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ON CONSTRUCTIONS OF NEW SUPER EDGE-MAGIC GRAPHS FROM SOME OLD ONES BY ATTACHING SOME PENDANTS

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Abstract. Baskoro et al. [1] provided some constructions of new super edge-magic graphs from some old ones by attaching 1, 2, or 3 pendant vertices and edges. In this paper, we introduce (q, m)-super edge-magic total labeling and we give a construction of new super edge-magic graphs by attaching n pendant vertices and edges under some conditions, for any positive integer n. Also, we give a constraint on our construction.

1. Introduction

Throughout this paper, we only consider finite simple graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. And we denote by (v, e)-*graph* G a graph with v vertices and e edges. For any positive integer k, we denote by P_k the path graph with k vertices. For any graph G and positive integer n, the graph consisting of n disjoint copies of G will be denoted by nG.

An *edge-magic total labeling* on a (v, e)-graph G is a bijection $\lambda : V(G) \cup E(G) \to \{1, 2, \ldots, v + e\}$ with the property that, given any edge xy,

$$\lambda(x) + \lambda(xy) + \lambda(y) = k$$

for some constant k. Here, k is called the *magic sum* of G. A graph with an edge-magic total labeling is called *edge-magic*. In particular, if $\lambda(V(G)) = \{1, 2, \ldots, v\}$, then λ is called *super edge-magic total labeling* and a graph with a super edge-magic total labeling is called *super edge-magic*.

In [1], one can find the following construction of new super edge-magic graphs from some old ones by adding 1, 2, or 3 pendants.

Theorem 1.1 ([1]). *From any super edge-magic (v, e)-graph G with the magic sum k, we can construct a new super edge-magic total graph from G by adding*

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one pendant incident to vertex x of G whose label $k-2v-1$. The magic constant of the new graph is $k' = k+2$.

Theorem 1.2 ([1]). *Let a (v,e)-graph G be super edge-magic total with the magic sum k and $k \geq 2v+3$. Then, a new graph formed from G by adding exactly two pendants incident to two distinct vertices x and y of G whose labels $k-2v$ and $k-2v-2$ respectively is super edge-magic total with the magic constant $k' = k+4$.*

Theorem 1.3 ([1]). *Let a (v,e)-graph G be super edge-magic total with the magic sum k and $k \geq 2v+3$. Then, a new graph formed from G by adding exactly three pendants incident to three distinct vertices x, y and z of G whose labels $k-2v, k-2v-1$ and $k-2v-2$ respectively is super edge-magic total with the magic constant $k' = k+6$.*

In this paper we introduce a new notion called by (q,m)-super edge-magic total labeling. Using this notion, we will give a generalized construction of the previous ones. At the last we will give a constraint on choosing a positive integer m in our construction.

2. Main results

2.1. Our construction

In [3], one can find the following lemma which gives a necessary and sufficient condition for a graph being super edge-magic.

Lemma 2.1 ([3]). *A (v,e)-graph G is super edge-magic if and only if there exists a bijection $\lambda : V(G) \to \{1,2,\ldots,v\}$ such that the set*

$$S = \{\lambda(x) + \lambda(y) \mid xy \in E(G)\}$$

consists of e consecutive integers. In such a case, λ extends to a super edge-magic labeling of G with a magic sum $k = v+e+s$, where $s = \min S$ and

$$S = \{k-(v+1), k-(v+2), \ldots, k-(v+e)\}.$$

Analogously, we define (q,m)-super edge-magic total labeling as following:

Let G be a (v,e)-graph, $\mu : V(G) \to \{1,2,\ldots,v\}$ be a bijection and let q be an indeterminate. Suppose that there is a positive integer m such that

$$S = \{\mu(x) + \mu(y) | xy \in E(G)\}$$

is equal to

$$\{k-(v+q), k-(v+q+m), \ldots, k-(v+q+(e-1)m)\},$$

where $k = v+q+(e-1)m+s$ and $s = \min S$. In this case, define a map $\lambda : V(G) \cup E(G) \to \{1,2,\ldots,v\} \cup \{v+q, v+q+m, \ldots, v+q+(e-1)m\}$ by

$$\begin{cases} \lambda(x) = \mu(x) & \text{for all } x \in V(G), \\ \lambda(xy) = k - (\mu(x)+\mu(y)) & \text{for all } xy \in E(G). \end{cases}$$

Then for all $xy \in E(G)$, $\lambda(x) + \lambda(y) + \lambda(xy) = k$.

