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LIE ALGEBRA AND OPERATIONAL TECHNIQUES ON THREE-VARIABLE HERMITE POLYNOMIALS

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Abstract. The present paper aims at harnessing the technique of Lie Algebra and operational methods to derive and interpret generating relations for the three-variable Hermite Polynomials $H_n(x, y, z)$ introduced recently in [2]. Certain generating relations for the polynomials related to $H_n(x, y, z)$ are also obtained as special cases.

1. Introduction

Dattoli et. al. ([2]-[4]) introduced and discussed the three variable Hermite polynomials:

$$H_n(x,y,z) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{z^r H_{n-2r}(x,y)}{(n-2r)!r!}, \tag{1.1}$$

satisfy the following differential equations

$$\left(x + 2(y+z)\frac{\partial}{\partial x}\right) Z_n(x,y,z) = Z_{n+1}(x,y,z), \tag{1.2}$$

$$\left(\frac{\partial}{\partial x}\right) Z_n(x,y,z) = nZ_{n-1}(x,y,z),$$

$$\left(x\frac{\partial}{\partial x} + 2(y+z)\frac{\partial^2}{\partial x^2} - n\right) Z_n(x,y,z) = 0, \;\; n = 0,1,2,..$$

where (see [7])

$$H_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2k} y^k}{(n-2k)!k!}, \tag{1.3}$$

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and hence

$$H_n(x,y,z) = n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{k=0}^{\left[\frac{n-2r}{2}\right]} \frac{z^r x^{n-2r-2k} y^k}{(n-2r-2k)!r!k!}. \tag{1.4}$$

Hermite polynomials arise in the study of classical boundary value problems in parabolic regions, through the use of parabolic coordinates, or in quantum mechanics as well as in other application areas. Indeed, due to the noticeable important of Hermite polynomials for application, it deserve some interest to get, in addition to the results obtained in [2, 3, 4, 7], further generating functions for these polynomials by group theoretic and operational methods. Recently, Pathan et al. [10] derived some implicit summation formulae and general symmetry identities for class of generalized polynomials associated with Hermite polynomials. Babusci et al. [1] have shown the combined use of generating function method and the theory of multivariable Hermite polynomials is naturally suited to evaluate integrals of Gaussian functions and of multiple products of Hermite polynomials. On other hand, the theory of special functions from the group-theoretic point of view, provides a unifying formalism to deal with the immense aggregate of the special functions and a collection of formulae such as the relevant differential equations, integral representations, recurrence formulae, composition theorems, etc., see for example [13, 14]. The first significant advance in the direction of obtaining generating relations by Lie-theoretic method is made by Miller [8,9], Rehana [12] and Weisner [15]- [17]. In this paper, we consider the three variable Hermite polynomials $H_n(x,y,z)$ and derive generating relations involving these polynomials and the associated Laguerre Polynomials $L_l^n(x)$ [11] by using representation $\uparrow_{\omega,\mu}$ of Lie algebra and operational representations.

2. Representation $\uparrow_{\omega,\mu}$ of $G(0,1)$ and Generating Relations

We note that the following isomorphism [8]

$$\zeta(0,1) \cong L\left[G(0,1)\right],$$

where $L\left[G(0,1)\right]$ is the Lie algebra of a complex four-dimensional Lie group $G(0,1)$, a multiplicative matrix group with elements ([8], p. 9)

$$g(a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (a,b,c,\tau) \in \mathbb{C}. \tag{2.1}$$

The group $G(0,1)$ is called the complex harmonic oscillator group (see [9],Chapter 10). A basis for $L[G(0,1)]$ is provided by the matrices (see [8],p.9)

$$j^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ j^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.2}$$

$$j^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with commutation relations

$$[j^3, j^\pm] = \pm j^\pm,\ [j^+, j^-] = -\varepsilon,\ [\varepsilon, j^\pm] = [\varepsilon, j^3] = \Theta. \tag{2.3}$$

The machinery constructed in ([8]; Chapters 1,2 and 4) will be applied to find a realization of the irreducible representation $\uparrow_{\omega,\mu}$ of $\zeta(0,1)$, where $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum S of $\uparrow_{\omega,\mu}$ is the set $S = \{-\omega + k, k - \text{a nonnegative integer}\}$. In particular, we are looking for the function $f_n(x,y,z;t) = Z_n(x,y,z)t^n$ such that

$$\begin{aligned} &J^3 f_n = n f_n,\ E f_n = \mu f_n, \\ &J^+ f_n = \mu f_{n+1},\ J^- f_n = (n+\omega) f_{n-1}, \\ &C_{0,1} f_n = (J^+J^- - EJ^3) f_n = \mu\omega f_n, \end{aligned} \tag{2.4}$$

for all $n \in S$. The commutation relations satisfied by the operators $j^\pm, j^3, E$ are

$$[J^3, J^\pm] = \pm J^\pm,\ [J^+, J^-] = -E,\ [J^\pm, E] = [J^3, E] = 0. \tag{2.5}$$

