Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 193–200
https://doi.org/10.4134/CKMS.c160059
pISSN: 1225-1763 / eISSN: 2234-3024

TWO DIMENSIONAL ARRAYS FOR ALEXANDER POLYNOMIALS OF TORUS KNOTS

Hyun-Jong Song

Abstract. Given a pair p, q of relative prime positive integers, we have uniquely determined positive integers x, y, u and v such that $vx - uy = 1$, $p = x + y$ and $q = u + v$. Using this property, we show that

$$\sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+(j-1)p} - \sum_{1\le k\le y, 1\le l\le u} t^{1+(k-1)q+(l-1)p}$$

is the Alexander polynomial $\Delta_{p,q}(t)$ of a torus knot $t(p, q)$. Hence the number $N_{p,q}$ of non-zero terms of $\Delta_{p,q}(t)$ is equal to $vx + uy = 2vx - 1$.

Owing to well known results in knot Floer homology theory, our expanding formula of the Alexander polynomial of a torus knot provides a method of algorithmically determining the total rank of its knot Floer homology or equivalently the complexity of its (1,1)-diagram. In particular we prove (see Corollary 2.8);

Let q be a positive integer$>$ 1 and let k be a positive integer. Then we have

$$\begin{array}{lrcl} (1) & N_{kq+1,q} & = & 2k(q-1)+1 \\ (2) & N_{kq+q-1,q} & = & 2(k+1)(q-1)-1 \\ (3) & N_{kq+2,q} & = & \frac{1}{2}k(q^2-1)+q \\ (4) & N_{kq+q-2,q} & = & \frac{1}{2}(k+1)(q^2-1)-q \end{array}$$

where we further assume q is odd in formula (3) and (4).

Consequently we confirm that the complexities of (1,1)-diagrams of torus knots of type $t(kq + 2, q)$ and $t(kq + q - 2, q)$ in [5] agree with $N_{kq+2,q}$ and $N_{kq+q-2,q}$ respectively.

1. Introduction

We exhibit a 2-dimensional expanding formula of the Alexander polynomial $\Delta_{p,q}(t)$ of a torus knot $t(p, q)$. Intriguing numerical conditions involved in such expansion naturally reveal the number $N_{p,q}$ of non-zero terms of $\Delta_{p,q}(t)$. Indeed they arise from derivation of a (1,1)-diagram of $t(p, q)$ as illustrated in Fig.1. For more details see [6].

Received March 18, 2016.

2010 *Mathematics Subject Classification.* 57M25.

Key words and phrases. torus knots, Alexander polynomials, knot Floer homology (1,1)-knots.

This work was supported by a Research Grant of Pukyong National University (2016).

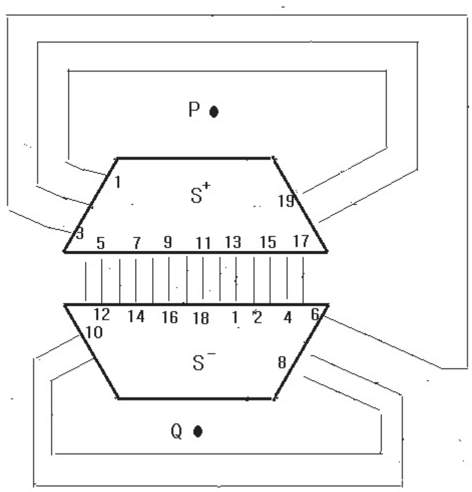

FIGURE 1. a (1,1)-diagram of $t(8,5)$

A (1,1)-diagram of a knot K in S^3 means a genus one Heegaard diagram of S^3 with a pair of points P, Q representing intersection of K and the associated Heegaard torus T such that the part of K in each solid torus determined by T constitutes a trivial arc therein. Moreover the meridian disk of the solid torus bounding each meridian circle in the Heegaard diagram is chosen to be disjoint from the trivial arc. For more systematic treatments of a (1,1)-decomposition of (S^3, K), see [3]. For a (1,1)-diagram of (S^3, K) the intersection number of the two meridian circles is said to be its complexity, A (1,1)-diagram is said to be minimal if it has the minimal complexity in the usual sense. In [3] Hayashi showed that a minimal (1,1)-diagram is uniquely determined up to isotopic type of the associated (1,1)-decomposition.

Nowadays knot Floer homology theorists pay much attention to (1,1)-diagrams. For computational aspect of knot Floer homology of (1,1)-diagrams, see [2], [8] or [9]. In particular it is well known that for a torus knot (or more generally a lens space surgery knot) the total rank of its knot Floer homology, the number of non-zero terms of its Alexander polynomial and the complexity of its minimal (1,1)-diagram, are same. See Theorem 1.2 of [7] and Corollary 3.4.5 of [8]. But for torus knots we can get those numbers quite easily and systematically from the expanding formula of their Alexander polynomials. For instance Figure 1 shows a minimal (1,1)-diagram of torus knot $t(8,5)$ with complexity 19. Indeed we show that $N_{5,8} = 19$ in Example 1. Matthew Hedden [4] kindly informed me that to his knowledge there was no such formula available.

