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MORE ON MAXIMAL, MINIMAL OPEN AND CLOSED SETS

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Abstract. In this paper, we introduce a notion of cleanly covered topological spaces along with two strong separation axioms. Some properties of cleanly covered topological spaces are obtained in term of maximal open sets including some similar properties of a topological space in term of maximal closed sets. Two strong separation axioms are also investigated in terms of minimal open and maximal closed sets.

1. Introduction

A field of study remains vivid and investigative till new contributions to the theory of the field are remain to add constantly. General topology is an example of such a field and its theory are enriching day by day by adding many new contributions from various directions. Following such trends of adding new contributions to general topology, Nakaoka and Oda [3] introduced and studied the concept of minimal open sets in a topological space. Dualizing the concept of minimal open sets, Nakaoka and Oda [4] introduced and studied the idea of maximal open sets. Thereafter, as consequences of maximal and minimal open sets, Nakaoka and Oda [5] introduced and studied notions of maximal and minimal closed sets. Nakaoka and Oda [5], also obtained some interrelations among four concepts: maximal open sets, minimal open sets, maximal closed sets, minimal closed sets. Several authors have carried out investigations on the notions of maximal and minimal open and closed sets from various directions. The author of the present paper in [2] also obtained some conditions for disconnectedness in terms of maximal and minimal open sets and observed that if a topological space has a set of double nature such as a set which is both maximal open and minimal open, then the space may be disconnected.

We simply write X to denote the topological space $(X, \mathscr{T})$. Let $\mathscr{U}$ be a collection of open sets of a topological space X. It is trivial that $\mathscr{U}$ is an open cover of X, if $X \in \mathscr{U}$. Consideration of such trivial open covers of a topological

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space X are meaningless from the view of covering properties of a topological space, specially when we need to consider finitely many open sets in an open cover of the topological space X. So throughout the paper, by an open cover of X, we mean only nontrivial open cover $\mathscr{U}$ of X. This means that if $\mathscr{U}$ is an open cover of X, then $G \neq X$ for each $G \in \mathscr{U}$. We note that in an open cover of X, there exist at least two open sets.

For a subset A of a topological space $(X, \mathscr{T})$, $Cl(A)$ denotes the closure of A with respect to the topological space $(X, \mathscr{T})$. By a proper open set of a topological space X, we mean an open set $G \neq \emptyset, X$ and by a proper closed set, we mean a closed set $E \neq \emptyset, X$. For a topological space $(X, \mathscr{T})$ and $A \subset X$, we write $(A, \mathscr{T}_A)$ to denote the subspace on A of $(X, \mathscr{T})$. Throughout the paper, R denotes the set of real numbers.

2. Maximal and minimal open and closed sets

We recall some definitions and results to use in the sequel.

Definition 2.1 (Nakaoka and Oda [4])**.** A proper open set U of X is said to be a maximal open set if any open set which contains U is X or U.

Theorem 2.2 (Nakaoka and Oda [4])**.** *If U is a maximal open set and W is an open set, then either $U \cup W = X$ or $W \subset U$. If W is a maximal open set distinct from U, then $U \cup W = X$.*

Definition 2.3 (Nakaoka and Oda [3])**.** A proper open set U of X is said to be a minimal open set if any open set which is contained in U is U or $\emptyset$.

Theorem 2.4 (Nakaoka and Oda [3])**.** *If U is a minimal open set and W is an open set, then either $U \cap W = \emptyset$ or $U \subset W$. If W is a minimal open set distinct from U, then $U \cap W = \emptyset$.*

Definition 2.5 (Howard and Tachtsis [1])**.** A cover $\mathscr{U}$ of X is said to be a minimal cover if for any $U \in \mathscr{U}$, $\mathscr{U} - \{U\}$ is not a cover of X. $\mathscr{U}$ is said to be a minimal open (resp. closed) cover if each member of $\mathscr{U}$ is open (resp. closed).

If $\mathscr{U}$ is a minimal open cover of X, then there do not exist distinct $U, V \in \mathscr{U}$ such that $V \subset U$. Also if an open cover $\mathscr{U}$ of X consists of two distinct open sets U, V such that $V \subset U$, then $\mathscr{U}$ is not a minimal open cover of X. Each minimal open cover of a compact space is finite and each open cover of a compact space has a finite minimal open cover.

