Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 147–163
https://doi.org/10.4134/CKMS.c160024
pISSN: 1225-1763 / eISSN: 2234-3024

FOCAL SURFACES AND EVOLUTES OF CURVES IN HYPERBOLIC SPACE

Ryota Hayashi, Shyuichi Izumiya, and Takami Sato

Abstract. We define de Sitter focal surfaces and hyperbolic focal surfaces of hyperbolic space curves. As an application of the theory of unfoldings of function germs, we investigate the singularities of these surfaces. For characterizing the singularities of these surfaces, we discover a new hyperbolic invariants and investigate the geometric meanings.

1. Introduction

In [2] the notion of horospherical surfaces of hyperbolic space curves has been introduced and investigated from the view point of the contact of curves with horospheres. The singularities of the horospherical surface of a hyperbolic space curve correspond to the points of the curve where the order of the contact with horospheres is at least 3. This fact induces the notion of osculating horospheres of hyperbolic space curves. This is one of the motivations for the study of the singularities of horospherical surfaces. In order to describe the contact of hyperbolic space curves with horospheres, a hyperbolic invariant σ_h of a hyperbolic space curve has been discovered and investigated in [2]. Here, a horosphere is one of the totally umbilical surfaces in hyperbolic 3-space. Other totally umbilical surfaces are an equidistant surface and a (hyperbolic) sphere. In the classical differential geometry in Euclidean space, the totally umbilical surfaces (i.e., spheres or planes) are the model surfaces. If we consider the contact of space curves (or surfaces) with the model surfaces, we can find invariants of curves (or surfaces) which explain the contact of curves with the model surfaces. In this paper we consider the contact of hyperbolic space curves with equidistant surfaces or spheres in hyperbolic 3-space, with the result that we introduce the notion of de Sitter focal surfaces and hyperbolic focal surfaces of hyperbolic space curves. Moreover, we investigate the singularities of those surfaces and discover a new invariant δ_h of a hyperbolic space curve

Received February 5, 2016.
2010 *Mathematics Subject Classification.* Primary 53A25; Secondary 58K35.
Key words and phrases. hyperbolic space, hyperbolic space curves, focal surfaces, evolutes.

for characterizing the singularities. As a consequence, we obtain the geometric meanings of this invariant.

On the other hand, the lightlike focal set along a spacelike submanifolds in Lorentz-Minkowski space has been introduced and investigated the singularities of these sets in [3]. Since a hyperbolic space curve is a spacelike submanifold with codimension 3 in Lorentz-Minkowski 4-space, we have the lightlike focal set along the curve. In [3] we defined certain projections from Lorentz-Minkowski $n+1$-space onto hyperbolic n-space or de Sitter n-space (cf., §5) and shown that the projection images of the lightlike focal set are equal to the hyperbolic focal set or the de Sitter focal set of the submanifold in hyperbolic n-space when the codimension of the submanifold is one. For a general codimensional case, there are no definitions of the hyperbolic (or, de Sitter) focal set of a submanifold in hyperbolic n-space so far as we know. In this paper we consider curves in hyperbolic 3-space which is the simplest case of the higher codimension case. As a consequence, the projection image of the lightlike focal set along a curve in hyperbolic 3-space is equal to the hyperbolic (or, de Sitter) focal set of the curve (cf., Theorem 5.1). We expect the same result in the general dimensional case.

In §2 we give the basic notions on Lorentz-Minkowski 4-space and hyperbolic 3-space. We define de Sitter focal surfaces and hyperbolic focal surfaces in §2. One of the main results is Theorem 2.1 which gives a classification of singularities of those surfaces. We define two families of height functions on hyperbolic space curves in §3. Those functions are useful for analyzing the singularities of focal surfaces. The geometric meanings of the above new invariants δ_h are investigated in §4. Moreover, the notion of de Sitter evolutes and hyperbolic evolutes of hyperbolic space curve is defined in §4. Another main result is Theorem 4.3 which explains the contact of hyperbolic space curves with the model surfaces by using the notion of de Sitter or hyperbolic evolutes. In §5 the relation of the hyperbolic (or, de Sitter) focal surface with the lightlike hyper surfaces along a hyperbolic space curve are investigated.

2. Basic notions and results

We adopt the Lorentzian model of the hyperbolic 3-space. Let

$$\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R},\ i = 0, 1, 2, 3\}$$

be a 4-dimensional vector space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i.$$

We call $(\mathbb{R}^4, \langle , \rangle)$ *Minkowski space.* We denote $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$,

$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define *the hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define *hyperbolic* 3*-space* by

$$H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 \geq 1\},$$

de Sitter 3*-space* by

$$S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and a *closed lightcone* with the vertex $\boldsymbol{a}$ by

$$LC_a = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x} - \boldsymbol{a}, \boldsymbol{x} - \boldsymbol{a} \rangle = 0\}.$$

We denote that

$$LC^*_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in LC_0 \mid x_0 > 0\}$$

and we call it *the future lightcone* at the origin. We can also define the notion of *the past lightcone*.

For any $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ by

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x^1_0 & x^1_1 & x^1_2 & x^1_3 \\ x^2_0 & x^2_1 & x^2_2 & x^2_3 \\ x^3_0 & x^3_1 & x^3_2 & x^3_3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$. We can easily show that $\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}\ \boldsymbol{x}_1\ \boldsymbol{x}_2\ \boldsymbol{x}_3)$, so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$, $i = 1, 2, 3$.

We have three kinds of surfaces in $H^3_+(-1)$ which are given by intersections of $H^3_+(-1)$ and hyperplanes in $\mathbb{R}^4_1$. A surface $H^3_+(-1) \cap HP(\boldsymbol{v}, c)$ is called a *sphere*, an *equidistant surface* or a *horosphere* if $H(\boldsymbol{v}, c)$ is spacelike, timelike or lightlike respectively. We write $SP^2(\boldsymbol{v}, c)$ as a sphere and $ES^2(\boldsymbol{v}, c)$ as an equidistant surface. Especially, $ES^2(\boldsymbol{v}, 0)$ is called a *hyperbolic plane*.

