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FINITE TOPOLOGICAL SPACES AND GRAPHS

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Abstract. We define a stratification and a partition of a finite topological space and define a partial order on the partition. Open subsets can be described completely in terms of this partially ordered partition. We associate a directed graph to the partially ordered partition of a finite topological space. This gives a one-to-one correspondence between finite topological spaces and a certain class of directed graphs.

1. Introduction

Let $(X, \mathcal{T})$ be a finite topological space. How can we describe its open subsets? One way may be to give a partition $\mathcal{M} \subset \mathcal{P}(X)$ of X such that every open subset is a union of elements of $\mathcal{M}$. (Then every closed subset is also a union of elements of $\mathcal{M}$.) Of course, $\mathcal{M} = \{\{x\} \mid x \in X\}$ is such a partition, which does not give any useful information on $\mathcal{T}$. We want to find the "coarsest" among such partitions.

We define inductively a filtration $Y^{(0)} = X \supset Y^{(1)} \supset \cdots$ and a stratification $X = X^{(0)} \cup X^{(1)} \cup \cdots$ of X in §2.1. $X^{(0)}$ is the union of minimal (nonempty) open subsets of $Y^{(0)}$ and $Y^{(1)} = Y^{(0)} \setminus X^{(0)}$. $X^{(1)}$ is the union of the minimal open subsets of the subspace $Y^{(1)}$ and $Y^{(2)} = Y^{(1)} \setminus X^{(1)}$, and so on. Then $\mathcal{M}$, the set of minimal open subsets of $Y^{(0)}, Y^{(1)}, \ldots$, is the desired partition of X. It turns out that $\mathcal{M}$ is the partition given by some equivalence relation: $x \sim y$ if every neighborhood (in X) of x contains y and vice versa.

To describe open subsets in terms of $\mathcal{M}$, we define a partial order $\leq$ on $\mathcal{M}$ by the reverse inclusion relation of their closures (in X): $U \leq V$ if $\overline{U} \supset \overline{V}$. Then $\widetilde{U} := \cup_{V \leq U} V$, where the union is over all $V \in \mathcal{M}$ with $V \leq U$, is the smallest open subset containing $U \in \mathcal{M}$ and these subsets form a basis for $\mathcal{T}$ (§2.2). (On the other hand, the closure of U is given by $\overline{U} = \cup_{U \leq V} V$.) Thus a union A of elements of $\mathcal{M}$ is open if and only if $U \subset A$ implies $V \subset A$ for all $V \leq U$.

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To the partially ordered set $(\mathcal{M}, \leq)$, we can associate a directed graph in an obvious way: vertices are elements of $\mathcal{M}$ and each pair (U, V) with $U < V$ represents a directed edge. (To avoid loops at every vertex, we use the strict partial order $<$.) This graph has many redundant edges which may be removed without losing any information on $\mathcal{T}$: we remove (U, V) if there is a directed path of length ≥ 2 from U to V. See §3.1. For example, if $X = \{1, 2, 3, 4, 5\}$ with $\mathcal{T} = \{\emptyset, \{1, 2\}, \{1, 2, 3, 4\}, X\}$, then the associated directed graph is

$$\{1, 2\} \to \{3, 4\} \to \{5\} .$$

Conversely, given any admissible directed graph (whose definition is given in §3.3), we can construct a finite topological space. This gives a one-to-one correspondence between finite topological spaces and admissible directed graphs (Theorem 4.1). Under this correspondence, automorphisms of finite topological spaces correspond to those of admissible directed graphs (§4.2). And the dual space corresponds to the dual graph (§4.3). We hope to find further applications of this correspondence.

In the last section, we give remarks on known results, especially those of [2] in comparison with ours. The author would like to thank the referee who kindly indicated results of [2].

2. Finite topological spaces

Let $(X, \mathcal{T})$ be a finite topological space. We give a description of topology $\mathcal{T}$ in terms of some stratification defined inductively and minimal open subsets of strata. By a *minimal* open subset, we always mean a minimal nonempty open subset. Note that any two minimal open subsets are either identical or disjoint since otherwise their intersection will be a smaller open subset.