Surprisingly in 1966 Carlitz [1] has already suggested a method of computing $N_{p,q}$ viewing $\Delta_{p,q}(t)$ as a generalization of a binary cyclotomic polynomial.

Indeed $\Delta_{p,q}(t)$ is equal to $\Phi_{pq}(t)$, the pq-th cyclotomic polynomial if p, q are distinct primes. His approach to $N_{p,q}$ is sightly different from ours. Through our expanding formula of $\Delta_{p,q}(t)$, we prove in Proposition 2.1 the Carlitz's four formulas stated without proof in [1]. Finally we show that the complexities of (1,1)-diagrams of torus knots $t(p, q)$ in [5] agree with $N_{p,q}$ in Corolary 2.8.

2. Proof of the main theorem

Proposition 2.1. *Given a pair p, q of relative prime positive integers, we have uniquely determined positive integers x, y, u and v such that $vx - uy = 1$, $p = x + y$ and $q = u + v$.*

The proof can be done with the following lemmas:

Lemma 2.2. *Let p, q be positive integers. If α_0, β_0 are integers such that $p\alpha_0 + q\beta_0 = 1$, then $p\alpha + q\beta = 1$ in Z if and only if $\alpha = \alpha_0 + qk, \beta = \beta_0 - pk$ for some integer k.*

Lemma 2.3. *Given a pair p, q of relative prime positive integers,*
(1) *there exist uniquely determined integers α, β such that $p\alpha + q\beta = 1$ and $0 < \alpha < q$.*
(2) *there exist uniquely determined positive integers x, y, u and v such that*

$$vx - uy = 1, \ p = x + y \quad \text{and} \quad q = u + v.$$

Proof. (1) Let α_0, β_0 be any integers such that $p\alpha_0 + q\beta_0 = 1$. Let α be the unique positive integer such that $\alpha_0 = qk + \alpha$ with $0 < \alpha < q$, and let $\beta = \beta_0 + pk$. Then we have $p\alpha + q\beta = 1$. From the first lemma, it follows that such α, β are uniquely determined.

(2) Let α, β be integers such that $p\alpha + q\beta = 1$ and $0 < \alpha < q$. Then $0 < -\beta < p$ since $q(-\beta) = p\alpha - 1 < p\alpha - 1 < pq$. Let $v = \alpha, y = -\beta$, and $x = p - y, u = q - v$). Then

$$p = x + y, \ q = u + v.$$

It follows from

$$vx - uy = v(p - y) - (q - v)y = pv + q(-y) = p\alpha + q\beta = 1$$

that $vx - uy = 1$. Since $pv + q(-y) = 1$ is such an expression in (1), the uniqueness of such expression follows from (1). □

As consequences of Proposition 2.1, we have:

Corollary 2.4. *With the notations in proposition 2.1, the following equations hold*;
(1) $pv + qx = pq + 1$,
(2) $pu + qy = pq - 1$,
(3) $qx - pu = 1$,
(4) $pv - qy = 1$.

Put

$$A_{p,q}(t) = \sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+(j-1)p} - \sum_{1\le k\le y, 1\le l\le u} t^{1+(k-1)q+(l-1)p}.$$

Theorem 2.5. *$A_{p,q}(t)$ is equal to $\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$, the Alexander polynomial $\Delta_{p,q}(t)$ of a torus knot $t(p,q)$.*

Proof. Put

$$\begin{aligned}
t^{p+q}A_{p,q}(t) &= \sum_{1\le i\le x, 1\le j\le v} t^{iq+jp} - t \sum_{1\le k\le y, 1\le l\le u} t^{kq+lp}\\
&\equiv P_1 - tN_1\\
t^{p}A_{p,q}(t) &= \sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+jp} - t \sum_{1\le k\le y, 1\le l\le u} t^{(k-1)q+lp}\\
&\equiv P_2 - tN_2\\
t^{q}A_{p,q}(t) &= \sum_{1\le i\le x, 1\le j\le v} t^{iq+(j-1)p} - t \sum_{1\le k\le y, 1\le l\le u} t^{kq+(l-1)p}\\
&\equiv P_3 - tN_3\\
A_{p,q}(t) &= \sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+(j-1)p} - t \sum_{1\le k\le y, 1\le l\le u} t^{(k-1)q+(l-1)p}\\
&\equiv P_4 - tN_4.
\end{aligned}$$