We now construct the explicit differential geometry on curves in $H^3_+(-1)$. Let $\boldsymbol{\gamma} : I \longrightarrow H^3(-1)$ be a regular curve. Since $H^3_+(-1)$ is a Riemannian manifold, we can reparametrise $\boldsymbol{\gamma}$ by the arc-length. Hence, we may assume that $\boldsymbol{\gamma}(s)$ is a unit speed curve. So we have the tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ with $\|\boldsymbol{t}(s)\| = 1$, where $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. In the case when $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, then we have a unit vector $\boldsymbol{n}(s) = \frac{\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)}{\|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|}$. Moreover, define $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$, then we have a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{e}(s)\}$ of $\mathbb{R}^4_1$ along $\boldsymbol{\gamma}$. By standard arguments, under the assumption that $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s) \rangle \neq -1$, we have the following *Frenet-Serret type formula*:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{t}(s), \\ \boldsymbol{t}'(s) = \kappa_h(s)\boldsymbol{n}(s) + \boldsymbol{\gamma}(s), \\ \boldsymbol{n}'(s) = -\kappa_h(s)\boldsymbol{t}(s) + \tau_h(s)\boldsymbol{e}(s), \\ \boldsymbol{e}'(s) = -\tau_h(s)\boldsymbol{n}(s), \end{cases}$$

where $\kappa_h(s) = \|\boldsymbol{t}'(s) - \boldsymbol{\gamma}(s)\|$ and $\tau_h(s) = -\frac{\det(\boldsymbol{\gamma}(s),\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}''(s),\boldsymbol{\gamma}'''(s))}{(\kappa_h(s))^2}$.

Since $\langle \boldsymbol{t}'(s)-\boldsymbol{\gamma}(s), \boldsymbol{t}'(s)-\boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle + 1$, the condition $\langle \boldsymbol{t}'(s), \boldsymbol{t}'(s)\rangle \neq -1$ is equivalent to the condition $\kappa_h(s) \neq 0$. Moreover, we can show that the curve $\boldsymbol{\gamma}(s)$ satisfies the condition $\kappa_h(s) \equiv 0$ if and only if there exists a lightlike vector $\boldsymbol{c}$ such that $\boldsymbol{\gamma}(s) - \boldsymbol{c}$ is a geodesic. Such a curve is called *an equidistant curve.*

Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve. We denote that $\kappa_h^k = (\kappa_h)^k$ and $\tau_h^k = (\tau_h)^k$. Then we define two maps as follows:

$$DF_\gamma : I \times J \longrightarrow S^3_1;\ DF_\gamma(s,\theta) = \frac{\cos\theta}{\sqrt{1-\kappa_h^2(s)}}(\kappa_h(s)\gamma(s) + \boldsymbol{n}(s)) + \sin\theta \boldsymbol{e}(s),$$

where $\kappa_h^2(s) < 1$ and $J = [0, 2\pi]$, which is called a *de Sitter focal surface* of $\boldsymbol{\gamma}$ and

$$HF_\gamma : I \times \mathbb{R} \longrightarrow H^3_+(-1);\ HF_\gamma(s,\theta) = \frac{\cosh\theta}{\sqrt{\kappa_h^2(s)-1}}(\kappa_h(s)\gamma(s) + \boldsymbol{n}(s)) + \sinh\theta \boldsymbol{e}(s),$$

where $\kappa_h^2(s) > 1$, which is called a *hyperbolic focal surface* of $\boldsymbol{\gamma}$.

In this paper we consider geometric meanings of the singularities of these two surfaces. In order to avoid the complicated situation, we assume that $\tau_h \neq 0$ for $\boldsymbol{\gamma}$. We introduce a hyperbolic invariant of γ as follows:

$$\delta_h(s) = (\kappa_h^2\tau_h^3 - \kappa_h\kappa_h''\tau_h + 2(\kappa_h')^2\tau_h + \kappa_h\kappa_h'\tau_h')(s).$$

The geometric meaning of this invariant will be discussed in §4. One of our main results is given as follows:

Theorem 2.1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h\tau_h \neq 0$.*

(A) *Suppose that $\kappa_h^2 < 1$. Then we have the following*:

(1) *The image of de Sitter focal surface DF_γ of $\boldsymbol{\gamma}$ is singular at (s_0, θ_0) if and only if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0).$$

(2) *The image of de Sitter focal surface DF_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at (s_0, θ_0) if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0),\ \kappa_h'(s_0) \neq 0 \text{ and } \delta_h(s_0) \neq 0.$$

(3) *The image of de Sitter focal surface DF_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at* (s_0, θ_0) *if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0),\ \ \kappa_h'(s_0) \neq 0,\ \ \delta_h(s_0) = 0 \ \textit{and}\ \delta_h'(s_0) \neq 0.$$

(B) *Suppose that* $\kappa_h^2 > 1$. *Then we have the followings*:

(1) *The image of hyperbolic focal surface HF_γ of $\boldsymbol{\gamma}$ is singular at* (s_0, θ_0) *if and only if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0).$$

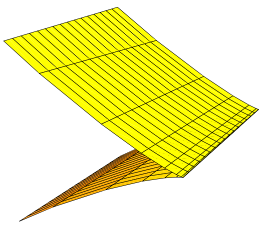
(2) *The image of hyperbolic focal surface HF_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the cuspidaledge $C \times \mathbb{R}$ at* (s_0, θ_0) *if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0),\ \ \kappa_h'(s_0) \neq 0 \ \textit{and}\ \delta_h(s_0) \neq 0.$$

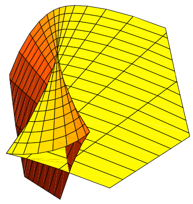
(3) *The image of hyperbolic focal surface HF_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the swallowtail SW at* (s_0, θ_0) *if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0),\ \ \kappa_h'(s_0) \neq 0,\ \ \delta_h(s_0) = 0 \ \textit{and}\ \delta_h'(s_0) \neq 0.$$

Here, $C \times \mathbb{R} = \{(x_1, x_2, x_3) \mid {x_1}^2 = {x_2}^3\}$ is the *cuspidaledge* (c.f., Fig. 1) and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the *swallowtail* (c.f., Fig. 2).



cuspidaledge
Fig.1.



swallowtail
Fig. 2.