Definition 2.6. A cover $\mathscr{U}$ of X is said to be disconnected if for each $U \in \mathscr{U}$, there exists a $V \in \mathscr{U}$ such that $U \cap V = \emptyset$.

Theorem 2.7. *A minimal open cover consists of a minimal open set is disconnected.*

Proof. Let $\mathscr{U}$ be a minimal open cover of X and let $U \in \mathscr{U}$ be a minimal open set. Since U is a proper open set and $\mathscr{U}$ is a cover of X, there exists at least one more element $V \in \mathscr{U}$. By Theorem 2.4, we have $U \cap V = \emptyset$ or $U \subsetneqq V$. $\mathscr{U}$ being a minimal open cover, $U \subsetneqq V$ is not possible. $\square$

Corollary 2.8. *A minimal open cover consists of only minimal open sets is disconnected.*

The following definition is due to the requirement of at least two proper open sets in an open cover of a topological space.

Definition 2.9. A topological space X is said to be cleanly covered if each open cover of X has a minimal open cover consisting of exactly two open sets.

So a cleanly covered topological space is a compact space. It is very easy to see that a compact space may not be a cleanly covered space.

Example 2.10 (Steen and Seebach [6], p. 77)**.** Let $X = [-1, 1]$ and $\mathscr{T}$ is the topology generated by the subbase $\{\emptyset\} \cup \{[-1, b) \mid b > 0\} \cup \{(a, 1] \mid a < 0\}$. Then any open cover of $(X, \mathscr{T})$ consists of two sets G, H such that $1 \in G$, $-1 \in H$ and $G \cup H = X$. So $(X, \mathscr{T})$ is cleanly covered. Note that the space consists of no maximal open as well as minimal open set.

Example 2.11 (Steen and Seebach [6], p. 48)**.** Let $X = [-1, 1]$ and $\mathscr{T} = \{G \mid (-1, 1) \subset G\} \cup \{G \subset X \mid 0 \notin G\}$. The topological space $(X, \mathscr{T})$ is compact but the space is not cleanly covered. It is interested to note that the space has uncountably many minimal open sets, and only two maximal open sets $[-1, 1)$ and $(-1, 1]$.

Theorem 2.12. *If each open cover of X contains a maximal open set, then X is cleanly covered.*

Proof. Let $\mathscr{U}$ be an open cover of X and $G \in \mathscr{U}$ be a maximal open set. If $H \in \mathscr{U}$ is another maximal open set distinct from G, then we have $G \cup H = X$ by Theorem 2.2. So $\{G, H\}$ constitutes subcover of $\mathscr{U}$ for X. Let G be the only maximal open set in $\mathscr{U}$. G being a proper open set, there exists at least one more open set $H \in \mathscr{U}$ distinct from G to cover X. If $H = X$, then $\mathscr{U}$ is a trivial open cover of X. So we suppose that $H \neq X$. By Theorem 2.2, we have $H \subsetneqq G$ or $H \cup G = X$. If for all $U \in \mathscr{U}$, we have $U \subsetneqq G$, then $\mathscr{U}$ can not become an open cover of X. So it follows that there exists an open set $H \in \mathscr{U}$ distinct from G such that $G \cup H = X$. $\square$

Recall that a collection $\mathscr{U}$ of subsets of X is called locally finite if each $x \in X$ has a neighborhood meeting only finitely many members of $\mathscr{U}$. Also if $\{U_\lambda \mid \lambda \in \Lambda\}$ is a locally finite collection of sets in X, then $Cl(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} Cl(U_\lambda)$.

Due to Nakaoka and Oda [3], a topological space X is a locally finite space if for each $x \in X$, there exists a finite open set U such that $x \in U$.

Theorem 2.13. *If $\{U_\lambda \mid \lambda \in \Lambda\}$ is a collection of distinct minimal open sets in a locally finite space X, then $Cl(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} Cl(U_\lambda)$.*

Proof. Since U_λ is minimal open for each $\lambda \in \Lambda$, we have $U_{\lambda_1} \cap U_{\lambda_2} = \emptyset$ for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. For each $x \in X$, there exists a finite open set U such that $x \in U$. Since U is a finite open set and $U_{\lambda_1} \cap U_{\lambda_2} = \emptyset$ for $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$, U intersects only finitely many members of $\{U_\lambda \mid \lambda \in \Lambda\}$. So $\{U_\lambda \mid \lambda \in \Lambda\}$ is locally finite and hence the result follows. □

Definition 2.14 (Nakaoka and Oda [5])**.** A proper closed set E of X is said to be a minimal closed set if any closed set which is contained in E is E or $\emptyset$.

