}\left(\frac{1}{q(s)}\right)\psi^{-1}\left(A_u-\int_{-\infty}^{s}g(w,u(w),u'(w))dw\right)ds=0.$$

Lemma 2.5. *Suppose that both* (B1) *and* (B2) *hold. Then*

(i) $T(u,v)\in P$ *for each* $(u,v)\in P$;

(ii) (x,y) *is a positive solution of* BVP(1.1) *if and only if* (u,v) *is a solution of the operator equation* $(u,v)=T(u,v)$ *in* P;

(iv) *T is completely continuous.*

Proof. The proofs of (i) and (ii) are simple, To prove (iii), it suffices to prove that T is continuous on P and for each bounded subset $D \subset P$, $T(D) = \{T(u,v) : (u,v) \in D\}$ is relative compact. We divide the proof into two steps:

Step 1. for $(u,v) \in P$, since both f and g are Carathéodory functions, we can prove that T is continuous at (u,v).

Step 2. Suppose that $D \in P$ is bounded. We need to prove that $T(D)$ is relative compact. It suffices to prove that $T(D)$ is equi-convergent as $t \to \pm\infty$ and $T(D)$ is equi-continuous in any subinterval $[A,B]$ in R.

The proofs are similar to those of the proofs of lemmas in [22,29,32,33] and are omitted. $\square$

3. Main results

In this section, we give the main results and their proofs under the assumptions (B1) and (B2). Let X, P, μ, ν be defined in Section 2.

Theorem 3.1. *Suppose that both* (B1) *and* (B2) *hold, e_1, e_2, c are positive numbers, $\theta \in L^1(R)$. Denote*

$$M = \min\left\{\int_{-k}^{0} \phi^{-1}\left(\frac{1}{p(s)}\int_s^0 \theta(w)dw\right)ds,\ \int_0^k \phi^{-1}\left(\frac{1}{p(s)}\int_0^s \theta(w)dw\right)ds\right\},$$
$$N = \min\left\{\int_{-k}^{0} \psi^{-1}\left(\frac{1}{q(s)}\int_s^0 \theta(w)dw\right)ds,\ \int_0^k \psi^{-1}\left(\frac{1}{q(s)}\int_0^s \theta(w)dw\right)ds\right\}.$$

Let Q, W and E be given by

$$Q = \max\left\{\frac{\phi\left(\frac{c}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds,\int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)}{\int_{-\infty}^{+\infty}\theta(s)ds},\right.$$
$$\left.\frac{\psi\left(\frac{c}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds,\int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)}{\int_{-\infty}^{+\infty}\theta(s)ds}\right\};$$

$$E = \max\left\{\frac{\phi\left(\frac{e_1}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds,\int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)}{\int_{-\infty}^{+\infty}\theta(s)ds},\right.$$

$$\left.\frac{\psi\left(\frac{e_1}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds,\int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)}{\int_{-\infty}^{+\infty}\theta(s)ds}\right\},$$

$$W = \min\left\{\phi\left(\frac{e_2}{\min\{\mu,\nu\}}\frac{1}{M}\right),\quad \psi\left(\frac{e_2}{\min\{\mu,\nu\}}\frac{1}{N}\right)\right\}.$$

If $Q \geq W$ *and*

(B6)

$$f\left(t, w, \tfrac{z}{\phi^{-1}(p(t))}\right) \leq Q\theta(t) \text{ for all } t \in \mathbb{R}, w \in [0, c], z \in [-c, c],$$

$$g\left(t, w, \tfrac{z}{\psi^{-1}(q(t))}\right) \leq Q\theta(t) \text{ for all } t \in \mathbb{R}, w \in [0, c], z \in [-c, c];$$

(B7)

$$f\left(t, w, \tfrac{z}{\phi^{-1}(p(t))}\right) \geq W\theta(t) \text{ for all } t \in [-k, k], z \in [-c, c], w \in \left[e_2, \tfrac{e_2}{\min\{\mu,\nu\}}\right],$$

$$g\left(t, w, \tfrac{z}{\psi^{-1}(q(t))}\right) \geq W\theta(t) \text{ for all } t \in [-k, k], z \in [-c, c], w \in \left[e_2, \tfrac{e_2}{\min\{\mu,\nu\}}\right];$$