3. Height functions

In this section we introduce a family of functions on a curve which is useful for the study of invariants of hyperbolic space curves. For a hyperbolic space curve $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$, we define a function $D : I \times S^3_1 \longrightarrow \mathbb{R}$ by $D(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle$. We call D a *de Sitter height function* on $\boldsymbol{\gamma}$. We denote that $d(s) = d_{v_0}(s) = D(s, \boldsymbol{v}_0)$ for any $\boldsymbol{v}_0 \in S^3_1$. We also define a function $H : I \times H^3(-1) \longrightarrow$

$\mathbb{R}$ by $H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle$. We call H a *hyperbolic height function* on $\boldsymbol{\gamma}$. We also denote that $h(s) = h_{v_0}(s) = H(s, \boldsymbol{v}_0)$. Then we have the following proposition.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \tau_h \neq 0$.*

Suppose that $\kappa_h^2 < 1$. Then we have the followings:

(1) $d'_{v_0}(s_0) = 0$ *if and only if there exist $\lambda, \mu, \nu \in \mathbb{R}$ such that*

$$\boldsymbol{v}_0 = \lambda \boldsymbol{\gamma}(s_0) + \mu \boldsymbol{n}(s_0) + \nu \boldsymbol{e}(s_0).$$

(2) $d'_{v_0}(s_0) = d''_{v_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{v_0} = \frac{\cos\theta}{\sqrt{1 - \kappa_h^2(s_0)}}(\kappa_h(s_0)\gamma(s_0) + \boldsymbol{n}(s_0)) + \sin\theta \boldsymbol{e}(s_0),$$

where $\theta \in [0, 2\pi)$.

(3) $d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$ *if and only if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\tau_h \kappa_h \sqrt{1 - \kappa_h^2}}(s_0)$$

and

$$\boldsymbol{v_0} = \frac{\cos\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{1 - \kappa_h^2(s_0)}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0) + \kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0) - \kappa_h'(s_0)\boldsymbol{e}(s_0)).$$

(4) $d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = d^{(4)}(s_0) = 0$ *if and only if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\tau_h \kappa_h \sqrt{1 - \kappa_h^2}}(s_0), \ \ \delta_h(s_0) = 0$$

and

$$\boldsymbol{v_0} = \frac{\cos\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{1 - \kappa_h^2(s_0)}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0) + \kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0) - \kappa_h'(s_0)\boldsymbol{e}(s_0)).$$

(5) $d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = d^{(4)}(s_0) = d^{(5)}(s_0) = 0$ *if and only if*

$$\tan\theta_0 = \frac{-\kappa_h'}{\tau_h \kappa_h \sqrt{1 - \kappa_h^2}}(s_0), \ \ \delta_h(s_0) = \delta_h'(s_0) = 0$$

and

$$\boldsymbol{v_0} = \frac{\cos\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{1 - \kappa_h^2(s_0)}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0) + \kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0) - \kappa_h'(s_0)\boldsymbol{e}(s_0)).$$

Proof. Since $d_{v_0}(s) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v}_0 \rangle$, we have the following calculations:

(a) $d'_{v_0}(s) = \langle \boldsymbol{t}(s), \boldsymbol{v}_0 \rangle$,
(b) $d''_{v_0}(s) = \langle \kappa_h(s)\boldsymbol{n}(s) + \gamma(s), \boldsymbol{v}_0 \rangle$,
(c) $d'''_{v_0}(s) = \langle (1 - \kappa_h^2(s))\boldsymbol{t}(s) + \kappa_h'(s)\boldsymbol{n}(s) + \kappa_h(s)\tau_h(s)\boldsymbol{e}(s), \boldsymbol{v}_0 \rangle$,
(d) $d^{(4)}_{v_0}(s) = \langle (1 - \kappa_h^2(s))\gamma(s) - 3\kappa_h(s)\kappa_h'(s)\boldsymbol{t}(s) + (\kappa_h(s) - \kappa_h^3(s) - \kappa_h(s)\tau_h^2(s) + \kappa_h''(s))\boldsymbol{n}(s) + (2\kappa_h'(s)\tau_h(s) + \kappa_h(s)\tau_h'(s)\boldsymbol{e}(s)), \boldsymbol{v}_0 \rangle$,

(e) $d^{(5)}_{v_0}(s) = \langle -5\kappa_h(s)\kappa'_h(s)\gamma(s) + (1 - 2\kappa_h^2(s) + \kappa_h^4(s) + \kappa_h^2(s)\tau_h^2(s) - 3(\kappa'_h)^2(s) - 4\kappa_h(s)\kappa''_h(s))\boldsymbol{t}(s) + (-6\kappa_h^2(s)\kappa'_h(s)) - 3\kappa_h(s)\tau_h(s)\tau'_h(s) + \kappa'_h(s) - 3\kappa'_h(s)\tau_h^2(s) + \kappa_h^3(s))\boldsymbol{n}(s) + (\kappa_h(s)\tau_h(s) + \kappa_h(s)\tau''_h(s) - \kappa_h^3(s)\tau_h(s) - \kappa_h^3(s)\tau_h(s) + 3\kappa'_h(s)\tau'_h(s) + 3\kappa''_h(s)\tau_h(s))\boldsymbol{e}(s), \boldsymbol{v}_0\rangle$,

By definition and the formula (a), the assertion (1) follows. By the formula (b), $d'_{v_0}(s) = d''_{v_0}(s_0) = 0$ if and only if $\boldsymbol{v}_0 = \mu\kappa_h(s_0)\gamma(s_0) + \mu\boldsymbol{n}(s_0) + \nu\boldsymbol{e}(s_0)$, and $\mu^2(1-(\kappa_h)^2(s_0))+\nu^2 = 1$. It follows that $\mu = \frac{\cos\theta}{\sqrt{1-\kappa_h^2(s_0)}}$, $\nu = \sin\theta$, where $0 \le \theta < 2\pi$. Therefore the assertion (2) holds. By the formula (c), $d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$ if and only if $\frac{\kappa'_h(s_0)\cos\theta}{\sqrt{1-\kappa_h^2(s_0)}} + \kappa_h(s_0)\tau_h(s_0)\sin\theta = 0$. Since $\kappa_h(s_0) \ne 0$ and $\tau_h(s_0) \ne 0$, there exists θ_0 such that $\tan\theta_0 = \frac{-\kappa'_h}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0)$. We have the assertion (3). By the formula (d), $d_{v_0}(s) = d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = d^{(4)}(s_0) = 0$ if and only if there exists θ_0 such that $\tan\theta_0 = \frac{-\kappa'_h}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s_0)$ and $\delta_h(s_0) = 0$. This means that the assertion (4) holds. By the similar arguments to the above, we can show the assertion (5). This completes the proof. □

For the hyperbolic height function H, we have the following proposition.