The number of possible solutions of Eq. (2.5) is tremendous. We assume that these operators take the form

$$J^+ = t\left[x + 2(y+z)\frac{\partial}{\partial x}\right],\ J^- = \frac{1}{t}\frac{\partial}{\partial t}, J^3 = t\frac{\partial}{\partial t},\ E = 1, \tag{2.6}$$

and note that these operators satisfy the commutation relations (2.5). We can assume $\omega = 0$ and $\mu = 1$ without any loss of generality for the theory of special functions. In terms of the functions $Z_n(x,y,z)$ relations (2.4) become

$$\begin{aligned} &\left(x + 2(y+z)\frac{\partial}{\partial x}\right) Z_n(x,y,z) = Z_{n+1}(x,y,z), \\ &\left(\frac{\partial}{\partial x}\right) Z_n(x,y,z) = n Z_{n-1}(x,y,z), \\ &\left(x\frac{\partial}{\partial x} + 2(y+z)\frac{\partial^2}{\partial x^2} - n\right) Z_n(x,y,z) = 0,\ n = 0,1,2,.. \end{aligned} \tag{2.7}$$

We see from (2.7) that $Z_n(x,y,z) = H_n(x,y,z)$ where $H_n(x,y,z)$ is given by (1.1). The functions $f_n(x,y,z;t) = H_n(x,y,z)t^n$, $n \in S$, form a basis for a realization of the representation $\uparrow_{1,0}$ of $\zeta(0,1)$. This realization of $\zeta(0,1)$ can be extended to a local multiplier representation $T(g)$, $g \in G(0,1)$, defined on F the space of all functions analytic in a neighborhood of the point $(x^0, y^0, z^0; t^0) = (1,1,1;1)$. Using operators (2.6), the local multiplier representation ([8], p. 17) takes the form

$$\begin{aligned} &[T(exp(a\varepsilon))f](x,y,z;t) = exp(a)f(x,y,z;t), \\ &\left[T(exp(bj^+))f\right](x,y,z;t) = exp(btx)f(x+2bt(y+z),y,z;t), \\ &\left[T(exp(cj^-))f\right](x,y,z;t) = f(x+\frac{c}{t},y,z;t), \\ &\left[T(exp(\lambda j^3))f\right](x,y,z;t) = f(x,y,z;te^{\lambda}), \end{aligned} \tag{2.8}$$

for $f \in F$. If $g \in G(0,1)$ has parameters (a,b,c,λ), then

$$T(g) = T(exp(a\varepsilon))T(exp(bj^+))T(exp(cj^-))T(exp(\lambda j^3)),$$

and therefore we obtain

$$[T(g)f](x,y,p,s;\tau) = exp(a+btx) \times f\left(x+2bt(y+z)+\frac{c}{t},y,z;te^{\lambda}\right). \tag{2.9}$$

The matrix elements of $T(g)$ with respect to the analytic basis

$$f_n(x,y,z;t) = H_n(x,y,z)t^n$$

are the functions $A_{lk}(g)$ uniquely determined by $\uparrow_{\omega,\mu}$ of $\zeta(0,1)$ and we obtain relations

$$[T(g)f_k](x,y,p,s;\tau) = \sum_{l=0}^{\infty} A_{lk}(g)f_l(x,y,p,s;\tau),\ k=0,1,2,...,$$

which simplify to the identity

$$\begin{aligned} &exp(a+\lambda k+btx)H_k\left(x+2bt(y+z)+\frac{c}{t},y,z\right) \\ &= \sum_{l=0}^{\infty} A_{lk}(g)H_l(x,y,z)t^{l-k},\ k=0,1,2,..., \end{aligned} \tag{2.10}$$

and the matrix element $A_{lk}(g)$ are given by [[8]; p. 87, Eq. (4.26)]

$$A_{lk}(g) = exp(a+k\lambda)c^{k-l}L_l^{(k-l)}(-bc),\ \{k,l\} >\geq 0. \tag{2.11}$$

Substituting (2.11) into (2.10), we obtain the following desired generating relation

$$exp(btx)H_k\left(x+2bt(y+z)+\frac{c}{t},y,z\right) = \sum_{l=0}^{\infty} c^{k-l}L_l^{(k-l)}(-bc)H_l(x,y,z)t^{l-k}, \tag{2.12}$$

$$\{b,c,t\} \in \mathbb{C},\ (l,k=0,1,2,...).$$

Now, we consider some applications of generating relation (2.12). Taking $b \mapsto 0$ in generating relation (2.12) and using the limit [[8], p. 88(4.29)]:

$$c^n L_l^n(bc)|_{b=0} \begin{cases} \binom{n+l}{n} c^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases} \tag{2.13}$$

the generating relation(2.12) reduces to the result:

$$H_k\left(x + ct^{-1}, y, z\right) = \sum_{l=0}^{k} \binom{k}{k-l} c^{k-l} H_l(x, y, z) t^{l-k}. \tag{2.14}$$