(B8)

$$f\left(t, w, \tfrac{z}{\phi^{-1}(p(t))}\right) \leq E\theta(t) \text{ for all } t \in \mathbb{R}, w \in [0, e_1], z \in [-c, c],$$

$$g\left(t, w, \tfrac{z}{\psi^{-1}(q(t))}\right) \leq E\theta(t) \text{ for all } t \in \mathbb{R}, w \in [0, e_1], z \in [-c, c];$$

then BVP(1.2) *has at least three pair of solutions* $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ *such that*

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} x_1(t) < e_1, \quad \sup_{t\in\mathbb{R}} y_1(t) < e_1,\\
&\min_{t\in[-k,k]} x_2(t) > e_2, \quad \min_{t\in[-k,k]} y_2(t) > e_2,\\
&\textit{either } \sup_{t\in\mathbb{R}} x_3(t) > e_1 \textit{ or } \sup_{t\in\mathbb{R}} y_3(t) > e_1,\\
&\textit{either } \min_{t\in[-k,k]} x_3(t) < e_2 \textit{ or } \min_{t\in[-k,k]} y_3(t) < e_2.
\end{aligned} \tag{1}$$

Proof. To apply Lemma 2.1, we prove that all hypotheses in Lemma 2.1 are satisfied.

By the definitions, it is easy to see that α_1, α_2 are nonnegative continuous concave functionals on the cone P, $\beta_1, \beta_2, \beta_3$ nonnegative continuous convex functionals on the cone P, and $\alpha_1(u, v) \leq \beta_2(u, v)$ for all $(u, v) \in P$. Lemma

2.5 implies that (u, v) is a pair of positive solutions of BVP(1.1) if and only if (u, v) is a fixed point the operator T in P and T is completely continuous.

We claim that

$$(2) \qquad \begin{array}{l} \sup_{t\in\mathbb{R}} |u(t)| \le \int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|, \\ \sup_{t\in\mathbb{R}} |v(t)| \le \int_{-\infty}^{+\infty} \frac{1}{\psi^{-1}(q(s))} ds \sup_{t\in\mathbb{R})} \psi^{-1}(q(t))|v'(t)| \end{array} \quad \text{for } (u,v) \in P.$$

In fact, for $(u, v) \in P$, we have for $t > T$ that

$$\begin{aligned} |u(t)| = & \left| u(T) + \int_T^t u'(s)ds \right| \le |u(T)| + \int_T^t \frac{1}{\phi^{-1}(p(s))} \phi^{-1}(p(s))|u'(s)|ds \\ & \le |u(T)| + \int_T^t \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)| \\ & \le |u(T)| + \int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|. \end{aligned}$$

Let $T \to -\infty$. It follows that

$$|u(t)| \le \int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|.$$

Then

$$\sup_{t\in\mathbb{R}} |u(t)| \le \int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds \sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|.$$

Similarly we get the other inequality in (3.2). It follows from (3.2) that

$$(3) \qquad ||(u, v)|| \le \max \left\{ \int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))} ds, \int_{-\infty}^{+\infty} \frac{1}{\psi^{-1}(q(s))} ds \right\} \beta_1(u, v)$$

for all $(u, v) \in P$.

From the above discussion, we see that (A1) and (A2) of Lemma 2.1 are satisfied.

Now we prove that (A3) in Lemma 2.1 holds. Corresponding to Lemma 2.1, choose

$$c_1 = e_1, \;\; c_2 = e_2, \;\; c_3 = \min\{\mu, \nu\}e_1, \;\; c_4 = \frac{e_2}{\min\{\mu, \nu\}}, \;\; c_5 = c.$$

One sees that $c_1 < c_2$. The remainder is divided into five steps.

Step 1. Prove that $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;

For $(u, v) \in \overline{P_{c_5}}$, we have $||(u, v)|| \le c$. Then

$0 \le u(t), \; v(t) \le c, \; -c \le \phi^{-1}(p(t))u'(t), \; \psi^{-1}(q(t))v'(t) \le c$ for all $t \in \mathbb{R}$.