Proposition 3.2. *Let* $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ *be a unit speed hyperbolic space curve with* $\kappa_h\tau_h \ne 0$.

Suppose that $\kappa_h^2 > 1$. *Then we have the followings*:

(1) $h'_{v_0}(s_0) = 0$ *if and only if there exist* $\lambda, \mu, \nu \in \mathbb{R}$ *such that*

$$\boldsymbol{v}_0 = \lambda\boldsymbol{\gamma}(s_0) + \mu\boldsymbol{n}(s_0) + \nu\boldsymbol{e}(s_0).$$

(2) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$ *if and only if*

$$\boldsymbol{v_0} = \frac{\cosh\theta}{\sqrt{\kappa_h^2(s_0) - 1}}(\kappa_h(s_0)\gamma(s_0) + \boldsymbol{n}(s_0)) + \sinh\theta\boldsymbol{e}(s_0).$$

(3) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = h'''_{v_0}(s_0) = 0$ *if and only if*

$$\tanh\theta_0 = \frac{-\kappa'_h}{\tau_h\kappa_h\sqrt{\kappa_h^2 - 1}}(s_0)$$

and

$$\boldsymbol{v_0} = \frac{\cosh\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{\kappa_h^2(s_0) - 1}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0) + \kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0) - \kappa'_h(s_0)\boldsymbol{e}(s_0)).$$

(4) $h'_{v_0}(s_0) = h''_{v_0}(s_0) = h'''_{v_0}(s_0) = h^{(4)}(s_0) = 0$ *if and only if*

$$\tanh\theta_0 = \frac{-\kappa'_h}{\tau_h\kappa_h\sqrt{\kappa_h^2 - 1}}(s_0),\ \delta_h(s_0) = 0$$

and

$$\boldsymbol{v_0}=\frac{\cosh\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{\kappa_h^2(s_0)-1}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0)+\kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0)-\kappa_h'(s_0)\boldsymbol{e}(s_0)).$$

(5) $h'_{v_0}(s_0)=h''_{v_0}(s_0)=h'''_{v_0}(s_0)=h^{(4)}(s_0)=h^{(5)}(s_0)=0$ *if and only if*

$$\tanh\theta_0=\frac{-\kappa_h'}{\tau_h\kappa_h\sqrt{\kappa_h^2-1}}(s_0),\ \ \delta_h(s_0)=\delta_h'(s_0)=0$$

and

$$\boldsymbol{v_0}=\frac{\cosh\theta_0}{\kappa_h(s_0)\tau_h(s_0)\sqrt{\kappa_h^2(s_0)-1}}(\kappa_h^2(s_0)\tau_h(s_0)\gamma(s_0)+\kappa_h(s_0)\tau_h(s_0)\boldsymbol{n}(s_0)-\kappa_h'(s_0)\boldsymbol{e}(s_0)).$$

Proof. Since $h_{v_0}(s)=\langle\gamma(s),\boldsymbol{v}_0\rangle$, we have the following calculations:

(a) $h'_{v_0}(s)=\langle\boldsymbol{t}(s),\boldsymbol{v}_0\rangle$,
(b) $h''_{v_0}(s)=\langle\kappa_h(s)\boldsymbol{n}(s)+\gamma(s),\boldsymbol{v}_0\rangle$,
(c) $h'''_{v_0}(s)=\langle(1-\kappa_h^2(s))\boldsymbol{t}(s)+\kappa_h'(s)\boldsymbol{n}(s)+\kappa_h(s)\tau_h(s)\boldsymbol{e}(s),\boldsymbol{v}_0\rangle$,
(d) $h^{(4)}_{v_0}(s)=\langle(1-\kappa_h^2(s))\gamma(s)-3\kappa_h(s)\kappa_h'(s)\boldsymbol{t}(s)+(\kappa_h(s)-\kappa_h^3(s)-\kappa_h(s)\tau_h^2(s)$ $+\kappa_h''(s))\boldsymbol{n}(s)+(2\kappa_h'(s)\tau_h(s)+\kappa_h(s)\tau_h'(s)\boldsymbol{e}(s)),\boldsymbol{v}_0\rangle$,
(e) $h^{(5)}_{v_0}(s)=\langle-5\kappa_h(s)\kappa_h'(s)\gamma(s)+(1-2\kappa_h^2(s)+\kappa_h^4(s)+\kappa_h^2(s)\tau_h^2(s)$ $-3(\kappa_h')^2(s)-4\kappa_h(s)\kappa_h''(s))\boldsymbol{t}(s)+(-6\kappa_h^2(s)\kappa_h'(s))-3\kappa_h(s)\tau_h(s)\tau_h'(s)$ $+\kappa_h'(s)-3\kappa_h'(s)(\tau_h)^2(s)+\kappa_h^3(s))\boldsymbol{n}(s)+(\kappa_h(s)\tau_h(s)+\kappa_h(s)\tau_h''(s)$ $-\kappa_h^3(s)\tau_h(s)-\kappa_h^3(s)\tau_h(s)+3\kappa_h'(s)\tau_h'(s)+3\kappa_h''(s)\tau_h(s))\boldsymbol{e}(s),\boldsymbol{v_0}\rangle$.

By definition and the formula (a), the assertion (1) follows. By the formula (b), $h'_{v_0}(s)=h''_{v_0}(s_0)=0$ if and only if $\boldsymbol{v}_0=\mu\kappa_h(s_0)\gamma(s_0)+\mu\boldsymbol{n}(s_0)+\nu\boldsymbol{e}(s_0)$, and $\mu^2(\kappa_h^2(s_0)-1)-\nu^2=1$. It follows that $\mu=\frac{\cosh\theta}{\sqrt{\kappa_h^2(s_0)-1}}$, $\nu=\sinh\theta$, where $\theta\in\mathbb{R}$. Therefore the assertion (2) holds. By the formula (c), $h'_{v_0}(s_0)=h''_{v_0}(s_0)=h'''_{v_0}(s_0)=0$ if and only if $\frac{\kappa_h'(s_0)\cosh\theta}{\sqrt{\kappa_h^2(s_0)-1}}+\kappa_h(s_0)\tau_h(s_0)\sinh\theta=0$. Since $\kappa_h(s_0)\neq0$ and $\tau_h(s_0)\neq0$, there exists θ_0 such that $\tanh\theta_0=\frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{\kappa_h^2-1}}(s_0)$. We have the assertion (3). By the formula (d), $h_{v_0}(s)=h'_{v_0}(s_0)=h''_{v_0}(s_0)=h'''_{v_0}(s_0)=h^{(4)}(s_0)=0$ if and only if there exists θ_0 such that $\tanh\theta_0=\frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{\kappa_h^2-1}}(s_0)$ and $\delta_h(s_0)=0$. This means that the assertion (4) holds. By the similar arguments to the above, we can show the assertion (5). This completes the proof. □

