Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 165–173
https://doi.org/10.4134/CKMS.c160052
pISSN: 1225-1763 / eISSN: 2234-3024

# ASYMPTOTIC FOLIATIONS OF QUASI-HOMOGENEOUS CONVEX AFFINE DOMAINS

Kyeonghee Jo

Abstract. In this paper, we prove that the automorphism group of a quasi-homogeneous properly convex affine domain in $\mathbb{R}^n$ acts transitively on the set of all the extreme points of the domain. This set is equal to the set of all the asymptotic cone points coming from the asymptotic foliation of the domain and thus it is a homogeneous submanifold of $\mathbb{R}^n$.

## 1. Introduction

A *quasi-homogeneous* affine domain is an open subset $\Omega$ of $\mathbb{R}^n$ which has a compact subset $K \subset \Omega$ and a subgroup $G$ of $\mathrm{Aut}(\Omega) < \mathrm{Aff}(n, \mathbb{R})$ such that $GK = \Omega$. Sometimes we say that $G$ acts on $\Omega$ *syndetically* in this case. An affine domain $\Omega$ is called *divisible* if there exists a discrete subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ so that $\Omega/\Gamma$ is a compact manifold.

Note that both homogeneous domains and divisible domains are quasi-homogeneous. There are quasi-homogeneous domains which are neither homogeneous nor divisible, but those are not so many in some sense. Every quasi-homogeneous convex affine domain has a very special foliation, an asymptotic foliation, which does not allow quasi-homogeneity to be very far from homogeneity. Divisible domains are distinguished from quasi-homogenous domains by the fact that their asymptotic foliations consist of just one leaf if they have no complete line. (This follows from the well-known fact by Vey [4] that a properly convex divisible domain is a cone.)

Every quasi-homogeneous convex affine domain has the unique maximal linear cone which can be contained in its interior by translation, we call it the asymptotic cone of the domain. The leaves of an asymptotic foliation of a quasi-homogeneous convex affine domain $\Omega$ are all cones which are translations of the interior of the asymptotic cone. We call each cone point of a leaf of the

Received March 11, 2016.
2010 *Mathematics Subject Classification.* 52A20, 53C12, 57N16, 57S25.
*Key words and phrases.* asymptotic cone, foliation, quasi-homogeneous domain.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(NRF-2011-0010632).

asymptotic foliation an asymptotic cone point. The set $S(\Omega)$ of all the asymptotic cone points of $\Omega$ is a topological submanifold of $\mathbb{R}^n$ whose codimension is equal to the dimension of the asymptotic cone. This paper shows that $S(\Omega)$ is homogeneous.

The author has proved in [2] that $S(\Omega)$ is the set of all the extreme points, and furthermore $S(\Omega)$ is the whole boundary of $\Omega$ and $\mathrm{Aut}(\Omega)$ acts transitively on the boundary if $\Omega$ is a strictly convex domain. In this paper, we extend this result to an arbitrary properly convex domain:

**Theorem 1.** *Let $\Omega$ be a properly convex affine domain in $\mathbb{R}^n$ and $G$ be a closed subgroup of $\mathrm{Aut}(\Omega)$ acting synthetically on $\Omega$. Then $G$ acts transitively on the set $S(\Omega)$ of all the asymptotic cone points of $\Omega$.*

From this theorem, we see that the dimension of the Lie group $\mathrm{Aut}(\Omega)$ is at least of $n - \dim \mathrm{AC}(\Omega)$, which is nonzero if $\Omega$ is not a cone.

For quasi-homogeneous convex affine domain $\Omega$ which is not properly convex, i.e., when $\Omega$ has a complete affine line, there is a natural number $k$ such that $\Omega = \mathbb{R}^k \times \Omega'$. In this case, $\mathrm{Aut}(\Omega)$ acts transitively on $\mathbb{R}^k \times S(\Omega')$.