Theorem 2.15 (Nakaoka and Oda [5])**.** *If E is a minimal closed set and F is any closed set, then either $E \cap F = \emptyset$ or $E \subset F$.*

Definition 2.16 (Nakaoka and Oda [5])**.** A proper closed set E of X is said to be a maximal closed set if any closed set which contains E is X or E.

Theorem 2.17 (Nakaoka and Oda [5])**.** *If E is a maximal closed set and F is any closed set, then either $E \cup F = X$ or $F \subset E$.*

Analogous to Theorem 2.7, Corollary 2.8, Theorem 2.12 and Theorem 2.13, we have Theorem 2.18, Corollary 2.19, Theorem 2.20 and Theorem 2.21 respectively. The proofs of them are omitted as proofs are very much similar to corresponding results already established.

Theorem 2.18. *A minimal closed cover consists of a minimal closed set is disconnected.*

Corollary 2.19. *A minimal closed cover consists of only minimal closed sets is disconnected.*

Theorem 2.20. *If each closed cover $\mathscr{F}$ of X contains a maximal closed set E, then there exists an $F \in \mathscr{F}$ such that $E \cup F = X$.*

Theorem 2.21. *If $\{U_\lambda \mid \lambda \in \Lambda\}$ is a collection of distinct minimal closed sets in a locally finite space X, then $Cl(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} U_\lambda$.*

Theorem 2.22. *Let A, E be closed sets in X such that $A \cap E \neq \emptyset$. Then $A \cap E$ is a minimal closed set in $(A, \mathscr{T}_A)$ if E is a minimal closed set in $(X, \mathscr{T})$.*

Proof. Similar to the proof of Theorem 3.14 [2]. □

3. Two strong separation axioms

We note that if there exist two minimal open sets U, V in a topological space X, then $U \cap V = \emptyset$ [3]. This observation motivates us to introduce the following two strong separation axioms.

Definition 3.1. A topological space X is said to be strongly regular if for each $x \in X$ and each closed set F with $x \notin F$, there exist distinct minimal open sets U, V such that $x \in U$ and $F \subset V$.

Clearly, a strongly regular topological space is regular.

Definition 3.2. A topological space X is said to be strongly normal if for each pair of disjoint closed sets E, F, there exist distinct minimal open sets U, V such that $E \subset U$ and $F \subset V$.

Let $\mathscr{T}$ be the topology on R generated by a basis to be form by the family $\{[a, b) \mid a, b \in R\}$. The right half-open interval topological space $(R, \mathscr{T})$ is both regular and normal. But the space is neither strongly regular nor strongly normal. So we conclude that a regular (resp. normal) topological space may not be a strongly regular (resp. strongly normal) space.

For a subset A of a topological space X, we define

$$MinInt(A) = \begin{cases} \emptyset \text{ if } A \text{ contains no minimal open set} \\ \bigcup\{G \mid G \text{ is a minimal open set contained in } A\}, \end{cases}$$

and

$$MaxCl(A) = \begin{cases} X \text{ if } A \text{ contained in no maximal closed set} \\ \bigcap\{E \mid E \text{ is a maximal closed set containing } A\}. \end{cases}$$

It follows that if G is a minimal open set, then $MinInt(G) = G$ and if E is a maximal closed set, then $MaxCl(E) = E$. If G, H are two distinct minimal open sets, then G, H are proper open sets distinct from $G \cup H$ and contained in $G \cup H$ which means that $G \cup H$ is not a minimal open set. So we conclude that the union of even finitely many distinct minimal open sets is not minimal open. Again if E, F are two distinct maximal closed sets, then E, F are proper closed sets distinct from $E \cap F$ and contain $E \cap F$ which means that $E \cap F$ is not a maximal closed set. So we conclude that the intersection of even finitely many distinct maximal closed sets is not maximal closed. Thus it follows that $MinInt(A)$ (resp. $MaxCl(A)$) may not be minimal open (resp. maximal closed).