Taking $c \mapsto 0$ in generating relation (2.12) and using the limit [[8],p.88(4.29)]:

$$c^n L_l^n(bc)|_{c=0} \begin{cases} \frac{(-b)^{-n}}{(-n)!} & \text{if } n \leq 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{2.15}$$

we get

$$exp(btx) H_k\left(x + 2bt(y+z), y, z\right) = \sum_{l=0}^{k} \frac{b^{k-l}}{(l-k)!} H_l(x, y, z) t^{l-k}. \tag{2.16}$$

3. Operational Identities and Generating Functions

First of all, since

$$\hat{D}_x^{2k} \frac{x^{n-2r}}{(n-2r)!} = \frac{x^{n-2r-2k}}{(n-2r-2k)!}, \; \hat{D}_x = \frac{\partial}{\partial x},$$

we infer from the series representation (1.4) and the definition of the classical Hermite polynomials $H_n(x)$ [11] the identity

$$H_n(x, y, z) = e^{y \frac{\partial^2}{\partial x^2}} \left\{ (-z)^{\frac{n}{2}} H_n\left(\frac{x}{2\sqrt{(-z)}} \right) \right\}. \tag{3.1}$$

Secondly, according to the identity

$$\hat{D}_x^{2r} \frac{x^{n-2k}}{(n-2k)!} = \frac{x^{n-2r-2k}}{(n-2r-2k)!}, \tag{3.2}$$

we find from (1.4) and (1.3) that

$$H_n(x, y, z) = e^{z \frac{\partial^2}{\partial x^2}} H_n\left(x, y\right), \tag{3.3}$$

or equivalently

$$H_n(x, y, z) = e^{y \frac{\partial^2}{\partial x^2}} H_n\left(x, z\right). \tag{3.4}$$

Again, According to the identity

$$\hat{D}_x^{2r+2k}x^n = \frac{n!x^{n-2r-2k}}{(n-2r-2k)!}, \tag{3.5}$$

we get from (1.4) that

$$H_n(x,y,z) = e^{y\frac{\partial^2}{\partial x^2}+z\frac{\partial^2}{\partial x^2}}\left\{x^n\right\}. \tag{3.6}$$

Further, according to the identity

$$\hat{D}_x^{r+k}\frac{x^{n-r-k}}{(n-r-k)!} = \frac{x^{n-2r-2k}}{(n-2r-2k)!}, \tag{3.7}$$

we find from (1.4) that

$$H_n(x,y,z) = n!\sum_{r=0}^{n}\sum_{k=0}^{\left[\frac{n-r}{2}\right]}\frac{(-1)^{r+k}(-n)_{r+k}z^r y^k}{r!k!}\hat{D}_x^{r+k}\{x^{n-r-k}\}, \tag{3.8}$$

which further can be handled to get the symbolic relation:

$$H_n(x,y,z) = \left(1 - z\hat{D}_x x^{-1} - y\hat{D}_x x^{-1}\right)^n\left\{x^n\right\}, \tag{3.9}$$

or equivalently, in the more compact form

$$H_n(x,y,z) = \left(x - z\hat{D}_x - y\hat{D}_x\right)^n. \tag{3.10}$$

First, in this section we show how readily new generating functions for the polynomials $H_n(x,y,z)$ can be derived from the corresponding known generating functions of polynomials having direct operational relations with the polynomials $H_n(x,y,z)$. In this regard the operational formulas in (3.1), (3.3) and (3.4) play the key role in obtaining such generating functions for the polynomials $H_n(x,y,z)$. First, let us consider the well-known generating function [11]

$$exp[2xt - t^2] = \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}. \tag{3.11}$$

Upon replacing x by $\frac{x}{2\sqrt{(-z)}}$ in (3.11) and applying the first equation in (3.1), one obtain by routine calculations the following generating function

$$exp\left[y\frac{\partial^2}{\partial x^2} + \frac{xt}{\sqrt{(-z)}} - t^2\right] = \left(\frac{1}{-z}\right)^{\frac{n}{2}}\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.12}$$