Definition. In the previous situations, λ is called a (q, m)-*super edge-magic total labeling*, G is called a (q, m)-*super edge-magic graph* and k is called the magic sum of G.

Example 2.2. All super edge-magic graphs are $(1, 1)$-super edge-magic graphs by Lemma 2.1.

Now, we introduce a generalized construction of the constructions induced by Theorems 1.1, 1.2, and 1.3.

Theorem 2.3. *Let G be a super edge-magic (v, e)-graph and λ be a super edge-magic total labeling with the magic sum k. Suppose that for fixed positive integers m and n, there is a (q, m)-super edge-magic total labeling λ' on nP_2 with the magic sum k'. If for all $i, j \in \{1, 2, \dots, n\}$, $\lambda'^{-1}(i)\lambda'^{-1}(j) \notin E(nP_2)$ and*

$$2v + s + (n-1)m - 2n + 1 \le k \le 3v + s - 2n,$$

where $s = \min\{\lambda'(u) + \lambda'(v) \mid uv \in E(nP_2)\}$, then a new graph $\tilde{G}$ formed from G by adding exactly n pendants incident to n distinct vertices whose labels are

$$k + 2n - (2v + s), k + 2n - (2v + s + m), \dots, k + 2n - (2v + s + (n-1)m),$$

respectively is super edge-magic. Moreover, there is a super edge-magic total labeling on $\tilde{G}$ with the magic sum $k + 2n$.

Proof. The hypothesis

$$2v + s + (n-1)m - 2n + 1 \le k \le 3v + s - 2n$$

implies that

$$k + 2n - (2v + s + (n-1)m) \ge 1 \text{ and } k + 2n - (2v + s) \le v.$$

So $k + 2n - (2v + s), k + 2n - (2v + s + m), \dots, k + 2n - (2v + s + (n-1)m)$ are vertex labels of G.

Since λ' is a (q, m)-super edge-magic total labeling on nP_2, the set

$$\begin{aligned} S' &= \{\lambda'(x) + \lambda'(y) \mid xy \in E(nP_2)\} \\ &= \{k' - (2n + q), k' - (2n + q + m), \dots, k' - (2n + q + (e-1)m)\}. \end{aligned}$$

For $i = 1, 2, \dots, n$, let u_i and v_i be the vertices of nP_2 such that

$$\lambda'(u_i) + \lambda'(v_i) = k' - (2n + q + (n-i)m) \text{ and } \lambda'(u_i) \in \{1, 2, \dots, n\}.$$

Since for all $i, j \in \{1, 2, \dots, n\}$, $\lambda'^{-1}(i)\lambda'^{-1}(j) \notin E(nP_2)$, $u_1, u_2, \dots, u_n$, $v_1, v_2, \dots, v_n$ are $2n$ distinct vertices of nP_2.

Let $x_1, x_2, \dots, x_n$ be vertices of G such that

$$\lambda(x_i) = k + 2n - (2v + s + (i-1)m)$$

for each $i = 1, 2, \dots, n$. Also, let $y_1, y_2, \dots, y_n$ be attached pendant vertices of $\tilde{G}$ incident to $x_1, x_2, \dots, x_n$, respectively. Define a map $\tilde{\lambda} : V(\tilde{G}) \cup E(\tilde{G}) \to$

$\{1, 2, \ldots, v + e + 2n\}$ by

$$\begin{cases} \tilde{\lambda}(x) = \lambda(x) & \text{for all } x \in V(G), \\ \tilde{\lambda}(y_i) = v + \lambda'(u_i) & \text{for all } i = 1, 2, \ldots, n, \\ \tilde{\lambda}(xy) = \lambda(xy) + 2n & \text{for all } xy \in E(G), \\ \tilde{\lambda}(x_i y_i) = v + \lambda'(v_i) & \text{for all } i = 1, 2, \ldots, n. \end{cases}$$