Then we have:

$$\begin{aligned}
(t^p-1)(t^q-1)A_{p,q}(t) &= \{(P_1-P_2)+(P_4-P_3)\} - t\{(N_1-N_2)+(N_4-N_3)\}\\
&\equiv P - tN.
\end{aligned}$$

Since

$$\begin{aligned}
P_1 - P_2 &= \sum_{1\le i\le x, 1\le j\le v} t^{iq+jp} - \sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+jp}\\
&= \sum_{1\le j\le v} t^{xq+jp} - \sum_{1\le j\le v} t^{jp}
\end{aligned}$$

and

$$\begin{aligned}
P_4 - P_3 &= \sum_{1\le i\le x, 1\le j\le v} t^{(i-1)q+(j-1)p} - \sum_{1\le i\le x, 1\le j\le v} t^{iq+(j-1)p}\\
&= \sum_{1\le j\le v} t^{(j-1)p} - \sum_{1\le j\le v} t^{xq+(j-1)p}
\end{aligned}$$

we have

$$\begin{aligned}
P &= \Big(\sum_{1\le j\le v} t^{xq+jp} - \sum_{1\le j\le v} t^{xq+(j-1)p}\Big) + \Big(\sum_{1\le j\le v} t^{(j-1)p} - \sum_{1\le j\le v} t^{jp}\Big)\\
&= t^{qx+pv} - t^{qx} + 1 - t^{pv}.
\end{aligned}$$

Replacing v and x in the right hand side of the above equation by u ad y respectively, we have

$$N = t^{qy+pu} - t^{qy} + 1 - t^{pu}.$$

Consequently we have

$$P - tN = t^{qx+pv} - t^{qy+pu+1} + (t^{pu+1} - t^{qx}) + (t^{qy+1} - t^{pv}) - t + 1.$$

By Corollary 2.4, the above equation is equal to $(t^{pq} - 1)(t - 1)$. $\square$

Corollary 2.6. *The number $N_{p,q}$ of all non-zero terms of $\Delta_{p,q}(t)$ is equal to $vx + uy = 2vx - 1$.*

Example 1. $N_{8,5} = 19$.

Proof. The desired x, u is uniquely determined by an equation:

$$5x - 8u = 1.$$

Hence we have $x = 5, y = 3 = u$ and $v = 2$; $N_{8,5} = vx + uy = 19$. $\square$

Example 2. $N_{55,21} = 545$.

Proof. Applying the Euclidean algorithm to an equation:

$$21x - 55u = 1$$

we have

$$8(x - 2u) - 13(3u - x) = 1.$$

Hence we get the claim from $x = 21, y = 34, u = 8$ and $v = 13$. $\square$

We provide proofs for the Carlitz's four formulas in [1] by means of our method.

Theorem 2.7.
(1) $N^+_{kq+1,q} = k(q-1) - 1$,
(2) $N^+_{kq+q-1,q} = k(q-1) + q - 1$,
(3) $N^+_{kq+2,q} = \frac{1}{4}k(q^2-1) + \frac{1}{2}(q+1)$,
(4) $N^+_{kq+q-2,q} = \frac{1}{4}(k+1)(q^2-1) - \frac{1}{2}(q-1)$,
where we assume q is odd in formula (3) *and* (4).

One would immediately notice that the Caritz's original formula (4) has a typographical error.

Proof. (1) An equation

$$qx - (kq + q - 1)u = 1$$

yields solutions $u = 1$, $x = k + 1$ and $v = q - 1$. Hence we have

$$N^+_{qk+q-1,q} = xv = k(q-1) + 1.$$

(2) An equation

$$qx - (kq + 1)u = 1$$

yields solutions $u = q-1$, $x = k(q-1)+1$: $v = 1$ and $y = k$. Hence we have

$$N^+_{qk+q-1,q} = xv = (k+1)(q-1) = k(q-1) + q - 1.$$

(3) Let $q = 2u_0 + 1$. Then an equation

$$qx - (kq+2)u = 1$$

is equivalent to

$$q(x-1) - qku - r(u-u_0) + (q-2u_0) = 1.$$

Since the last term of the left hand side of the above equation is equal to 1, we have

$$q(x-ku-1) = 2(u-u_0).$$

Since q must divide $u - u_0$ and $|u-u_0| < q$, it is possible only when $u = u_0$. And hence $x = ku_0 + 1$ and $v = q - u_0$. Since $u_0 = \frac{q-1}{2}$, we have

$$\begin{aligned} N^+_{kq+2,q} = xv &= (ku_0+1)(q-u_0) \\ &= q + qku_0 - ku_0^2 - u_0 \\ &= q + (qk-1)\frac{q-1}{2} - k\frac{(q-1)^2}{4} \\ &= \frac{1}{4}k(q^2-1) + \frac{1}{2}(q+1). \end{aligned}$$