2.1. Stratification

We define inductively a finite stratification $X = X^{(0)} \cup X^{(1)} \cup \cdots$ of X by locally closed subsets and a finite decreasing filtration $Y^{(0)} \supset Y^{(1)} \supset \cdots$ by closed subsets of X such that $Y^{(k)} = X^{(k)} \cup X^{(k+1)} \cup \cdots$ and $X^{(k)} = Y^{(k)} \setminus Y^{(k+1)}$ as follows.

First, let $Y^{(0)} = X$. Suppose we have defined $Y^{(k)}$ which is closed in X. Let $X^{(k)}$ be the (disjoint) union of minimal open subsets of the subspace $Y^{(k)}$. Then $X^{(k)}$ is dense in $Y^{(k)}$ since any nonempty open subset contains a minimal one. Let $Y^{(k+1)} = Y^{(k)} \setminus X^{(k)}$, which is closed in X. Thus $Y^{(k)} = X^{(k)} \cup X^{(k+1)} \cup \cdots$. Note that since X is finite, we have $Y^{(n)} = X^{(n)}$ and $Y^{(n+1)} = \emptyset$ for some n in the end.

Each stratum $X^{(k)}$ is locally closed, in other words, is open in its closure $Y^{(k)}$. The subspace topology of each stratum is easy to describe.

Proposition 2.1. *The subspace topology of $X^{(k)}$ is generated by its minimal open subsets: every open subset is a finite disjoint union of minimal open subsets. Every open subset is also closed. In particular, minimal open subsets are minimal closed subsets.*

Proof. Follows from the fact that distinct minimal open sets are disjoint and there are only finitely many minimal open sets. □

2.2. Minimal open subsets of strata

Let $\mathcal{M}$ be the set of minimal open subsets of $Y^{(0)}, Y^{(1)}, \ldots$ (hence of $X^{(0)}$, $X^{(1)}, \ldots$). Let us call $\mathcal{M}$ the *partition of X associated to $\mathcal{T}$*. We define a relation $\le$ on $\mathcal{M}$, which will be shown to be a partial order soon, as follows: for $U, V \in \mathcal{M}$, we define $U \le V$ if $\overline{U} \supset V$ where $\overline{U}$ denotes the closure of U in X. As usual, we will write $U < V$ if $U \le V$ and $U \ne V$.

Let us say $U \in \mathcal{M}$ is of *level k* if U is a minimal open subset of $Y^{(k)}$. The level of $U \in \mathcal{M}$, which will be denoted by $\ell(U)$, is the largest integer k such that $U \subset Y^{(k)}$.

Proposition 2.2. *Let $U, V \in \mathcal{M}$ with $U < V$. Then we have $\ell(U) < \ell(V)$.*

Proof. Suppose U and V are of level k and k', respectively. We have $\overline{U} \subset Y^{(k)}$ since $Y^{(k)}$ is closed in X. Hence $V \subset Y^{(k)}$ and $k \le k'$. Suppose $k = k'$. Then both U and V are minimal open subsets of $X^{(k)}$. Since a minimal open subset of a stratum is closed in that stratum, we have $\overline{U} \cap X^{(k)} = U$ hence $\overline{U} \cap V = \emptyset$, which is a contradiction. □

Corollary 2.3. *The relation $\le$ on $\mathcal{M}$ is a partial order.*

Proof. The only non-trivial partial order axiom to be checked for $\le$ is its antisymmetry: $U \le V$ and $V \le U$ imply $U = V$. Suppose $U \le V$ and $V \le U$ but $U \ne V$. Then by the last proposition we have $\ell(U) < \ell(V)$ and $\ell(V) < \ell(U)$. □

Open and closed subsets of X are disjoint union of elements of $\mathcal{M}$. We can give more precise description using the partial order on $\mathcal{M}$.

Proposition 2.4. *Every nonempty open subset of X is a finite disjoint union of elements of $\mathcal{M}$. The same is true for closed subsets of X.*

Proof. Any open (resp. closed) subset U of X is a disjoint union $U = \cup_k (X^{(k)} \cap U)$ of open (resp. closed) subsets of strata. By Proposition 2.1 these are finite disjoint union of elements of $\mathcal{M}$. □

For $U \in \mathcal{M}$, $\overline{U}$ is the smallest closed subset of X containing U. Also, there exists the smallest open subset of X that contains U since there are only finitely many such open subsets. Let us denote this open set by $\widetilde{U}$.