So (B6) implies for $t \in \mathbb{R}$ that

$$f(t, v(t), v'(t)) = f\left(t, v(t), \frac{1}{\phi^{-1}(p(t))}\phi^{-1}(p(t))v'(t)\right) \le Q\theta(t), t \in \mathbb{R},$$

$$g(t, u(t), u'(t)) = g\left(t, u(t), \frac{1}{\psi^{-1}(q(t))}\psi^{-1}(q(t))u'(t)\right) \le Q\theta(t), t \in \mathbb{R}.$$

Lemma 2.3 implies

$$A_v \in \left[0, \int_{-\infty}^{+\infty} f(w, v(w), v'(w))dw\right].$$

The definition of T_1 implies that

$$\phi^{-1}(p(t))(T_1 v)'(t) = \phi^{-1}\left(A_v - \int_{-\infty}^{t} f(w, v(w), v'(w))dw\right).$$

Then

$$\begin{aligned}\phi^{-1}(p(t))|(T_1 v)'(t)| &\le \phi^{-1}\left(\int_{-\infty}^{+\infty} f(w, v(w), v'(w))dw\right)\\ &\le \phi^{-1}\left(\int_{-\infty}^{+\infty} Q\theta(s)ds\right).\end{aligned}$$

It follows that

$$\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|(T_1 v)'(t)| \le \phi^{-1}\left(\int_{-\infty}^{+\infty} Q\theta(s)ds\right).$$

Similarly we get that

$$\sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|(T_2 u)'(t)| \le \psi^{-1}\left(\int_{-\infty}^{+\infty} Q\theta(s)ds\right).$$

Then (3.3) implies that

$$\begin{aligned}||T(u, v)|| = &\max\left\{\int_{-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(s))}ds, \int_{-\infty}^{+\infty} \frac{1}{\psi^{-1}(q(s))}ds\right\}\\ &\times \max\left\{\phi^{-1}\left(\int_{-\infty}^{+\infty} Q\theta(s)ds\right), \psi^{-1}\left(\int_{-\infty}^{+\infty} Q\theta(s)ds\right)\right\} \le c.\end{aligned}$$

So $T(\overline{P_{c_5}}) \subseteq \overline{P_{c_5}}$. This completes the proof of (A3)(i) of Lemma 2.1.

Step 2. Prove that $\{(u, v) \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) \,|\, \alpha_1(u, v) > c_2\} \neq \emptyset$ and

$\alpha_1(T(u, v)) > c_2$ for every $(u, v) \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5)$;

It is easy to show that $\{(u, v) \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) \,|\, \alpha(u, v) > c_2\} \neq \emptyset$. For $(u, v) \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5)$, one has that

$$\alpha_1(u, v) = \min\left\{\min_{t\in[-k,k]} u(t), \min_{t\in[-k,k]} v(t)\right\} \ge e_2,$$

$$\beta_3(u, v) = \max\left\{\sup_{t\in\mathbb{R}} u(t), \sup_{t\in\mathbb{R}} v(t)\right\} \le \frac{e_2}{\min\{\mu, \nu\}},$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|, \sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|v'(t)|\right\} \le c.$$

Then

$$e_2 \le u(t), v(t) \le \frac{e_2}{\min\{\mu,\nu\}},\ t \in [-k,k],$$
$$\phi^{-1}(p(t))|u'(t)|, \psi^{-1}(q(t))|v'(t)| \le c, t \in \mathbb{R}.$$

Thus (B7) implies for $t \in [-k,k]$ that

$$f(t,,v(t),v'(t)) = f\left(t, v(t), \frac{1}{\phi^{-1}(p(t))}\phi^{-1}(p(t))v'(t)\right) \ge W\theta(t), t \in [-k,k],$$
$$g(t,u(t),u'(t)) = g\left(t, u(t), \frac{1}{\psi^{-1}(q(t))}\psi^{-1}(q(t))u'(t)\right) \ge W\theta(t), t \in [-k,k].$$

We can prove that $\lim_{t\to-\infty}(T_1v)'(t) \ge 0$ and $\lim_{t\to+\infty}(T_1v)'(t) \le 0$. Hence there exists $t_0 \in R$ such that $(T_1v)'(t_)) = 0$. Then the definition of T_1 implies that

$$\phi((T_1v)'(t)) = \frac{1}{p(t)}\begin{cases} -\int_{t_0}^{t} f(w,v(w),v'(w))dw, & t \ge t_0, \\ \int_{t}^{t_0} f(w,v(w),v'(w))dw, & t \le t_0. \end{cases}$$