Example 3.3 (Mukharjee [2])**.** For $a \in R$, we define

$$\mathscr{T} = \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a], [a, \infty)\}.$$

For $b \in R$ with $a < b$, we have $MinInt((-\infty, b)) = (-\infty, a]$ which is not a minimal open set and $MaxCl((b, \infty)) = (a, \infty)$ which is not a maximal closed set.

The following two are duality principle due to Nakaoka and Oda [5]:

(i) F is minimal closed in X if and only if $X - F$ is maximal open in X.
(ii) F is maximal closed in X if and only if $X - F$ is minimal open in X.

Theorem 3.4. *For a subset A of X, $X - MinInt(A) = MaxCl(X - A)$.*

Proof. Firstly, suppose that $MinInt(A) = \emptyset$. This means that A contains no minimal open set. If possible, suppose that $X - A$ contains a maximal closed set E. Then $X - E$ is a minimal open set contained in A, a contradiction. So in this case we have $X - MinInt(A) = MaxCl(X - A)$. Now suppose that A

contains a minimal open set G. Then $X - A \subset X - G$. Also $X - G$ is a maximal closed set. So we get $MaxCl(X - A) \neq X$ if and only if $MinInt(A) \neq \emptyset$. It is easy to see that if $\{G\}$ is the collection of all minimal open sets contained in A, then $\{X - G\}$ is the collection of all maximal closed sets containing $X - A$ and vice-versa. So we have

$$\begin{aligned} X - MinInt(A) &= X - \bigcup\{G \mid G \text{ is minimal open contained in } A\} \\ &= \bigcap\{X - G \mid X - G \text{ is maximal closed containing } X - A\} \\ &= MaxCl(X - A). \end{aligned}$$

□

Theorem 3.5. *For a subset A of X, $MinInt(A)$ is minimal open if and only if A contains one and only one minimal open set.*

Proof. The sufficiency follows easily. Let $MinInt(A)$ be minimal open and A contain two minimal open sets G, H. Then $MinInt(A) = G \cup H \subset A$. Since $G, H \subset G \cup H$ and $MinInt(A)$ is minimal open, we have $G = G \cup H$ and $H = G \cup H$. $G = G \cup H$ implies that $H \subset G$ and $H = G \cup H$ implies that $G \subset H$. So we have $G = H$. □

Theorem 3.6 is a dual of Theorem 3.5. The proof of the theorem is omitted as the proof is similar to that of Theorem 3.5.

Theorem 3.6. *For a subset A of X, $MaxCl(A)$ is maximal closed if and only if A contained in one and only maximal closed set.*

Theorem 3.7. *A topological space X is strongly regular if and only if for each open set G and each $x \in G$, there exist a minimal open set U and a maximal closed set E such that $x \in U \subset MaxCl(U) \subset E \subset G$.*

Proof. Firstly, let X be a strongly regular topological space and G be an open set. For $x \in G$, we obtain by strong regularity of X two distinct minimal open sets U, V such that $x \in U, X - G \subset V$. U, V being distinct minimal open sets, we have $U \cap V = \emptyset$. Now $U \cap V = \emptyset \Rightarrow U \subset X - V$. Since V is minimal open, $X - V$ is maximal closed and so $MaxCl(X - V) = X - V$. Thus $MaxCl(U) \subset MaxCl(X - V) = X - V \subset G$. Putting $E = X - V$, we see that E is maximal closed and $x \in U \subset MaxCl(U) \subset E \subset G$.

Conversely, let E be a closed set and $x \in X$ such that $x \notin E$. As $X - E$ is an open set with $x \in X - E$, there exist a minimal open set U and a maximal closed set F such that $x \in U \subset MaxCl(U) \subset F \subset X - E$. We put $V = X - F$. Then V is minimal open with $E \subset V$ and $U \cap V = \emptyset$. □

Theorem 3.8. *A topological space X is strongly normal if and only if for a closed set E and for an open set G with $E \subset G$, there exist a minimal open set U and a maximal closed set F such that $E \subset U \subset MaxCl(U) \subset F \subset G$.*

Proof. Similar to the proof of Theorem 3.7. □

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