Proposition 2.5. *For $U, V \in \mathcal{M}$, we have $U \le V$ if and only if $U \subset \widetilde{V}$.*

Proof. For $x \in V$, $\widetilde{V}$ is also the smallest open subset of X containing x by the last proposition. Hence if $U \subset \widetilde{V}$, then $V \subset \overline{U}$. The converse is proved similarly. □

Corollary 2.6. *Let $U \in \mathcal{M}$. Then we have $\overline{U} = \cup_{V \in \mathcal{M}, U \leq V} V$ and $\widetilde{U} = \cup_{V \in \mathcal{M}, V \leq U} V$. Thus a subset S of X is open (resp. closed) if and only if for any $U \in \mathcal{M}$ with $U \subset S$, we have $V \subset S$ for all $V \in \mathcal{M}$ with $V \leq U$ (resp. $U \leq V$).*

From Proposition 2.4, it is also clear that points in one $U \in \mathcal{M}$ can not be distinguished from each other topologically. More precisely, we have the following. Let us define a relation $\sim$ on X by $x \sim y$ if every neighborhood (in X) of x contains y and vice versa. Clearly, $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$.

Corollary 2.7. *The relation $\sim$ is an equivalence relation. The equivalence classes are precisely elements of $\mathcal{M}$.*

Proof. If $x \in U \in \mathcal{M}$, then $\overline{\{x\}} = \overline{U}$. And for $U, V \in \mathcal{M}$, we have $U = V$ if and only if $\overline{U} = \overline{V}$ by Corollary 2.6 and Proposition 2.2. □

3. Admissible directed graphs

Given a finite topological space $(X, \mathcal{T})$, we will define a directed graph $\mathcal{G} = \mathcal{G}(X, \mathcal{T})$. Actually, the graph is constructed from the finite partially ordered set $(\mathcal{M}, \leq)$. The pair $(X, \mathcal{T})$ can be reconstructed from the graph $\mathcal{G}$. This gives a one-to-one correspondence between finite topological spaces and a certain class of directed graphs.

3.1. Definition of the graph $\mathcal{G}(X, \mathcal{T})$

Let $\mathcal{G}_0 = \mathcal{G}_0(X, \mathcal{T})$ be the directed graph whose vertices and directed edges are given as follows. The vertices are elements of $\mathcal{M}$. Two vertices $U, V \in \mathcal{M}$ are connected by a directed edge from U to V if $U < V$.

The graph $\mathcal{G}_0$ contains many redundant edges, which may be discarded without losing any information on $(X, \mathcal{T})$, or equivalently, on $(\mathcal{M}, \leq)$. Let $\mathcal{G} = \mathcal{G}(X, \mathcal{T})$ be the directed graph obtained from $\mathcal{G}_0$ by discarding some edges as follows. Let $U, V \in \mathcal{M}$ with $U < V$. We remove the edge from U to V if there exists a directed path from U to V of length ≥ 2, in other words, if there exist $U_0 = U, U_1, \ldots, U_k = V \in \mathcal{M}$ with $k \geq 2$ such that $U_0 < U_1 < \cdots < U_k$.

3.2. Reconstruction of $(X, \mathcal{T})$ from $\mathcal{G}(X, \mathcal{T})$

The directed graph $\mathcal{G}(X, \mathcal{T})$ determines the partially ordered set $\mathcal{M}$ completely, hence determines the topology $\mathcal{T}$ on X by Corollary 2.6. Moreover, $\mathcal{G}(X, \mathcal{T})$ determines the level function on $\mathcal{M}$. Intuitively, this is obvious since the level is defined using the topology of X.