## 2. Hilbert metric

A convex affine domain $\Omega$ is called *properly convex* if it does not contain any complete line. We can define on any properly convex affine domain a complete continuous metric which is invariant under the action of $\mathrm{Aut}(\Omega) = \{g \in \mathrm{Aff}(n, \mathbb{R}) \mid g(\Omega) = \Omega\}$, the group of affine automorphisms of $\Omega$. This metric is called the *Hilbert metric* and denoted by $\mathrm{d_H}$.

**Definition 2.** Let $\Omega$ be a properly convex domain in $\mathbb{R}^n$. For any two different points $p_1, p_2 \in \Omega$, we define $\mathrm{d_H}(p_1, p_2)$ to be the logarithm of the absolute value of the cross ratio of $(s_1, p_1, p_2, s_2)$, where $s_1$ and $s_2$ are the points in which the line $\overleftrightarrow{p_1p_2}$ intersects $\partial\Omega$ such that $p_1$ lies in the line segment $\overline{s_1p_2}$. For $p_1 = p_2$, we define $\mathrm{d_H}(p_1, p_2) = 0$.

Here the cross ratio of $(s_1, p_1, p_2, s_2)$ is defined by

$$\frac{(p_2 - s_1)(s_2 - p_1)}{(p_1 - s_1)(s_2 - p_2)}.$$

If one of $s_1$ and $s_2$ is an infinite boundary, i.e., one of them is not in $\mathbb{R}^n$, then the two terms containing it are cancelled.

Since every properly convex affine domain is projectively equivalent to a bounded convex domain and the cross ratio is invariant by projective transformations, we may use such bounded domains in calculating the Hilbert metric to avoid the case that $s_1$ or $s_2$ is contained in the infinite boundary.

Now let's see the action of $\mathrm{Aut}(\Omega)$ on $\Omega$.

**Proposition 3** (Jo, [2])**.** *Let $\Omega$ be a properly convex affine domain in $\mathbb{R}^n$ and $G$ be a closed subgroup of $\mathrm{Aff}(n, \mathbb{R})$ preserving $\Omega$. Then $G$ acts properly on $\Omega$ and each $G$-orbit is a closed subset of $\Omega$.*

The Hilbert metric $\mathrm{d_H}$ on a properly convex domain $\Omega$ descends to the orbit space $G\setminus\Omega$ if $G$ is a closed subgroup of $\mathrm{Aut}(\Omega)$ by Proposition 3, i.e., we obtain a metric $\mathrm{d_G}$ on the orbit space $G\setminus\Omega$ as follows:

$$\mathrm{d_G}(Gx, Gy) = \inf_{g,g'\in G} \mathrm{d_H}(gx, g'y). \tag{2.1}$$

Since $\mathrm{d_H}$ is invariant by projective automorphism, we have

$$\mathrm{d_G}(Gx, Gy) = \inf_{g\in G} \mathrm{d_H}(x, gy).$$

We can also prove that the distance between the two orbits $Gx$ and $Gy$ is realized by the Hilbert distance between two points $x_0 \in Gx$ and $y_0 \in Gy$ in $\Omega$.

**Lemma 4** (Jo, [2])**.** *Let $G$ be a closed subgroup of $\mathrm{Aff}(n,\mathbb{R})$. Then for each pair $(Gx, Gy) \in G\setminus\Omega$, there exists $g_{x,y} \in G$ such that $d_G(Gx, Gy) = d_H(x, g_{x,y}y)$.*

## 3. Asymptotic foliation

Every quasi-homogeneous convex affine domain contains a cone invariant under the action of linear parts of their automorphism groups, which is called an asymptotic cone. This terminology was originally introduced by Vey in [4].

**Definition 5.** Let $\Omega$ be a convex domain in $\mathbb{R}^n$. The *asymptotic cone* of $\Omega$ is defined as follows:

$$\mathrm{AC}(\Omega) := \{u \in \mathbb{R}^n \mid x + tu \in \Omega \ \text{ for all } x \in \Omega, t \geq 0\}.$$

By the convexity of $\Omega$, for any $x_0 \in \Omega$,

$$\mathrm{AC}(\Omega) = \mathrm{AC}_{x_0}(\Omega) := \{u \in \mathbb{R}^n \mid x_0 + tu \in \Omega \ \text{ for all } t \geq 0\}.$$