Then

$$(T_1v)(t) = \begin{cases} \int_{t}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\int_{t_0}^{s} f(w,v(w),v'(w))dw\right)ds, & t \ge t_0, \\ \int_{-\infty}^{t} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{t_0} f(w,v(w),v'(w))dw\right)ds, & t \le t_0. \end{cases}$$

Hence

$$\begin{aligned}\sup_{t\in\mathbb{R}}(T_1v)(t) &= (T_1v)(t_0) \\ &= \begin{cases} \int_{t_0}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\int_{t_0}^{s} f(w,v(w),v'(w))dw\right)ds, \\ \int_{-\infty}^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{t_0} f(w,v(w),v'(w))dw\right)ds. \end{cases}\end{aligned}$$

Case 1. $t_0 \ge 0$.
Then

$$\begin{aligned}\sup_{t\in\mathbb{R}}(T_1v)(t) &= \int_{-\infty}^{t_0} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{t_0} f(w,v(w),v'(w))dw\right)ds \\ &\ge \int_{-k}^{0} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{0} f(w,v(w),v'(w))dw\right)ds \\ &\ge \int_{-k}^{0} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{0} W\theta(w)dw\right)ds.\end{aligned}$$

Case 2. $t_0 \leq 0$.
Then

$$\begin{aligned}\sup_{t\in\mathbb{R}}(T_1v)(t) &= \int_{t_0}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\int_{t_0}^{s} f(w,v(w),v'(w))dw\right) ds \\ &\geq \int_0^k \phi^{-1}\left(\frac{1}{p(s)}\int_0^s f(w,v(w),v'(w))dw\right) ds \\ &\geq \int_0^k \phi^{-1}\left(\frac{1}{p(s)}\int_0^s W\theta(w)dw\right) ds.\end{aligned}$$

We get that

$$\begin{aligned}\sup_{t\in\mathbb{R}}(T_1v)(t) \geq \phi^{-1}(W)\min\Bigg\{&\int_{-k}^{0}\phi^{-1}\left(\frac{1}{p(s)}\int_s^0 \theta(w)dw\right) ds, \\ &\int_0^k \phi^{-1}\left(\frac{1}{p(s)}\int_0^s \theta(w)dw\right) ds\Bigg\}.\end{aligned}$$

Similarly we get

$$\begin{aligned}\sup_{t\in\mathbb{R}}(T_2u)(t) \geq \psi^{-1}(W)\min\Bigg\{&\int_{-k}^{0}\psi^{-1}\left(\frac{1}{q(s)}\int_s^0 \theta(w)dw\right) ds, \\ &\int_0^k \psi^{-1}\left(\frac{1}{q(s)}\int_0^s \theta(w)dw\right) ds\Bigg\}.\end{aligned}$$

Since $T(u,v) = (T_1v, T_2u) \in P$, we get

$$\begin{aligned}&\alpha_1(T(u,v)) \\ =\ &\min\left\{\min_{t\in[-k,k]}(T_1v)(t), \min_{t\in[-k,k]}(T_2u)(t)\right\} \\ \geq\ &\min\left\{\mu\sup_{t\in\mathbb{R}}(T_1v)(t), \nu\sup_{t\in\mathbb{R}}(T_2u)(t)\right\} \\ \geq\ &\min\{\mu,\nu\}\min\left\{\sup_{t\in\mathbb{R}}(T_1v)(t), \sup_{t\in\mathbb{R}}(T_2u)(t)\right\} \\ \geq\ &\min\{\mu,\nu\}\min\Bigg\{\phi^{-1}(W)\min\Bigg\{\int_{-k}^{0}\phi^{-1}\left(\frac{1}{p(s)}\int_s^0 \theta(w)dw\right) ds, \\ &\int_0^k \phi^{-1}\left(\frac{1}{p(s)}\int_0^s \theta(w)dw\right) ds\Bigg\}, \\ &\psi^{-1}(W)\min\left\{\int_{-k}^{0}\psi^{-1}\left(\frac{1}{q(s)}\int_s^0 \theta(w)dw\right) ds,\ \int_0^k \psi^{-1}\left(\frac{1}{q(s)}\int_0^s \theta(w)dw\right) ds\right\}\Bigg\} \\ \geq\ &e_2.\end{aligned}$$

It follows that $\alpha_1(T(u,v)) > c_2$ for every $(u,v) \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5)$. This completes the proof of (A3)(ii) of Lemma 2.1.