Then for all $xy \in E(G)$,

$$\tilde{\lambda}(x) + \tilde{\lambda}(y) + \tilde{\lambda}(xy) = \lambda(x) + \lambda(y) + \lambda(xy) + 2n = k + 2n$$

and for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} &\tilde{\lambda}(x_i) + \tilde{\lambda}(y_i) + \tilde{\lambda}(x_i y_i) \\ &= \lambda(x_i) + (v + \lambda'(u_i)) + (v + \lambda'(v_i)) \\ &= k + 2n - (2v + s + (i-1)m) + k' - (2n + q + (n-i)m) + 2v \\ &= k + 2n, \end{aligned}$$

since $k' = 2n + q + (n-1)m + s$. Hence $\tilde{\lambda}$ is a super edge-magic total labeling on $\tilde{G}$ and the magic sum is $k + 2n$. □

2.2. Generalizations of Baskoro's results

Here we will give a constraint on the constructions using Theorem 2.3. To see the constraint, we first introduce some lemmas.

In [4], one can find the following lemma.

Lemma 2.4 ([4]). *For a positive integer n, nP_2 is edge-magic if and only if n is odd.*

Lemma 2.5. *For a positive integer n, nP_2 is super edge-magic if and only if n is odd. Moreover, if n is odd, then there is a super edge-magic total labeling λ such that for all $i, j \in \{1, 2, \ldots n\}$, $\lambda^{-1}(i)\lambda^{-1}(j) \notin E(nP_2)$.*

Proof. By Lemma 2.4, if nP_2 is super edge-magic, then n is odd.

Conversely, suppose that n is odd. Let $n = 2k + 1$ for some positive integer k. Let $\{x_i y_i \,|\, i = 1, 2, \ldots, n\}$ be the set of all edges of nP_2. Define a map $\lambda' : V(nP_2) \to \{1, 2, \ldots, 2n\}$ by

$$\lambda'(x_i) = i \text{ and } \lambda'(y_i) = \begin{cases} 3k - \frac{i-5}{2} & \text{if } i = 1, 3, \ldots, 2k+1, \\ 4k - \frac{i-6}{2} & \text{if } i = 2, 4, \ldots, 2k. \end{cases}$$

One can see that λ is a bijection. Observe that the set

$$\begin{aligned} S &= \{\lambda'(x) + \lambda'(y) \,|\, xy \in E(nP_2)\} \\ &= \{\lambda'(x_i) + \lambda'(y_i) \,|\, i = 1, 2, \ldots, n\} \\ &= \{3k+3, 4k+4, 3k+4, 4k+5, \ldots, 5k+2, 4k+2, 5k+3, 4k+3\} \\ &= \{3k+3, 3k+4, \ldots, 4k+3, 4k+4, \ldots, 5k+3\}. \end{aligned}$$

Thus by Lemma 2.1, λ' induces a super edge-magic total labeling λ and hence nP_2 is super edge-magic and λ is a desired labeling. $\square$

Lemma 2.6. *For any positive integer n, nP_2 is $(q, 2)$-super edge-magic. Moreover, there is $(q, 2)$-super edge-magic total labeling λ such that for all $i, j \in \{1, 2, \ldots, n\}$, $\lambda^{-1}(i)\lambda^{-1}(j) \notin E(nP_2)$.*

Proof. Let $\{x_i y_i \mid i = 1, 2, \ldots, n\}$ be the set of all edges of nP_2. Define a map $\lambda' : V(nP_2) \to \{1, 2, \ldots, 2n\}$ by

$$\lambda'(x_i) = i \text{ and } \lambda'(y_i) = n + i.$$

Then λ' is a bijection and the set

$$\begin{aligned} S &= \{\lambda'(x) + \lambda'(y) \mid xy \in E(nP_2)\} \\ &= \{n + 2, n + 4, \ldots, 3n\}. \end{aligned}$$

Since $k = 2n + q + 2(n - 1) + (n + 2) = q + 5n$, the set

$$S = \{k - (2n + q), k - (2n + q + 2), \ldots, k - (2n + q + 2(n - 1))\}.$$

Hence λ' induces a $(q, 2)$-super edge-magic total labeling λ which we desired. $\square$

With Lemmas 2.5, 2.6, and Theorem 2.3, we obtain the following corollary immediately.