Even though the asymptotic cone $\mathrm{AC}(\Omega)$ of a properly convex affine domain $\Omega$ is possibly empty, but it is nonempty if $\Omega$ is quasi-homogeneous, thanks to the following lemma:

**Lemma 6** (Jo, [2])**.** *Let $\Omega$ be a quasi-homogeneous properly convex affine domain in $\mathbb{R}^n$. Then $\Omega$ is not bounded and the boundary $\partial\Omega$ does not have any bounded face with dimension $k > 0$.*

Note that $\mathrm{AC}(\Omega)$ is a properly convex closed cone in $\mathbb{R}^n$ if $\Omega$ is properly convex. We will denote the interior of $\mathrm{AC}(\Omega)$ relative to its linear span by $\mathrm{AC}^\circ(\Omega)$.

*Remark* 7. We see the following facts immediately.

(i) The asymptotic cone of $\Omega$ is the maximal closed cone which can be contained in $\Omega$.

(ii) $\overline{\mathrm{AC}(\Omega)}$, the closure of the asymptotic cone in $\mathbb{RP}^n$, is a convex hull of the union of the origin $o$ and the infinite boundary $\partial_\infty\Omega \subset \partial\mathbb{R}^n$, when $\mathrm{AC}(\Omega)$ is considered as a projective domain.

Divisible domains are special class of quasi-homogeneous domains. The asymptotic cone $\mathrm{AC}(\Omega)$ of a divisible convex domain $\Omega$ equals $\overline{\Omega}$ if $\Omega$ does not have any complete line, which is an immediate consequence of the following result:

**Theorem 8** (Vey, [4]). *A divisible properly convex affine domain is a cone.*

The following theorem, which was proved in [2], is very important to understand quasi-homogeneous domains.

**Theorem 9** (Jo, [2]). *Let $\Omega$ be a properly convex quasi-homogeneous affine domain in $\mathbb{R}^n$. Then $\Omega$ admits a parallel foliation by cosets of the asymptotic open cone $\mathrm{AC}^\circ(\Omega)$ of $\Omega$.*

Let $L$ be the linear span of $\mathrm{AC}(\Omega)$, i.e., the linear subspace of $\mathbb{R}^n$ which is generated by $\mathrm{AC}(\Omega)$, and $\Omega_x$ the intersection of $\Omega$ and the affine subspace $x+L$, i.e., $\Omega_x = \Omega \cap (x+L)$. Then Theorem 9 implies that $\Omega_x$ is the translation of $\mathrm{AC}^\circ(\Omega)$ for all $x \in \Omega$. That is, there exists a continuous map $s : \Omega \to \partial\Omega$ such that $\Omega_x = \mathrm{AC}^\circ(\Omega) + s(x)$ for each $x \in \Omega$. Note that $S(\Omega)$ is a proper subset of $\partial\Omega$ if AC is not 1-dimensional.

**Definition 10.**

(i) We call the foliation of a properly convex quasi-homogeneous domain $\Omega$ by cosets of $\mathrm{AC}^\circ(\Omega)$ *the asymptotic foliation of* $\Omega$.

(ii) We call $p \in \partial\Omega$ an *asymptotic cone point* of a properly convex affine domain $\Omega$ if $p$ is a cone point of $\Omega_x$ for some $x \in \Omega$, that is, $p = s(x)$ for some $x \in \Omega$.

We see that $S(\Omega)$, the set of all the asymptotic cone points of $\Omega$, is an $\mathrm{Aut}(\Omega)$-invariant set. Actually $S(\Omega)$ is the set of all the extreme points of $\Omega$ in $\mathbb{R}^n$:

**Proposition 11** (Jo, [2]). *Let $\Omega$ be a quasi-homogeneous properly convex affine domain in $\mathbb{R}^n$. Then $S(\Omega)$ is equal to the set of all the extreme points of $\Omega$, and the set of all the infinite extreme points is the set of all the extreme points of the infinite closed face*

$$\partial_\infty\Omega = \overline{\Omega} \cap \mathbb{RP}^{n-1}_\infty = \overline{\mathrm{AC}(\Omega)} \cap \mathbb{RP}^{n-1}_\infty,$$