Step 3. Prove that $\{(u,v) \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) | \beta_2(u,v) < c_1\} \neq \emptyset$ and $\beta_2(T(u,v)) < c_1$ for every $(u,v) \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5)$;

It is easy to show that $\{(u,v) \in P(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) : \alpha_2(u,v) < c_1\} \neq \emptyset$. For $(u,v) \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5)$, one has that

$$\alpha_2(u,v) = \min\left\{\min_{n\in[-k,k]} u(t),\ \min_{t\in[-k,k]} v(t)\right\} \geq c_3,$$

$$\beta_2(u,v) = \max\left\{\sup_{t\in\mathbb{R}} u(t), \sup_{t\in\mathbb{R}} v(t)\right\} \leq c_1,$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|, \sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|v'(t)|\right\} \leq c_5.$$

Then

$$0 \leq u(t), v(t) \leq e_1, \quad \phi^{-1}(p(t))|u'(t)|, \psi^{-1}(q(t))|v'(t)| \leq c, \quad t \in \mathbb{R}.$$

Thus (B8) implies that

$$f(t, v(t), v'(t)) \leq E\theta(t), g(t, u(t), u'(t)) \leq E\theta(t), \quad t \in \mathbb{R}.$$

Similarly to Step 1, we have that

$$\beta_2(T(u,v)) = \max\{\max_{t\in R} |(T_1v)(t)|,\ \max_{t\in R} |(T_2u)(t)|\} \leq e_1 = c_1.$$

It follows that $\beta_2(T(u,v)) < c_1$. This completes the proof of (A3)(iii) of Lemma 2.1.

Step 4. Prove that $\alpha_1(T(u,v)) > c_2$ for $(u,v) \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T(u,v)) > c_4$;

For $(u,v) \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T(u,v)) > c_4$, we have that

$$\alpha_1(u,v) = \min\left\{\min_{t\in[-k,k]} u(t),\ \min_{t\in[-k,k]} v(t)\right\} \geq c_2 = e_2$$

$$\beta_1(u,v) = \max\left\{\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|y'(t)|, \sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|v'(t)|\right\} \leq c_5$$

and

$$\beta_3(T(u,v)) = \max\left\{\sup_{t\in\mathbb{R}}(T_1v)(t), \sup_{t\in\mathbb{R}}(T_2u)(t)\right\} > \frac{e_2}{\min\{\mu,\nu\}} = c_4.$$

Then

$$\begin{aligned}\alpha_1(T(u,v)) &= \min\left\{\min_{t\in[-k,k]}(T_1v)(t), \min_{t\in[-k,k]}(T_2u)(t)\right\} \\ &\geq \min\left\{\mu\sup_{t\in\mathbb{R}}(T_1v)(t), \nu\sup_{t\in\mathbb{R}}(T_2u)(t)\right\} \\ &\geq \min\{\mu,\nu\}\beta_2(T(u,v)) > \min\{\mu,\nu\}\frac{e_2}{\min\{\mu,\nu\}} = e_2 = c_2.\end{aligned}$$

This completes the proof of (A3)(iv) of Lemma 2.1.

Step 5. Prove that $\beta_2(T(u,v)) < c_1$ for each $(u,v) \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(T(u,v)) < c_3$;

For $(u,v) \in Q(\beta_1,\beta_2;c_1,c_5)$ with $\alpha_2(T(u,v)) < c_3$, we have that

$$\beta_2(u,v) = \min\left\{\min_{t\in[-k,k]} u(t),\ \min_{t\in[-k,k]} v(t)\right\} \le e_1$$

and

$$\beta_1(u,v) = \max\left\{\sup_{t\in\mathbb{R}} \phi^{-1}(p(t))|u'(t)|, \sup_{t\in\mathbb{R}} \psi^{-1}(q(t))|v'(t)|\right\} \le c_5$$

and

$$\alpha_2(T(u,v)) = \min\left\{\min_{t\in[-k,k]}(T_1v)(t),\ \min_{t\in[-k,k]}(T_2u)(t)\right\} < c_3 = \min\{\mu,\nu\}e_1.$$