In order to prove Theorem 2.1, we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book[1]. Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s)=F_{x_0}(s,x_0)$. We say that f has an *A_k-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\leq p\leq k$, and $f^{(k+1)}(s_0)\neq0$. We also say that f has an *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0)=0$ for all $1\leq p\leq k$. Let F be an unfolding of f and $f(s)$ has an A_k-singularity ($k\geq1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s,x_0))(s_0)=\sum_{j=0}^{k-1}\alpha_{ji}(s-s_0)^j$ for $i=1,\ldots,r$. Then F is called an *$\mathcal{R}^+$-versal unfolding* if the $(k-1)\times r$ matrix of coefficients $(\alpha_{ji})_{j=1,\dots,k-1;i=1,\dots,r}$ has rank $k-1$ $(k-1\le r)$. We introduce an important set concerning the unfoldings relative to the above notions. The *bifurcation set* of F is the set

$$\mathcal{B}_F=\left\{x\in\mathbb{R}^r \mid \exists s \text{ with } \frac{\partial F}{\partial s}(s,x)=\frac{\partial^2 F}{\partial s^2}(s,x)=0\right\}.$$

Then we have the following classification (cf., [1]).

Theorem 3.3. *Let $F:(\mathbb{R}\times\mathbb{R}^r,(s_0,x_0))\to\mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has an A_k singularity at s_0. Suppose that F is an $\mathcal{R}^+$-versal unfolding.*
(1) *If $k=3$, then $\mathcal{B}_F$ is locally diffeomorphic to $C\times\mathbb{R}^{r-2}$.*
(2) *If $k=4$, then $\mathcal{B}_F$ is locally diffeomorphic to $SW\times\mathbb{R}^{r-3}$.*

For the proof of Theorem 2.1, we have the following propositions.

Proposition 3.4. *Let $\gamma:I\longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve. Then we have the following*:
(1) *Let $D:I\times S^3_1\longrightarrow\mathbb{R}$ be the de Sitter height function on $\boldsymbol{\gamma}(s)$. If d_{v_0} has an A_k-singularity $(k=3,4)$ at s_0 and $\kappa_h'(s_0)\neq 0$, then D is an $\mathcal{R}^+$-versal unfolding of d_{v_0}.*
(2) *Let $H:I\times H^3_+(-1)\longrightarrow\mathbb{R}$ be the hyperbolic height function on $\boldsymbol{\gamma}(s)$. If h_{v_0} has an A_k-singularity $(k=3,4)$ at s_0 and $\kappa_h'(s_0)\neq 0$, then H is an $\mathcal{R}^+$-versal unfolding of h_{v_0}.*

Proof. We have

$$D(s,\boldsymbol{v})=-v_0x_0(s)+v_1x_1(s)+v_2x_2(s)+v_3x_3(s),$$

where v_i and $x_i(s)$ denote respectively the coordinates of $\boldsymbol{v}$ and $\boldsymbol{\gamma}(s)$. Since $v_3=\sqrt{v_0^2-v_1^2-v_3^2+1}$, we have

$$\frac{\partial D}{\partial v_i}(s,\boldsymbol{v})=\pm\frac{v_i}{v_3}x_3(s)\mp x_i(s),\ (i=0,1,2),$$

$$\frac{\partial^j D}{\partial s^{j-1}\partial v_i}(s,\boldsymbol{v})=\pm\frac{v_i}{v_3}x_0^{(j-1)}(s)\mp x_i^{(j-1)}(s),\ (i=0,1,2,j=2,3,4),$$

so that we consider the following matrix:

$$A=\begin{pmatrix} -x_0'(s_0)+\dfrac{v_0}{v_3}x_3'(s_0) & x_1'(s_0)-\dfrac{v_1}{v_3}x_3'(s_0) & x_2'(s_0)-\dfrac{v_2}{v_3}x_3'(s_0)\\ -x_0''(s_0)+\dfrac{v_0}{v_3}x_3''(s_0) & x_1''(s_0)-\dfrac{v_1}{v_3}x_3''(s_0) & x_2''(s_0)-\dfrac{v_2}{v_3}x_3''(s_0)\\ -x_0'''(s_0)+\dfrac{v_0}{v_3}x_3'''(s_0) & x_1'''(s_0)-\dfrac{v_1}{v_3}x_3'''(s_0) & x_2'''(s_0)-\dfrac{v_2}{v_3}x_3'''(s_0)\end{pmatrix}.$$

We denote that

$$\boldsymbol{a}=\begin{pmatrix} x_0'(s_0)\\ x_0''(s_0)\\ x_0'''(s_0)\end{pmatrix},\boldsymbol{b_i}=\begin{pmatrix} x_i'(s_0)\\ x_i''(s_0)\\ x_i'''(s_0)\end{pmatrix},\ (i=1,2,3).$$

Then we have

$$\begin{aligned}\det A &= \frac{v_0}{v_3}\det(\boldsymbol{b_3}\ \boldsymbol{b_1}\ \boldsymbol{b_2}) + \frac{v_1}{v_3}\det(\boldsymbol{a}\ \boldsymbol{b_3}\ \boldsymbol{b_2}) + \frac{v_2}{v_3}\det(\boldsymbol{a}\ \boldsymbol{b_1}\ \boldsymbol{b_3}) - \frac{v_3}{v_3}\det(\boldsymbol{a}\ \boldsymbol{b_1}\ \boldsymbol{b_2})) \\ &= \frac{v_0}{v_3}\det(\boldsymbol{b_1}\ \boldsymbol{b_2}\ \boldsymbol{b_3}) - \frac{v_1}{v_3}\det(\boldsymbol{a}\ \boldsymbol{b_2}\ \boldsymbol{b_3}) - \frac{v_2}{v_3}\det(\boldsymbol{b_1}\ \boldsymbol{a}\ \boldsymbol{b_3}) - \frac{v_3}{v_3}\det(\boldsymbol{b_1}\ \boldsymbol{b_2}\ \boldsymbol{a})).\end{aligned}$$