*when we see $\Omega$ as a subset of $\mathbb{RP}^n$, where $\mathbb{RP}^{n-1}_\infty = \mathbb{RP}^{n-1} \setminus \mathbb{R}^n = \partial\mathbb{R}^n$.*

**Example 12.** Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid y > x^2, z > 0\}$. Then $\Omega$ is homogeneous and AC and $S(\Omega)$ are as follows :

$$\mathrm{AC}^\circ(\Omega) = \{(0,y,z) \in \mathbb{R}^3 \mid y > 0, z > 0\},$$

$$S(\Omega) = \{(x,y,0) \in \mathbb{R}^3 \mid y = x^2\}.$$

By Theorem 9, the leaves of its asymptotic foliation are all translations of this cone $\mathrm{AC}^\circ(\Omega)$. Actually for each $p = (x_0, y_0, z_0) \in \Omega$ the leaf containing $p$ is

$$\begin{aligned}\Omega_p &= \{(x_0, x_0^2 + y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\} \\ &= (x_0, x_0^2, 0) + \{(0, y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\} \\ &= s(p) + \{(0, y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\}.\end{aligned}$$

That is, any intersection of $\Omega$ and a plane parallel to $yz$-plane is $\{(x_0, x_0^2+y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\}$ for some $x_0 \in \mathbb{R}$.

Let $\mathrm{Lin}(\Omega)$ be the image of $\mathrm{Aut}(\Omega)$ by the canonical homomorphism from $\mathrm{Aff}(n,\mathbb{R})$ to $\mathrm{GL}(n,\mathbb{R})$. Then $\mathrm{Lin}(\Omega)$ preserves $\mathrm{AC}(\Omega)$ since $\mathrm{AC}(\Omega)$ is the maximal closed cone which can be contained in $\Omega$. So the restriction map of $\mathrm{Lin}(\Omega)$ to the linear span $L$ of $\mathrm{AC}(\Omega)$ becomes a subgroup $G$ of $\mathrm{Aut}(\mathrm{AC}(\Omega)) < \mathrm{GL}(L)$. Obviously $G$ acts on $\mathrm{AC}^\circ(\Omega)$ transitively if $\Omega$ is homogeneous.

**Theorem 13** (Jo, [2] and [3])**.** *Let $\Omega$ be a quasi-homogeneous properly convex affine domain in $\mathbb{R}^n$. Then*

(i) $\mathrm{AC}^\circ(\Omega)$ *is (quasi)-homogeneous if $\Omega$ is (quasi)-homogeneous,*
(ii) $\mathrm{AC}(\Omega)$ *is one-dimensional if and only if $\Omega$ is strictly convex,*
(iii) $\mathrm{Aut}(\Omega)$ *acts transitively on $\partial\Omega$ if $\Omega$ is strictly convex,*
(iv) $S(\Omega) = \partial\Omega$ *if and only if $\Omega$ is strictly convex,*
(v) $\Omega$ *is affinely equivalent to an $n$-dimensional paraboloid if and only if it is strictly convex.*

## 4. Proof of Theorem 1

We have seen in the previous section that $\mathrm{Aut}(\Omega)$ acts transitively on $S(\Omega) = \partial\Omega$ if $\mathrm{AC}(\Omega)$ is one-dimensional. Actually we can prove that $\mathrm{Aut}(\Omega)$ acts transitively on $S(\Omega)$ for any quasi-homogeneous properly convex affine domain $\Omega$, which is exactly Theorem 1.