(4) Let $q = 2u_0 - 1$. Then an equation

$$qx - (kq+q-2)u = 1$$

is equivalent to

$$q(x+1) - (kq+q)u + 2(u-u_0) + (2u_0-q) = 1.$$

Since the last term of the left hand side of the above equation is equal to 1, we have

$$q\{x-(k+1)u+1\} = 2(u_0-u).$$

By the argument in proof of (3), we have $u = u_0$, $x = (k+1)u_0 - 1$ and $v = q - u_0$. Since $u_0 = \frac{q+1}{2}$, we have

$$\begin{aligned} N^+_{kq+q-2,q} = xv &= \{(k+1)u_0 - 1\}(q-u_0) \\ &= \{(k+1)q+1\}u_0 - (k+1)u_0^2 - q \\ &= \{(k+1)q+1\}\frac{q+1}{2} - (k+1)\frac{(q+1)^2}{4} - q \\ &= \frac{1}{4}(k+1)(q^2-1) - \frac{1}{2}(q-1). \end{aligned}$$

□

Since $N_{p,q} = 2N^+_{p,q} - 1$, from Theorem 2.7 we have:

Corollary 2.8. *Let q be a positive integer > 1. Then for any positive integer k we have*
(1) $N_{kq+1,q} = 2k(q-1)+1$,
(2) $N_{kq+q-1,q} = 2(k+1)(q-1)-1$,
(3) $N_{kq+2,q} = \frac{1}{2}k(q^2-1)+q$,
(4) $N_{kq+q-2,q} = \frac{1}{2}(k+1)(q^2-1)-q$,
where we further assume q is odd in formula (3) *and* (4).

In [5] Kim and Kim introduced a (1,1)-diagram $D(a,b,c,r)$ parameterized by four nonnegative integers a, b, c, and r such that its complexity is denoted by $d = 2a+b+c$. For more details see Section 4 of [5]. And they claimed in Corollary 1 of page 1117 that for all odd $q > 1$ and $k \geq 1$, a torus knot $t(kq+2,q)$ admits a (1,1)-diagram $D(\frac{q-1}{2}, 1, \frac{k(q^2-1)}{2}, r)$, and a torus knot $t(kq+q-2,q)$ admits a (1,1)-diagram $D(\frac{q-1}{2}, 1, \frac{(k+1)(q^2-1)-4q}{2}, s)$.

Thus we have:
the complexcity of $D(\frac{q-1}{2}, 1, \frac{k(q^2-1)}{2}, r)$ is equal to

$$\begin{aligned} d = 2a+b+c &= (q-1)+1+\frac{k(q^2-1)}{2} \\ &= \frac{1}{2}k(q^2-1)+q \\ &= N_{kq+2,q} \end{aligned}$$

and
the complexcity of $D(\frac{q-1}{2}, 1, \frac{(k+1)(q^2-1)-4q}{2}, s)$ is equal to

$$\begin{aligned} d = 2a+b+c &= (q-1)+1+\frac{(k+1)(q^2-1)-4q}{2} \\ &= \frac{1}{2}(k+1)(q^2-1)-q \\ &= N_{kq+q-2,q}. \end{aligned}$$

References

[1] L. Carlitz, *The number of terms in the cyclotomic polynomial $F_{pq}(x)$*, Amer. Math. Monthly **73** (1966), no. 9, 979–981.
[2] H. Goda, H. Mastuda, and T. Morifuji, *Knot Floer homology of $(1,1)$-knots*, Geom. Dedicata **112** (2005), no. 1, 197–214.
[3] C. Hayashi, *1-genus 1-bridge splittings for knots*, Osaka J. Math. **41** (2004), no. 2, 371–426.
[4] M. Hedden, *A private communication.*
[5] S.-H. Kim and Y.-K. Kim, *Torus knots and 3-manifolds*, J. Knot Theory Ramifications **13** (2004), no. 8, 1103–1119.
[6] S. Y. Lee and H.-J. Song, *Pointed rail road systems for $(1,1)$-diagrams of torus knots*, in preparation.
[7] P. S. Ozsvath and Z. Szabo, *On knot Floer homology and lens space surgeries*, Topology **44** (2005), no. 6, 1281–1300.
[8] B. A. Racz, *Geometry of $(1,1)$-Knots and Knot Floer Homology*, Thesis (Ph.D.)-Princeton University, 2015.

[9] J. Rasmussen, *Knot Polynomials and Knot Homologies*, Geometry and topology of manifolds, 261280, Fields Inst. Commun., 47, Amer. Math. Soc., Providence, RI, 2005.

Hyun-Jong Song
Department of Applied Mathematics
Pukyong National University
Pusan 608-737, Korea
E-mail address: hjsong@pknu.ac.kr