Proposition 3.1. *If the level of $U \in \mathcal{M}$ is k, then we have $\widetilde{U} \cap X^{(j)} \neq \emptyset$ for $j \leq k$ and there exists a chain $U_0 < U_1 < \cdots < U_k = U$ in $\mathcal{M}$ such that $\ell(U_j) = j$ for $j \leq k$. Thus $\ell(U)$ is the length of the longest directed path that ends at U.*

Proof. For $j \leq k$, we have $\widetilde{U} \cap X^{(j)} \neq \emptyset$ since $X^{(j)}$ is dense in $Y^{(j)}$. For the second statement, choose any minimal open subset U_{k-1} of $X^{(k-1)}$ contained in $\widetilde{U} \cap X^{(k-1)}$. Then clearly, $\widetilde{U}_{k-1} \subset \widetilde{U}$ by the minimality. Repeating with $\widetilde{U}_{k-1}$ in place of $\widetilde{U}$, we obtain a desired chain. □

3.3. Admissible directed graphs

Let $\mathcal{G}$ be a directed graph. A *directed cycle* is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$ such that $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$ and (v_n, v_1) are directed edges of $\mathcal{G}$. If the sequence $v_1, v_2, \ldots, v_n$ with $n \geq 3$ is such that $(v_1, v_2), \ldots, (v_{n-1}, v_n)$ and (v_1, v_n) are directed edges of $\mathcal{G}$, then let us call the sequence an *almost directed cycle*.

Definition 3.2. Let us say a directed graph is *admissible* if

(1) it is finite and simple, i.e., it has no loops nor multiple edges,
(2) its vertices are disjoint finite sets and
(3) it contains no directed nor almost directed cycle.

Proposition 3.3. *The directed graph $\mathcal{G}(X, \mathcal{T})$ associated to a finite topological space $(X, \mathcal{T})$ is admissible.*

Proof. It contains no directed cycle since its directed edges are defined in terms of the strict partial order $<$ on $\mathcal{M}$. It contains no almost directed cycle of length ≥ 3 simply because we have discarded the edges in the opposite direction from such cycles of $\mathcal{G}_0(X, \mathcal{T})$ to obtain $\mathcal{G}(X, \mathcal{T})$. □

Conversely, we can construct a finite topological space from an admissible directed graph as in the following.

3.4. From $\mathcal{G}$ to $(X, \mathcal{T})$

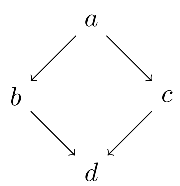
Let $\mathcal{G}$ be an admissible directed graph and let $\mathcal{M}$ be the set of its vertices. Let $<$ be the relation on $\mathcal{M}$ defined as follows: $U < V$ if there exists a directed path from U to V in $\mathcal{G}$. Clearly, $<$ is a strict partial order on $\mathcal{M}$ by the admissibility of $\mathcal{G}$. Let us write as usual, $U \leq V$ for $U, V \in \mathcal{M}$ if $U < V$ or $U = V$.

Let $X = \cup_{U \in \mathcal{M}} U$. And for each $U \in \mathcal{M}$, let $\widetilde{U} := \cup_{V \leq U} V$. Then the set of $\widetilde{U}$ for $U \in \mathcal{M}$ is a basis for a topology $\mathcal{T}$ on X. Really, if $x \in \widetilde{U} \cap \widetilde{V}$, then $\widetilde{W} \subset \widetilde{U} \cap \widetilde{V}$ where $W \in \mathcal{M}$ is such that $x \in W$. We have the following. Compare with Proposition 2.4 and Corollary 2.6.

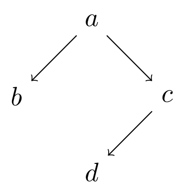
Proposition 3.4. *Let $\mathcal{G}$ be an admissible directed graph, $\mathcal{M}$ be the set of its vertices and let $(X, \mathcal{T})$ be the associated finite topological space. Any non-empty* *open (or closed) subset of X is a disjoint union of elements of $\mathcal{M}$. For $U \in \mathcal{M}$, $\widetilde{U} = \cup_{V \le U} V$ is the smallest open subset containing U and the closure $\overline{U}$ of U in X is $\cup_{U \le V} V$.*

Proof. The first two statements are obvious from the definition of the topology. Let $U \in \mathcal{M}$ and let $C = \cup_{U \le V} V$. Then C is closed. Really, if $W \in \mathcal{M}$ is contained in $X \backslash C$, then $\widetilde{W} \subset X \backslash C$ since otherwise there exists $V \in \mathcal{M}$ with $U \le V$ and $V \le W$. Suppose $V \in \mathcal{M}$ is such that $V \not\subset \overline{U}$. Then $V \cap \overline{U} = \emptyset$ and $\widetilde{V} \subset X \backslash \overline{U}$ since $\widetilde{V}$ is the smallest open subset containing V. Thus $U \not\le V$. □

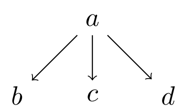
Example 3.5. Let $X = \{a, b, c, d\}$. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ be topologies on X corresponding to the following graphs, respectively. (We have labeled vertices by $a, \ldots$ in place of $\{a\}, \ldots$ for simplicity.)