*Proof of Theorem 1.* It suffices to prove the present theorem for the case that the dimension of $\mathrm{AC}(\Omega)$ is at least 2, thanks to Theorem 13. By Theorem 9, there exist a continuous map $s : \Omega \to S(\Omega) \subset \partial\Omega$ and an one parameter group of homeomorphisms of $\Omega$ with the following equation:

$$c_t(x) = s(x) + e^t(x - s(x)) \text{ for } t \in \mathbb{R},\ x \in \Omega.$$

Note that $\mathrm{d_H}\big(c_t(x), c_t(y)\big) = \mathrm{d_H}(x, y)$ if $s(x) = s(y)$. We will show first that $c_t$ is strictly distance decreasing for $t > 0$ if $x$ and $y$ are not in the same leaf of the asymptotic foliation, that is, for any two points $x, y \in \Omega$ such that $s(x) \neq s(y)$

$$\mathrm{d_H}\big(c_t(x), c_t(y)\big) < \mathrm{d_H}(x, y) \text{ for all } t > 0.$$

Let $V$ be a 2-dimensional linear subspace of the linear span $L$ of $\mathrm{AC}(\Omega)$, which is generated by $v_x = x - s(x)$ and $v_y = y - s(y)$. Then $V \cap \mathrm{AC}(\Omega)$ is affinely equivalent to a quadrant and thus it is bounded by two rays $\mathbf{r_1}$ and $\mathbf{r_2}$. Let $l$ be the line connecting $x$ and $y$ and $l'$ be the line connecting two points

$c_t(x) = s(x) + e^t v_x$ and $c_t(y) = s(x) + e^t v_y$. Then both $l$ and $l'$ lie in the 3-dimensional affine subspace $s(x) + W$ of $\mathbb{R}^n$, where $W$ is the 3-dimensional linear subspace of $\mathbb{R}^n$ generated by three linearly independent vectors

$$s(y) - s(x), v_x, v_y.$$

Note that $(s(x) + W) \cap \Omega$ is the 3-dimensional section of $\Omega$ and it has a 2-dimensional asymptotic foliation whose leaves are parallel translation of $V \cap \mathrm{AC}(\Omega)$, even though it may not be quasi-homogeneous.

If we consider the line $l_V$ connecting $v_x$ and $v_y$ in $V$, $l_V$ intersects the boundary rays $\mathbf{r_1}$ and $\mathbf{r_2}$ at

$$\eta_1 = \mu_1 v_x + (1 - \mu_1) v_y$$

and

$$\eta_2 = \mu_2 v_x + (1 - \mu_2) v_y$$

for some two real numbers $\mu_1, \mu_2$. Note that $\mu_1, \mu_2$ could be $\infty$ or $-\infty$ in case that $l_V$ doesn't meet one of two rays $\mathbf{r_1}$ and $\mathbf{r_2}$. We may assume that $\mu_1 < 0 < 1 < \mu_2$ by exchanging the name of the rays $\mathbf{r_1}$ and $\mathbf{r_2}$ if necessary, because either $\mu_1 < 0 < 1 < \mu_2$ or $\mu_2 < 0 < 1 < \mu_1$ holds depending only upon the relative position of $v_x$ and $v_y$ (see Figure 1).

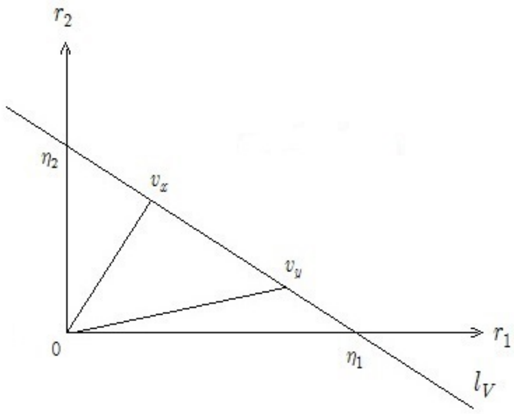


FIGURE 1. Case of $\mu_1 < 0 < 1 < \mu_2$

To find boundary points at which $l$ or $l'$ meets $\partial\Omega$, we parametrize the line $l_V$ and the line $l_S$ connecting $s(x)$ and $s(y)$ like this:

$$v_\lambda = \lambda v_x + (1 - \lambda) v_y,$$

$$s_\lambda = \lambda s(x) + (1 - \lambda) s(y)$$

(Note that $v_{\mu_1} = \eta_1, v_0 = v_y, v_1 = v_x, v_{\mu_2} = \eta_2$ and $v_\lambda \in \mathrm{AC}(\Omega)$ only for $\mu_1 \leq \lambda \leq \mu_2$).