Since we have

$$\begin{aligned}&(\boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \wedge \boldsymbol{\gamma}''')(s_0) \\ &= (-\det(\boldsymbol{b_1}\ \boldsymbol{b_2}\ \boldsymbol{b_3}), -\det(\boldsymbol{a}\ \boldsymbol{b_2}\ \boldsymbol{b_3}), -\det(\boldsymbol{b_1}\ \boldsymbol{a}\ \boldsymbol{b_3}), -\det(\boldsymbol{b_1}\ \boldsymbol{b_2}\ \boldsymbol{a})),\end{aligned}$$

$$\begin{aligned}\det A &= \langle (\frac{v_0}{v_3}, \frac{v_1}{v_3}, \frac{v_2}{v_3}, \frac{v_3}{v_3}), (\boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \wedge \boldsymbol{\gamma}''')(s_0)\rangle \\ &= \frac{\cos\theta_0(\kappa_h'(s_0))^2\kappa_h(s_0)}{v_3\kappa_h(s_0)\tau_h(s_0)\sqrt{1-\kappa_h^2(s_0)}} \neq 0 \text{ if } \kappa_h'(s_0) \neq 0.\end{aligned}$$

Thus, we have $\operatorname{rank} A = 3$ if $\kappa_h'(s_0) \neq 0$. If d_{v_0} has an A_k-singularity $(k = 3, 4)$ at s_0 and $\kappa_h'(s_0) \neq 0$, then D is $\mathcal{R}^+$-versal unfolding of d_{v_0}.

For the proof of the assertion (2), we have the similar arguments to the assertion (1), so that we omit it. This completes the proof. □

Proof of Theorem 2.1. First, we consider the assertion (A). By Proposition 3.1(2), the bifurcation set $\mathcal{B}_D$ of the de Sitter height function D of $\boldsymbol{\gamma}$ is the image of the de Sitter focal surface of $\boldsymbol{\gamma}$. The singularities of the bifurcation set are corresponding to the points of Proposition 3.1(3), so that the assertion (1) holds. It also follows from Proposition 3.1(4) and (5) that d_{v_0} has the A_3-type singularity (respectively, the A_4-type singularity) at $s = s_0$ if and only if $\tan\theta_0 = \kappa_h'/(\kappa_h\tau_h\sqrt{1-\kappa_h^2})(s_0)$ and $\delta_h(s_0) \neq 0$ (respectively, $\tan\theta_0$ =-$\kappa_h'/(\kappa_h\tau_h\sqrt{1-\kappa_h^2})(s_0)$ and $\delta_h'(s_0) \neq 0$). By Theorem 3.3 and Proposition 3.4, we have the assertions (A), (2) and (3).

For the proof of the assertion (B), we apply Proposition 3.2, Theorem 3.3 and Proposition 3.4 similar to the assertion (A). This completes the proof. □

4. Invariants of hyperbolic space curves

In the previous sections we found that the function

$$\delta_h(s) = (\kappa_h^2\tau_h^3 - \kappa_h\kappa_h''\tau_h + 2(\kappa_h')^2\tau_h + \kappa_h\kappa_h'\tau_h')(s)$$

on γ has a special meaning. Here, we try to understand the geometric meaning of this invariant. We have the following propositions.

Proposition 4.1. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\tau_h\kappa_h \neq 0$ and $(\kappa_h)^2(s) \neq 1$. Then we have the following*:

(A) *Suppose that $\kappa_h^2(s) < 1$. Let $\theta(s)$ be a function defined by $\tan\theta(s) = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s)$. Then the following conditions are equivalent*:

(1) *$DF_\gamma(s, \theta(s))$ is a constant vector,*

(2) *$\delta_h(s) \equiv 0$,*

(3) $\mathrm{Im}(\gamma) \subset ES^2(\boldsymbol{v}, c)$.

(B) *Suppose that* $\kappa_h^2(s) > 1$. *Let* $\theta(s)$ *be a function defined by* $\tanh\theta(s) = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{\kappa_h^2-1}}(s)$. *Then the following conditions are equivalent*:

(1) $HF_\gamma(s, \theta(s))$ *is a constant vector,*

(2) $\delta_h(s) \equiv 0$,

(3) $\mathrm{Im}(\gamma) \subset SP^2(\boldsymbol{v}, c)$.

Proof. First, we consider the assertion (A). By definition, we have

$$DF_\gamma(s,\theta(s)) = \frac{\cos\theta(s)}{\kappa_h(s)\tau_h(s)\sqrt{1-\kappa_h^2(s)}}(\kappa_h^2(s)\tau_h(s)\gamma(s) + \kappa_h(s)\tau_h(s)\boldsymbol{n}(s) - \kappa_h'(s)\boldsymbol{e}(s)).$$

By straightforward calculation, we have

$$\begin{aligned}\theta'(s) &= \frac{1}{1+\dfrac{(-\kappa_h')^2}{\kappa_h^2\tau_h^2(1-\kappa_h^2)}}\left(\frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}\right)'(s)\\ &= \frac{(1-\kappa_h^2)(-\kappa_h\kappa_h''\tau_h + (\kappa_h')^2\tau_h + \kappa_h(s)\kappa_h'(s)\tau_h'(s)) - \kappa_h^2(\kappa_h')^2\tau_h}{\sqrt{1-\kappa_h^2}(\kappa_h^2\tau_h^2(1-\kappa_h^2)+(\kappa_h')^2)}(s)\end{aligned}$$

and

$$\begin{aligned}&\frac{dDF_\gamma(s,\theta(s))}{ds}(s)\\ &= \frac{1}{\kappa_h^2(s)\tau_h^2(s)(1-\kappa_h^2(s))}\Big(\big(-\sin\theta(s)\theta'(s)(\kappa_h^2(s)\tau_h(s)\boldsymbol{\gamma}(s)\\ &\quad+\kappa_h(s)\tau_h(s)\boldsymbol{n}(s)-\kappa_h'(s)\boldsymbol{e}(s))+\cos\theta(s)((2\kappa_h(s)\kappa_h'(s)\tau_h(s)+\kappa_h^2(s)\tau_h'(s))\boldsymbol{\gamma}(s)\\ &\quad+(2\kappa_h'(s)\tau_h(s)+\kappa_h(s)\tau_h'(s))\boldsymbol{n}(s)-\kappa_h''(s)\boldsymbol{e}(s))\big)\kappa_h(s)\tau_h(s)\sqrt{1-\kappa_h^2(s)}\\ &\quad-\cos\theta(s)(\kappa_h^2(s)\tau_h(s)\gamma(s)+\kappa_h(s)\tau_h(s)\boldsymbol{n}(s)-\kappa_h'(s)\boldsymbol{e}(s))\big((\kappa_h'(s)\tau_h(s)\\ &\quad+\kappa_h(s)\tau_h'(s)\sqrt{1-\kappa_h^2(s)}\big)-\frac{\kappa_h^2(s)\kappa_h'(s)\tau_h(s)}{\sqrt{1-\kappa_h^2(s)}}\Big).\end{aligned}$$