(Graph 1)



(Graph 2)



(Graph 3)

Then we have

$\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$,

$\mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$,

$\mathcal{T}_3 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, X\}$.

Example 3.6. Suppose $X = a \cup b \cup c \cup d$ is a partition of a finite set X. Then the graphs in the previous example corresponds to similar topologies on X. For example, the topology associated to the first graph is:

$$\mathcal{T}_1 = \{\emptyset, a, a \cup b, a \cup c, a \cup b \cup c, X\}$$

4. Correspondence between spaces and graphs

4.1. Correspondence

Given a finite topological space $(X, \mathcal{T})$, we have defined an admissible directed graph $\mathcal{G}(X, \mathcal{T})$ in §3.1. Conversely, given an admissible directed graph $\mathcal{G}$, we have defined a finite topological space $(X, \mathcal{T})$ in §3.4. Let us denote these two maps by $\Phi : (X, \mathcal{T}) \mapsto \mathcal{G}$ and $\Psi : \mathcal{G} \mapsto (X, \mathcal{T})$, respectively.

Theorem 4.1. *The two maps Φ and Ψ are inverse to each other and induce a one-to-one correspondence between finite topological spaces and admissible directed graphs.*

Proof. It is clear from Corollary 2.6 that $\Psi \circ \Phi$ is the identity map. See §3.2. Suppose $\mathcal{G}$ is an admissible directed graph and let $(X, \mathcal{T}) = \Psi(\mathcal{G})$ and $\mathcal{G}' =$

$\Phi \circ \Psi(\mathcal{G})$. Let $\mathcal{M}$ (resp. $\mathcal{M}'$) be the set of vertices of $\mathcal{G}$ (resp. $\mathcal{G}'$) equipped with the partial order of §3.4 (resp. of §2.2).

Note that $(X, \mathcal{T}) = \Psi(\mathcal{G}) = \Psi \circ \Phi \circ \Psi(\mathcal{G}) = \Psi(\mathcal{G}')$. By Proposition 3.4, any non-empty open or closed subset of X is a disjoint union of elements of $\mathcal{M}$, and also a union of elements of $\mathcal{M}'$. In particular, for each $U \in \mathcal{M}$, $U = \widetilde{U} \cap \overline{U}$ is a disjoint union of elements of $\mathcal{M}'$. Thus $\mathcal{M}'$ is a partition of X which is finer than $\mathcal{M}$. Exchanging the role of $\mathcal{M}$ and $\mathcal{M}'$, we have $\mathcal{M} = \mathcal{M}'$ as sets (ignoring partial orders on them for a moment).

Suppose (U, V) is an edge in $\mathcal{G}$: a directed edge from $U \in \mathcal{M}$ to $V \in \mathcal{M}$. Then $U \subset \widetilde{V}$ by definition and (U, V) is an edge in $\mathcal{G}_0(X, \mathcal{T})$ by Proposition 3.1. (See §3.1 for definition of $\mathcal{G}_0(X, \mathcal{T})$.) Suppose this (U, V) is not an edge in $\mathcal{G}(X, \mathcal{T}) = \mathcal{G}'$. This happens only when there is a directed path of length $k \geq 2$ in $\mathcal{G}_0(X, \mathcal{T})$ from U to V. (See §3.1.) Suppose $(U, U_1), (U_1, U_2), \dots, (U_{k-1}, V)$ is such a path. Then we have $\widetilde{U} \subset \widetilde{U_1} \subset \cdots \subset \widetilde{V}$ and this implies $U < U_1 < \cdots < V$ in the partially ordered set $\mathcal{M}$. Hence there exists a directed path of length ≥ 2 from U to V in $\mathcal{G}$ contradicting the admissibility. Hence (U, V) is an edge in $\mathcal{G}'$.