Then the two points in $\partial\Omega$ where the line $l$ intersect are

$$\xi_1 = \lambda_1 x + (1-\lambda_1)y = s_{\lambda_1} + v_{\lambda_1}$$

and

$$\xi_2 = \lambda_2 x + (1-\lambda_2)y = s_{\lambda_2} + v_{\lambda_2}$$

with real numbers $\lambda_1$ and $\lambda_2$ such that $\mu_1 \leq \lambda_1 < 0$ and $1 < \lambda_2 \leq \mu_2$. This is because if $\lambda < \mu_1$ or $\lambda > \mu_2$, $s_\lambda + v_\lambda$ is outside of $\overline{\Omega}$ by convexity of $\overline{\Omega}$. One can prove $\mu_1 < \lambda_1$ or $\lambda_2 < \mu_2$ because both $s_{\mu_1} + \mathbf{r_1}$ and $s_{\mu_2} + \mathbf{r_2}$ cannot meet $\overline{\Omega}$ simultaneously. (Note that $s_\lambda = \lambda s(x) + (1-\lambda)s(y)$ is not in $\overline{\Omega}$ if $\lambda < 0$ or $\lambda > 1$.) We may assume that $\mu_1 < \lambda_1$.

Now we consider the line $l'$ connecting two points $c_t(x) = s(x) + e^t v_x$ and $c_t(y) = s(y) + e^t v_y$ for $t > 0$. Since $v_{\lambda_1} \in \mathrm{AC}^\circ(\Omega)$ and $v_{\lambda_2} \in \mathrm{AC}(\Omega)$,

$$\begin{aligned}
\zeta_1 &= \lambda_1 c_t(x) + (1-\lambda_1)c_t(y) \\
&= (\lambda_1 s(x) + (1-\lambda_1)s(y)) + e^t(\lambda_1 v_x + (1-\lambda_1)v_y) \\
&= s_{\lambda_1} + e^t v_{\lambda_1} \\
&= \xi_1 + (e^t - 1)v_{\lambda_1}
\end{aligned}$$

is in $\Omega$ for all $t > 0$, and

$$\begin{aligned}
\zeta_2 &= \lambda_2 c_t(x) + (1-\lambda_2)c_t(y) \\
&= (\lambda_2 s(x) + (1-\lambda_2)s(y)) + e^t(\lambda_2 v_x + (1-\lambda_2)v_y) \\
&= s_{\lambda_2} + e^t v_{\lambda_2} \\
&= \xi_2 + (e^t - 1)v_{\lambda_2}
\end{aligned}$$

is in $\overline{\Omega}$ for all $t > 0$. Therefore we get two real numbers $\lambda_1', \lambda_2'$ satisfying

$$\mu_1 < \lambda_1' < \lambda_1 < 0, 1 < \lambda_2 \leq \lambda_2' \leq \mu_2$$

such that the line $l'$ intersects $\partial\Omega$ at two points

$$\begin{aligned}
\xi_1' &= \lambda_1' c_t(x) + (1-\lambda_1')c_t(y) \\
&= \lambda_1' s(x) + (1-\lambda_1')s(y) + e^t(\lambda_1' v_x + (1-\lambda_1')v_y) \\
&= s_{\lambda_1'} + e^t v_{\lambda_1'},
\end{aligned}$$

and

$$\xi_2' = s_{\lambda_2'} + e^t v_{\lambda_2'}.$$

By an easy calculation, we can finally prove

$$\mathrm{d}_H\big(c_t(x), c_t(y)\big) < \mathrm{d}_H(x,y) \text{ for all } t > 0 \text{ and } x, y \in \Omega \text{ with } s(x) \neq s(y).$$

using $\lambda_1' < \lambda_1$ and $\lambda_2 \leq \lambda_2'$. (Note that the cross ratio of $(\xi_1, y, x, \xi_2)$ is equal to that of $(\zeta_1, c_t(y), c_t(x), \zeta_2)$ and the cross ratio of $(\xi_1', c_t(y), c_t(x), \xi_2')$ is less than that of $(\zeta_1, c_t(y), c_t(x), \zeta_2)$.)