Similarly , from the divergent generating function of Bateman [[11], p. 198(3)]

$$(1-2xt)^{\lambda}{}_2F_0\left[\frac{1}{2}\lambda, \frac{1}{2}\lambda + \frac{1}{2}; --; -4t^2/(1-2xt)^2\right] \cong \sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!}. \tag{3.13}$$

we get the generating function

$$e^{y\frac{\partial^2}{\partial x^2}}\left(1-\frac{x}{\sqrt{(-z)}}\right)^{-a}{}_2F_0\left[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};--;-4t^2/\left(1-\frac{x}{\sqrt{(-z)}}\right)^2\right] \tag{3.14}$$

$$\cong\sum_{n=0}^{\infty}(a)_nH_n(x,y,z)\frac{t^n}{n!}.$$

Next , we will show that starting from the operational representations of the polynomials $H_n(x,y,z)$ in the previous section , we can establish other new generating functions for the polynomials $H_n(x,y,z)$ [5]. In the identity of (3.3) multiply throughout by $\frac{t^n}{n!}$, sum and then employ the generating relation [7]

$$exp[xt-yt^2]=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}, \tag{3.15}$$

to get

$$exp\left[z\frac{\partial^2}{\partial x^2}+xt-yt^2\right]=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.16}$$

In the same manner, from the operational identity in (3.4) one can derive the following generating functions

$$exp\left[y\frac{\partial^2}{\partial x^2}+xt+zt^2\right]=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.17}$$

Similarly, from identity (3.4), we get

$$exp\left[y\frac{\partial^2}{\partial x^2}+xt-yt^2\right]=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.18}$$

Again, from the operational identity in (3.6) and (3.10) one can derive the following generating functions

$$exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^2}{\partial x^2}+xt\right)=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.19}$$

and

$$exp\left(\left[x-z\hat{D}_x-y\hat{D}_x\right]t\right)=\sum_{n=0}^{\infty}H_n(x,y,z)\frac{t^n}{n!}. \tag{3.20}$$

respectively. Again , by starting from equation (3.6) and (3.10) multiplying throughout by t^n and exploiting the previous outlined method, we can show that

$$e^{\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^2}{\partial x^2}\right)}(1-xt)^{-1}=\sum_{n=0}^{\infty}H_n(x,y,z)t^n, \tag{3.21}$$

and

$$\left(1-\left[x-z\hat{D}_x-y\hat{D}_x\right]t\right)=\sum_{n=0}^{\infty}H_n(x,y,z)t^n, \tag{3.22}$$

respectively. The previously outlined procedure offers a useful tool for the derivation of other families of generating functions for the polynomials $H_n(x,y,z)$, for example, bilinear and bilateral generating functions. Next, let us consider the generating function

$$f(x,y,w,u,z,v|t)=\sum_{n=0}^{\infty}H_n(x,y,z)\times H_n(w,u,v)\frac{t^n}{n!}, \tag{3.23}$$

which according to Equation (3.6) yields the following bilinear generating function

$$\begin{aligned} &exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^2}{\partial x^2}+u\frac{\partial^2}{\partial w^2}+v\frac{\partial^2}{\partial w^2}+xwt\right) \\ &\qquad=\sum_{n=0}^{\infty}H_n(x,y,z)\times H_n(w,u,v)\frac{t^n}{n!}. \end{aligned} \tag{3.24}$$

Also, we can apply an analogous procedure to get the further bilinear generating function

$$\begin{aligned} &(1-xwt)^{-1}exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^2}{\partial x^2}+u\frac{\partial^2}{\partial w^2}+v\frac{\partial^2}{\partial w^2}\right) \\ &\qquad=\sum_{n=0}^{\infty}H_n(x,y,z)\times H_n(w,u,v)t^n. \end{aligned} \tag{3.25}$$

In [6] the following operational identity for the associated Laguerre polynomials $L_n^m(x,y)$ have been introduced

$$(1-y\hat{D}_x)^n(y-\hat{D}_x^{-1})^m=L_n^m(x,y). \tag{3.26}$$

Let us consider the generating relation

$$f(x,y,z,u,v|t,w)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}L_n^m(u,v)\times H_n(x,y,z)\frac{t^n w^m}{n!m!}, \tag{3.27}$$

Now, directly from (3.10) and (3.26) by employing the previously outlined method leading to the bilinear generating functions, we obtain from (3.27) the following bilateral generating function

$$\begin{aligned} &exp\left[\left(x-z\hat{D}_x-y\hat{D}_x\right)\left(1-y\hat{D}_x\right)t+\left(y-\hat{D}_x^{-1}\right)w\right] \\ &\qquad=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}L_n^m(u,v)\times H_n(x,y,z)\frac{t^n w^m}{n!m!}. \end{aligned} \tag{3.28}$$

Conclusion

We have considered the problem of framing the three-variable Hermite polynomials $H_n(x, y, z)$ into the context of the representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\zeta(0, 1)$ of the complex harmonic group $G(0, 1)$ and the representations of operational identities. Generating relations involving $H_n(x, y, z)$ are obtained by using Millerś and operational technique. The study of $H_n(x, y, z)$ for applications as well as for its connections with various Lie algebras is an interesting problem for further research.

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