Conversely, suppose (U, V) is an edge in $\mathcal{G}'$. Then we have $U \subset \widetilde{V}$ in X and $U < V$ in $\mathcal{M}$. Hence there exists a directed path from U to V in $\mathcal{G}$. If this directed path is of length ≥ 2, i.e., is not a single edge, then by the same argument as above there exists a directed path of length ≥ 2 from U to V in $\mathcal{G}'$, which is a contradiction to the admissibility of $\mathcal{G}' = \mathcal{G}(X, \mathcal{T})$. □

4.2. Automorphisms

An automorphism of a graph is a permutation of its vertices which preserves the edge relation. Let $\mathcal{G}$ be an admissible directed graph and $\mathcal{M}$ be the set of its vertices. An automorphism of the admissible directed graph $\mathcal{G}$ is the automorphism of the underlying directed graph induced from a bijection of the set $X = \cup_{U \in \mathcal{M}} U$ which preserves $\mathcal{M}$ and the edge relation of $\mathcal{G}$.

Proposition 4.2. *Let $(X, \mathcal{T})$ be a finite topological space and $\mathcal{G}$ be the corresponding admissible directed graph. Then an automorphism of $(X, \mathcal{T})$ induces an automorphism of $\mathcal{G}$ and vice versa.*

Proof. It is clear that an automorphism of $(X, \mathcal{T})$ preserves the stratification and the filtration of X defined in §2.1. Hence it induces automorphisms of $\mathcal{G}_0(X, \mathcal{T})$ and $\mathcal{G}(X, \mathcal{T})$.

Conversely, let σ be an automorphism of $\mathcal{G}$ induced from a bijection of the set X, which will be denoted by the same σ. Then for $U \in \mathcal{M}$, we have

$$\sigma(\widetilde{U}) = \sigma(\cup_{V \leq U} V) = \cup_{V \leq U} \sigma(V) = \cup_{W \leq \sigma(U)} W = \widetilde{\sigma(U)} .$$

Since $\{\widetilde{U}\}_{U \in \mathcal{M}}$ is a basis for $\mathcal{T}$, σ induces a bijection of the set $\mathcal{T}$, hence an automorphism of $(X, \mathcal{T})$. □

4.3. Duality

For a finite topological space $(X, \mathcal{T})$, the set $\mathcal{T}^\vee$ of closed subsets also satisfies the axioms for a topology. Let us say this topology $\mathcal{T}^\vee$ is *dual* to the original topology $\mathcal{T}$.

For an admissible directed graph $\mathcal{G}$, the directed graph $\mathcal{G}^\vee$ obtained from $\mathcal{G}$ by reversing the orientation of all edges is also admissible. Let us say $\mathcal{G}^\vee$ is *dual* to $\mathcal{G}$. The duality $(\ ) \mapsto (\ )^\vee$ commutes with the correspondences Φ and Ψ. In other words, we have the following.

Proposition 4.3. *Let $(X, \mathcal{T})$ be a finite topological space and $\mathcal{G}$ be the corresponding admissible directed graph. Then the dual topological space $(X, \mathcal{T}^\vee)$ corresponds to the dual admissible directed graph $\mathcal{G}^\vee$.*

Proof. The set of $\widetilde{U} = \cup_{V \leq U} V$ for $U \in \mathcal{M}$ is a basis for $\mathcal{T}$. Clearly, the set of $\overline{U} = \cup_{U \leq V} V$ for $U \in \mathcal{M}$ is a "basis" for closed sets of $(X, \mathcal{T})$, hence is a basis for $\mathcal{T}^\vee$. The two sets $\widetilde{U}$ and $\overline{U}$ are exchanged under the duality $\mathcal{G} \leftrightarrow \mathcal{G}^\vee$. □

5. Remarks

A correspondence between finite topological spaces and digraphs (= directed graphs) is also considered in [2], which we summarize briefly and compare with our results. In this section we follow notations and terminologies therein.