On the other hand, we see that $g(\Omega_x) = \Omega_{gx}$ since $\mathrm{AC}(\Omega)$ is invariant under the action of the linear parts of $G$. This implies

$$s(gx) = gs(x) \text{ for all } g \in G. \tag{4.1}$$

Since $G$ is closed in $\mathrm{Aff}(n, \mathbb{R})$, $G$ acts on $\Omega$ properly by Proposition 3. Now we can define $c_t$ on the orbit space $G \setminus \Omega$ by (4.1). In fact, for all $g \in G$ and $y \in \Omega$

$$\begin{aligned} c_t(gy) &= s(gy) + e^t(gy - s(gy)) \\ &= gs(y) + e^t(gy - gs(y)) \\ &= e^t gy + (1 - e^t)gs(y) \\ &= g(e^t y + (1 - e^t)s(y)) \\ &= g(s(y) + e^t(y - s(y)) \\ &= gc_t(y) \end{aligned}$$

since every affine transformation preserves a convex combination. Note that Lemma 4 implies that there exists $g_{x,y} \in G$ such that

$$\mathrm{d_G}(Gx, Gy) = \mathrm{d_H}(x, g_{x,y}y), \tag{4.2}$$

where $\mathrm{d_G}$ is defined in (2.1).

It is obvious that for $t > 0$, the homeomorphism $c_t$ of $G \setminus \Omega$ is distance decreasing. By the way, the fact that $G$ acts on $\Omega$ synthetically implies that $G \setminus \Omega$ is a compact metric space and it is well known that every distance decreasing surjection from a compact metric space onto itself is an isometry.

Suppose there exists a pair $(x, y)$ such that $s(x) \neq gs(y)(= s(gy))$ for all $g \in G$. Then we have

$$\mathrm{d_G}(Gx, Gy) = \mathrm{d_H}(x, g_{x,y}y) > \mathrm{d_H}\big(c_t(x), c_t(g_{x,y}y)\big) = \mathrm{d_H}\big(c_t(x), g_{x,y}c_t(y)\big)$$

and this implies

$$\mathrm{d_G}(Gx, Gy) > \mathrm{d_G}\big(c_t(Gx), c_t(Gy)\big),$$

which is a contradiction. Therefore we can conclude that for any pair $(x, y)$, there is $g \in G$ such that $s(x) = gs(y)$, which completes the proof. □

The following is an immediate consequence of Theorem 1, since $S(\Omega)$ is equal to the set of all the extreme points of $\Omega$ by Proposition 11.

**Corollary 14.** *Let $\Omega$ be a properly convex affine domain in $\mathbb{R}^n$ and $G$ be a closed subgroup of $\mathrm{Aut}(\Omega)$ acting synthetically on $\Omega$. Then $G$ acts transitively on the set of all the extreme points of $\Omega$.*

*Remark* 15. Theorem 1 implies that the Lie group $G$ is at least of dimension $n - \dim \mathrm{AC}(\Omega)$, which is nonzero if $\Omega$ is not a cone. Actually, $\mathrm{Aut}(\Omega)$ has a dimension greater than 0 for every quasi-homogeneous convex affine domain, because positive homotheties are all in the automorphism group of a cone.

## References

[1] J. P. Benzécri, *Sur les variétés localement affines et projectives*, Bull. Soc. Math. France **88** (1960), 229–332.
[2] K. Jo, *Quasi-homogeneous domains and convex affine manifolds*, Topology Appl. **134** (2003), no 2, 123–146.
[3] ______, *Differentiability of quasi-homogeneous convex affine domains*, J. Korean Math. Soc. **42** (2005), no. 3, 485–498.
[4] J. Vey, *Sur les automorphismes affines des ouverts convexes saillants*, Ann. Scuola Norm. Sup. Pisa **24** (1970), no. 3, 641–665.

Kyeonghee Jo
Division of Liberal Arts and sciences
Mokpo National Maritime University
Mokpo 530-729, Korea
*E-mail address*: khjo@mmu.ac.kr