Since $\sin\theta = \frac{-\kappa_h'(s)\cos\theta}{\kappa_h(s)\tau_h(s)\sqrt{1-(\kappa_h)^2(s)}}$, we have

$$\begin{aligned}&\frac{dDF_\gamma(s,\theta(s))}{ds}\\ &= \frac{\cos\theta(s)\big(\kappa_h(s)\kappa_h'(s)\gamma(s)+\kappa_h'(s)\boldsymbol{n}(s)+\kappa_h(s)\tau_h(s)(1-\kappa_h^2(s)\boldsymbol{e}(s))\big)}{\kappa_h(s)\tau_h(s)\sqrt{1-\kappa_h^2(s)}(\kappa_h^2(s)\tau_h^2(s)(1-\kappa_h^2(s))+(\kappa_h')^2(s))}(\kappa_h^2(s)\tau_h^3(s)\\ &\quad-\kappa_h(s)\kappa_h''(s)\tau_h(s)+2\kappa_h^2(s)\tau_h(s)+\kappa_h(s)\kappa_h'(s)\tau_h'(s)).\end{aligned}$$

Therefore, $dDF_\gamma(s,\theta(s))/ds \equiv 0$ if and only if $\delta_h(s) \equiv 0$. This means that the conditions (1) and (2) are equivalent. Assume that the condition (1) holds. Then we have $\langle\boldsymbol{\gamma}(s), DF_\gamma(s,\theta(s))\rangle = -\cos\theta(s)/\sqrt{1-\kappa_h^2(s)}$, which is constant. The condition (3) holds. For the converse, we assume that

$\langle \gamma(s), \boldsymbol{v} \rangle = c$ for a constant vector $\boldsymbol{v}$ and a real number c. Since $d_v(s) = c$, we have $\boldsymbol{v} = DF_\gamma(s, \theta(s))$ by Proposition 3.1, so that the condition (1) holds.

For the proof of the assertion (B), we use $\tanh \theta(s)$ instead of $\tan \theta(s)$ in the assertion (A). By the similar calculations to the assertion (A), we can prove the assertion (B), so that we omit it. This completes the proof. □

Therefore, the equidistant surfaces and the (hyperbolic) spheres are the model surfaces in this case. We consider the contact of curves with these model surfaces. Let $F : H^3_+(-1) \longrightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a regular curve. We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have contact of *at least order* k for $t = t_0$ if the function $g(t) = F \circ \boldsymbol{\gamma}(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k)}(t_0) = 0$. If $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have contact of at least order k for $t = t_0$ and satisfies the condition that $g^{(k+1)}(t_0) \neq 0$, then we say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *contact of order* k for $t = t_0$. If an equidistant surface $ES(\boldsymbol{v}_0, c)$ (respectively, a sphere $SP(\boldsymbol{v}_0, c)$) and a hyperbolic space curve $\boldsymbol{\gamma}$ have contact of at least order 3 for a point s_0, we call $ES(\boldsymbol{v}_0, c)$ (respectively, $SP(\boldsymbol{v}_0, c)$) the *osculating equidistant surface* (respectively, *osculating sphere*) of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$. Then we have the following proposition.

Proposition 4.2. *Let $\gamma : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \tau_h \neq 0$. Then we have the followings*:

(1) *There exists the osculating equidistant surface of $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ if and only if $\kappa_h^2(s_0) < 1$.*

(2) *Suppose that $\kappa_h^2(s_0) < 1$. Then the osculating equidistant surface and $\boldsymbol{\gamma}$ have contact of order 4 for $s = s_0$ if and only if $\delta_h(s_0) = 0$ and $\delta'_h(s_0) \neq 0$.*

(3) *There exists the osculating sphere of $\boldsymbol{\gamma}$ at a point $\boldsymbol{\gamma}(s_0)$ if and only if $\kappa_h^2(s_0) > 1$.*

(4) *Suppose that $\kappa_h^2(s_0) > 1$. Then the osculating sphere and $\boldsymbol{\gamma}$ have contact of 4 for $s = s_0$ if and only if $\delta_h(s_0) = 0$ and $\delta'_h(s_0) \neq 0$.*

Proof. Let $\mathfrak{D} : H^3_+(-1) \times S^3_1 \longrightarrow \mathbb{R}$ be a function defined by $\mathfrak{D}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in S^3_1$, $\mathfrak{d}_{v_0}(\boldsymbol{x}) = \mathfrak{D}(\boldsymbol{x}, \boldsymbol{v}_0)$ is a submersion and $\mathfrak{d}_{v_0}^{-1}(c)$ is an equidistant surface. Moreover, any equidistant surface can be realized as a level set of $\mathfrak{d}_{v_0}$ for some $\boldsymbol{v}_0 \in S^3_1$. For any $\boldsymbol{\gamma}$, we have $\mathfrak{d}_{v_0} \circ \boldsymbol{\gamma}(s) = d_{v_0}(s)$, here $d_{v_0}(s) = D(s, \boldsymbol{v}_0)$. Therefore, $h_{v_0}^{-1}(c)$ is an osculating horosphere of $\boldsymbol{\gamma}$ at $\boldsymbol{\gamma}(s_0)$ if and only if $d_{v_0}(s_0) = c$ and $d'_{v_0}(s_0) = d''_{v_0}(s_0) = d'''_{v_0}(s_0) = 0$. By Proposition 3.1, this condition is equivalent to the condition that

$$\boldsymbol{v_0} = \frac{\cos\theta_0}{\sqrt{1 - \kappa_h^2(s_0)}}(\kappa_h(s_0)\gamma(s_0) + \boldsymbol{n}(s_0)) + \sin\theta_0 \boldsymbol{e}(s_0),$$

where $\tan \theta_0 = \frac{-\kappa'_h}{\kappa_h \tau_h \sqrt{1-(\kappa_h)^2}}(s_0)$ and $c = D(s_0, \boldsymbol{v}_0)$. The existence of such $\boldsymbol{v}_0 \in S^3_1$ and $\theta_0 \in [0, 2\pi)$ is equivalent to the condition that $\kappa_h^2(s_0) < 1$. The assertion (2) follows from the assertions (3) and (4) of Proposition 3.1.