5.1. Some results from [2]

Fix a finite set X. Let $\mathcal{T}_X$ be the set of topologies on X and let $\mathcal{G}_X^T$ be the set of *transitive* digraphs with the vertex set X. Given $T \in \mathcal{T}_X$, let $f(X, T) \in \mathcal{G}_X^T$ be the digraph whose set of edges is $\{xy \mid x \in \overline{\{y\}},\ x \neq y\}$. Conversely, given $(X, G) \in \mathcal{G}_X^T$ (where G is the set of edges), let $g(X, G) \in \mathcal{T}_X$ be the topology generated by the subbasis of closed sets $x\!\downarrow := \{y \mid yx \in G\} \cup \{x\}$. This gives a one-to-one correspondence between $\mathcal{T}_X$ and $\mathcal{G}_X^T$ such that $x\!\downarrow = \overline{\{x\}}$ [2, Theorem 11, Corollary 12]. Under this correspondence, T_0-topologies correspond to *acyclic* transitive digraphs [2, Corollary 17].

Let $T \in \mathcal{T}_X$ be a T_0-topology on X and let (X, G) be the corresponding acyclic transitive digraph. Then $x\!\uparrow = \{y \mid xy \in G\} \cup \{x\}$ is the minimum open that contains x for each $x \in X$, and they form the minimum basis for T [2, Remark 27, Theorem 30].

5.2. Comparison

Now let $(X, \mathcal{T})$ be a finite topological space and let $\mathcal{M}$ be the partition of X associated to $\mathcal{T}$ (§2.2). We have the admissible digraph $\mathcal{G}(X, \mathcal{T})$ with the vertex set $\mathcal{M}$ (§3.1). For a fixed partition $\mathcal{M}$ of a finite set X, topologies on X with the associated partition $\mathcal{M}$ correspond bijectively to admissible digraphs with the vertex set $\mathcal{M}$ (Theorem 4.1). On the other hand, $f(X, \mathcal{T})$ of [2] is a transitive digraph with the vertex set X. And topologies on X correspond bijectively to transitive digraphs with the vertex set X.

The partition $\mathcal{M}$ associated to $\mathcal{T}$ is the discrete partition, i.e., $\mathcal{M} = \{\{x\} \mid x \in X\}$, if and only if $(X, \mathcal{T})$ is a T_0-space by Corollary 2.7. In this case, the graph $\mathcal{G}_0(X, \mathcal{T})$ defined in §3.1 is equal to the graph $f(X, \mathcal{T})$ with the direction of all edges reversed. And the description of the minimal open (resp. closed) set containing $x \in X$ in terms of the partial order $\leq$ on $\mathcal{M}$ in Corollary 2.6 and Proposition 3.4 is the same as the one given in [2]. (They are $x\uparrow$ and $x\downarrow$ given above). Thus our work extends some results of [2] to general finite spaces.

5.3. Characterization of the partition $\mathcal{M}$ associated to the topology $\mathcal{T}$

Let $(X, \mathcal{T})$ be a finite topological space and $\mathcal{M}$ be the partition of X associated to $\mathcal{T}$. Let us say a partition of X *generates* the topology $\mathcal{T}$ if every open subset (equivalently, every closed subset) is a union of elements of the partition. For example, $\mathcal{M}$ generates $\mathcal{T}$ by Proposition 2.4. Moreover, it is the coarsest partition that generates $\mathcal{T}$:

Proposition 5.1. *Any partition $\mathcal{M}'$ of X which generates $\mathcal{T}$ is a refinement of $\mathcal{M}$: each $U \in \mathcal{M}$ is a union of elements of $\mathcal{M}'$.*

Proof. Recall the filtration $Y^{(0)} \supset Y^{(1)} \supset \cdots$ of X by closed subsets. Elements of $\mathcal{M}$ are minimal open subsets of $Y^{(0)}, Y^{(1)}, \ldots$ Any closed subset (hence any open subset) of $Y^{(k)}$ is a union of elements of $\mathcal{M}'$. In particular, each minimal open subset of $Y^{(k)}$ is a union of elements of $\mathcal{M}'$. □

This topological characterization of $\mathcal{M}$ gives another proof of the following fact, which we have used in §4.2.

Corollary 5.2. *An automorphism of $(X, \mathcal{T})$ preserves the associated partition $\mathcal{M}$.*

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