Since $T(u,v) \in P$, we get

$$\begin{aligned}\beta_2(T(u,v)) &= \max\left\{\sup_{t\in\mathbb{R}}(T_1v)(t), \sup_{t\in\mathbb{R}}(T_2u)(t)\right\}\\ &\le \frac{1}{\min\{\mu,\nu\}}\min\left\{\min_{t\in[-k,k]}(T_1v)(t),\ \min_{t\in[-k,k]}(T_2u)(t)\right\}\\ &< \frac{1}{\min\{\mu,\nu\}}\min\{\mu,\nu\}e_1 = c_1.\end{aligned}$$

This completes the proof of (A3)(v) of Lemma 2.1.

Then Lemma 2.1 implies that T has at least three fixed points (x_1,y_1), (x_2,y_2) and (x_3,y_3) satisfying

$$\beta_2(x_1,y_1) < c_1,\ \alpha_1(x_2,y_2) > c_2,\ \beta_2(x_3,y_3) > c_1,\ \alpha_1(x_3,y_3) < c_2.$$

So BVP(1.2) has three pairs of positive solutions $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) satisfying (3.1). The proof of Theorem 3.1 is completed. $\square$

4. An example

Now, we present an example to illustrate the main result: Theorem 3.1.

Example 4.1. Example 4.1. Choose

$$\theta(t) = \begin{cases}\frac{1}{\sqrt{|t|}}, & |t|<1,\\ \frac{1}{t^2}, & |t|\ge 1,\end{cases}\quad p(t) = \begin{cases}\sqrt{|t|}, & |t|<1,\\ t^2, & |t|\ge 1,\end{cases}\quad q(t) = \begin{cases}\sqrt[3]{|t|}, & |t|<1,\\ t^3, & |t|\ge 1.\end{cases}$$

Consider the following BVP of the second order differential system

$$\begin{cases}(p(t)x'(t))' + \theta(t)F(t,y(t),y'(t)) = 0, & t\in\mathbb{R},\\ (q(t)y'(t))' + \theta(t)G(t,x(t),x'(t)) = 0, & t\in\mathbb{R},\\ \lim_{t\to\pm\infty} x(t) = 0,\\ \lim_{t\to\pm\infty} y(t) = 0.\end{cases} \tag{4.1}$$

Corresponding to BVP(1.1), we have that $\phi(x) = \psi(x) = \phi^{-1}(x) = \psi^{-1}(x) = x$ and

$$f(t,u,v) = |t|^{-\frac{1}{2}}e^{-|t|}F(t,u,v), \quad g(t,u,v) = |t|^{-\frac{1}{2}}e^{-|t|}G(t,u,v).$$

It is easy to see that

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds = 6, \quad \int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{1}{q(s)}\right) ds = 4.$$

Suppose that

$$F(t,u,v) = f_0(u) + \frac{m}{1+p(t)^2v^2}, \; G(t,u,v) = g_0(u) + \frac{n}{1+q(t)^2v^2}, \; m,n \geq 0.$$

One sees that

$$f\left(t,w,\frac{z}{p(t)}\right) = \theta(t)\left[f_0(w) + \frac{m}{1+z^2}\right], \; g\left(t,w,\frac{z}{q(t)}\right) = \theta(t)\left[g_0(w) + \frac{n}{1+z^2}\right].$$

Here

$$f_0(u) = g_0(u) = \begin{cases} \frac{1}{36}u, & u \in [0,2], \\ \frac{1}{18} + \frac{\frac{84100+\frac{28000}{3}}{2} - \frac{1}{18}}{1000-2}(u-2), & u \in [2,1000], \\ \frac{84100+\frac{28000}{3}}{2}, & u \in [1000,336400], \\ \frac{84100+\frac{28000}{3}}{2}e^{u-336400}, & u \geq 336400. \end{cases}$$