Corollary 2.7. *Let a (v, e)-graph G be super edge-magic with the magic sum k and let $n \in \mathbb{N}$. If $2v + n + 1 \leq k \leq 3v - n + 2$, then a new graph formed from G by attaching exactly $2n - 1$ pendants incident to $2n - 1$ distinct vertices whose labels are $k - 2v + n - 2, k - 2v + n - 3, \ldots, k - 2v - n$ respectively is super edge-magic total with the magic sum $k + 4n - 2$.*

Corollary 2.8. *Let a (v, e)-graph G be super edge-magic with the magic sum k and let $n \in \mathbb{N}$. If $2v + n + 1 \leq k \leq 3v - n + 2$, then a new graph formed from G by attaching exactly n pendants incident to n distinct vertices whose labels are $k - 2v + n - 2, k - 2v + n - 4, \ldots, k - 2v - n$ respectively is super edge-magic total with the magic sum $k + 2n$.*

We can understand Corollary 2.7 as a generalization of Theorem 1.1 and 1.3. We can also understand Corollary 2.8 as a generalization of Theorem 1.2.

Corollaries 2.7 and 2.8 give some types of examples of constructions using Theorem 2.3. So finding another type is a natural problem. We obtain the following proposition which gives some constraint on choosing a positive integer m in Theorem 2.3.

Proposition 2.9. *Let m be a positive integer. There is a (q, m)-super edge-magic total labeling λ_n on nP_2 such that for all $i, j \in \{1, 2, \ldots, n\}$, $\lambda_n^{-1}(i)\lambda_n^{-1}(j) \notin E(nP_2)$, for some positive integer n greater than 1 if and only if $m = 1$ or 2.*

Proof. By Lemmas 2.5 and 2.6, if $m = 1$ or 2, then there is a (q, m)-super edge-magic total labeling which we desired. Suppose that there is a (q, m)-super edge-magic total labeling λ_n on nP_2. Since λ_n is a (q, m)-super edge-magic total labeling on nP_2, the set

$$\begin{aligned} S &= \{\lambda_n(x) + \lambda_n(y) \mid xy \in E(nP_2)\} \\ &= \{k - (2n + q), k - (2n + q + m), \ldots, k - (2n + q + (n - 1)m)\}, \end{aligned}$$

where $k = 2n + q + (n - 1)m + s$ and $s = \min S$. For $i = 1, 2, \ldots, n$, let x_i and y_i be the vertices of nP_2 such that

$$\lambda_n(x_i) + \lambda_n(y_i) = k - (2n + q + (n - i)m) \text{ and } \lambda_n(x_i) \in \{1, 2, \ldots, n\}.$$

Then

$$\lambda_n(x_1) + \lambda_n(y_1) = k - (2n + q + (n - 1)m) \text{ and } \lambda_n(x_n) + \lambda_n(y_n) = k - (2n + q).$$

Therefore

$$(\lambda_n(x_n) + \lambda_n(y_n)) - (\lambda_n(x_1) + \lambda_n(y_1)) = m(n - 1).$$

On the other hands,

$$(\lambda_n(x_n) - \lambda_n(x_1)) + (\lambda_n(y_n) - \lambda_n(y_1)) \leq (n - 1) + (n - 1) = 2(n - 1).$$

Hence $m \leq 2$. □

Remark 2.10. In this remark we use the notations in Theorem 2.3.

(1) For any super edge-magic total labeling λ' on nP_2, s is a constant since $k' = \frac{3(3n+1)}{2}$. Therefore when we construct a new graph using Theorem 2.3, the vertices incident with new pendants are invariant although the super edge-magic total labeling on nP_2 is replaced.

(2) For any $(q, 2)$-super edge-magic labeling λ' on nP_2 the sum of all vertex labeling is $1 + 2 + \cdots + n = n(2n + 1)$. On the other hands, the sum of all vertex labeling is

$$s + (s + 2) + (s + 4) + \cdots + (s + 2(n - 1)) = n(s + n - 1).$$

Thus s is a constant $n + 2$. Hence when we construct a new graph using Theorem 2.3, the vertices incident with new pendants are invariant although the $(q, 2)$-super edge-magic total labeling on nP_2 is replaced.

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