Use Theorem 3.1. One sees that (B1) and (B2) hold. Choose constants $k = 1, e_1 = 2, e_2 = 10^3, c = 336400$, then by direct computation and estimation, we get that

$$\mu = \frac{\int_{-\infty}^{-k} \phi^{-1}\left(\frac{1}{p(s)}\right) ds}{2\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{1}{p(s)}\right) ds} = \frac{1}{12},$$

$$\nu = \frac{\int_{-\infty}^{-k} \psi^{-1}\left(\frac{1}{q(s)}\right) ds}{2\int_{-\infty}^{+\infty} \psi^{-1}\left(\frac{1}{q(s)}\right) ds} = \frac{1}{16},$$

$$\begin{aligned} Q &= \frac{\max\left\{\phi\left(\frac{c}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds, \int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right), \psi\left(\frac{c}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds, \int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)\right\}}{\int_{-\infty}^{+\infty}\theta(s)ds} \\ &\geq \frac{c}{36} = \frac{84100}{9}; \\ E &= \frac{\max\left\{\phi\left(\frac{e_1}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds, \int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right), \psi\left(\frac{e_1}{\max\left\{\int_{-\infty}^{+\infty}\frac{1}{\phi^{-1}(p(s))}ds, \int_{-\infty}^{+\infty}\frac{1}{\psi^{-1}(q(s))}ds\right\}}\right)\right\}}{\int_{-\infty}^{+\infty}\theta(s)ds} \\ &\geq \frac{e_1}{36} = \frac{1}{18}, \end{aligned}$$

$$W = \min\left\{\phi\left(\frac{e_2}{\min\{\mu,\nu\}}\frac{1}{\min\left\{\int_{-k}^0\phi^{-1}\left(\frac{1}{p(s)}\int_s^0\theta(w)dw\right)ds,\ \int_0^k\phi^{-1}\left(\frac{1}{p(s)}\int_0^s\theta(w)dw\right)ds\right\}}\right),\right.$$
$$\left.\psi\left(\frac{e_2}{\min\{\mu,\nu\}}\frac{1}{\min\left\{\int_{-k}^0\psi^{-1}\left(\frac{1}{q(s)}\int_s^0\theta(w)dw\right)ds,\ \int_0^k\psi^{-1}\left(\frac{1}{q(s)}\int_0^s\theta(w)dw\right)ds\right\}}\right)\right\}$$
$$\le \frac{28}{3}e_2 = \frac{28000}{3}.$$

It is easy to show that $Q > W$ and

(B6)

$$f\left(t,w,\frac{z}{p(t)}\right) \le \frac{84100}{9}\theta(t) \text{ for all } t\in R,\ w\in[0,336400],$$
$$z\in[-336400,336400],$$
$$g\left(t,w,\frac{z}{q(t)}\right) \le \frac{84100}{9}\theta(t) \text{ for all } t\in R,\ w\in[0,336400],$$
$$z\in[-336400,336400];$$

(B7)

$$f\left(t,w,\frac{z}{p(t)}\right) \ge \frac{28000}{3}\theta(t) \text{ for all } t\in[-1,1],\ z\in[-336400,336400],$$
$$w\in[1000,16000],$$
$$g\left(t,w,\frac{z}{q(t)}\right) \ge \frac{28000}{3}\theta(t) \text{ for all } t\in[-1,1],\ z\in[-336400,336400],$$
$$w\in[1000,16000];$$

(B8)

$$f\left(t,w,\frac{z}{p(t)}\right) \le \frac{1}{18}\theta(t) \text{ for all } t\in R, w\in[0,2], z\in[-336400,336400],$$
$$g\left(t,w,\frac{z}{q(t)}\right) \le \frac{1}{18}\theta(t) \text{ for all } t\in R, w\in[0,2], z\in[-336400,336400];$$

Then applying Theorem 3.1 BVP(4.1) has at least three solutions (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that

$$\sup_{t\in\mathbb{R}} x_1(t) < 1,\quad \sup_{t\in\mathbb{R}} y_1(t) < 1,$$
$$\min_{t\in[-1,1]} x_2(t) > 1000,\quad \min_{t\in[-1,1]} y_2(t) > 1000,$$
$$\text{either } \max_{t\in\mathbb{R}} x_3(t) > 1 \text{ or } \max_{t\in\mathbb{R}} y_3(t) > 1,$$
$$\text{either } \min_{t\in[-1,1]} x_3(t) < 1000, \text{ or } \min_{t\in[-1,1]} y_3(t) < 1000.$$

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