For the assertions (3) and (4), we consider a function $\mathfrak{H} : H^3_+(-1) \times H^3_+(-1) \longrightarrow \mathbb{R}$ defined by $\mathfrak{D}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$ instead of $\mathfrak{D}$. Then the assertions follow from Proposition 3.2 by the similar arguments to the proof of assertions (1) and (2). This completes the proof. □

We now define $DE_\gamma(s) = DF_\gamma(s, \theta(s))$, where $\tan\theta(s) = \frac{-\kappa_h'}{\kappa_h\tau_h\sqrt{1-\kappa_h^2}}(s)$ for a unit speed hyperbolic space curve $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ with $\tau_h \neq 0$ and $0 < \kappa_h^2(s) < 1$. We call DE_γ a *de Sitter evolute* of $\boldsymbol{\gamma}$. We also define $HE_\gamma(s) = HF_\gamma(s, \theta(s))$, where $\tanh\theta(s) = -\kappa_h'/\kappa_h\tau_h\sqrt{\kappa_h^2 - 1}(s)$ for a unit speed hyperbolic space curve $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ with $\tau_h \neq 0$ and $1 < \kappa_h^2(s)$. We call HE_γ a *hyperbolic evolute* of $\boldsymbol{\gamma}$. By Propositions 4.1 and 4.2, the de Sitter evolute (respectively, the hyperbolic evolute) is the locus of the centers of the osculating equidistant surfaces (respectively, spheres). We have the following theorem. We define $C(2,3,4) = \{(t^2, t^3, t^4) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$, which is called a $(2,3,4)$-*cusp*.

Theorem 4.3. *Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve with $\kappa_h\tau_h \neq 0$.*

(A) *Suppose that $\kappa_h^2 < 1$. Then we have the following*:

(1) *The image of the de Sitter evolute DE_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the line at s_0 if the osculating equidistant surface and $\boldsymbol{\gamma}$ have contact of order 3 for $s = s_0$.*

(2) *The image of the de Sitter evolute DE_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the $(2,3,4)$-cusp $C(2,3,4)$ at s_0 if the osculating equidistant surface and $\boldsymbol{\gamma}$ have contact of order 4 for $s = s_0$.*

(B) *Suppose that $\kappa_h^2 > 1$. Then we have the followings*:

(1) *The image of the hyperbolic evolute HE_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the line at s_0 if the osculating sphere and $\boldsymbol{\gamma}$ have contact of order 3 for $s = s_0$.*

(2) *The image of the hyperbolic evolute HE_γ of $\boldsymbol{\gamma}$ is locally diffeomorphic to the $(2,3,4)$-cusp $C(2,3,4)$ at s_0 if the osculating sphere and $\boldsymbol{\gamma}$ have contact of order 4 for $s = s_0$.*

Proof. For the proof of the assertion (A). We apply Theorem 2.1(A). Then the image of the de Sitter focal surface is locally diffeomorphic to $C \times \mathbb{R}$ if $\delta_h(s_0) \neq 0$. By Proposition 4.2, this means that the osculating equidistant surface and $\boldsymbol{\gamma}$ have contact of order 3 for $s = s_0$. Since the locus of the singularities of CE is locally diffeomorphic to the line, the assertion (1) holds. Since the locus of singularities of SW is $C(2,3,4)$, the assertion (2) holds by Theorem 2.1 and Proposition 4.2. The assertion (B) also holds by the similar reason to the assertion (A). This completes the proof. □

We can summarize the results as follows:

$\kappa_h^2 < 1$	$\delta_h(s_0) \neq 0$	$\delta_h(s_0) = 0$, $\delta_h'(s_0) \neq 0$
DF_γ	$C \times \mathbb{R}$	SW
DE_γ	$\mathbb{R}$	$C(2,3,4)$
The contact with equidistant surfaces	the order 3	the order 4

$\kappa_h^2 > 1$	$\delta_h(s_0) \neq 0$	$\delta_h(s_0) = 0,\ \delta_h'(s_0) \neq 0$
HF_γ	$C \times \mathbb{R}$	SW
HE_γ	$\mathbb{R}$	$C(2,3,4)$
The contact with spheres	the order 3	the order 4

5. Lightlike hypersurfaces in Minkowski 4-space along hyperbolic space curves

In [3] the notion of lightlike hypersurfaces along spacelike submanifolds with general codimension in Minkowski space has been introduced and investigated the singularities. Since a curve in $H^3_+(-1)$ is a spacelike curve in $\mathbb{R}^4_1$ with codimension 3, we have the lightlike hypersurface in $\mathbb{R}^4_1$ along the curve. In this section we investigate the relationship of the hyperbolic (or de Sitter) focal surface of a curve in $H^3_+(-1)$ with the lightlike hypersurface along the curve in $\mathbb{R}^4_1$. Let $\boldsymbol{\gamma} : I \longrightarrow H^3_+(-1)$ be a unit speed hyperbolic space curve. Following the definition of lightlike hypersurfaces along spacelike submanifolds in [3], the *lightlike hypersurface along* $C = \boldsymbol{\gamma}(I)$ is a map $\mathbb{LH}_C : I \times [0, 2\pi) \times R \longrightarrow \mathbb{R}^4_1$ defined by

$$\mathbb{LH}_C(s,\phi,t) = \boldsymbol{\gamma}(s) + t(\boldsymbol{\gamma}(s) + \cos\phi \boldsymbol{n}(s) + \sin\phi \boldsymbol{e}(s)).$$

We have the followings:

$$\frac{\partial \mathbb{LH}_C}{\partial s} = (1 + t(1 - \kappa_h(s)\cos\phi)\boldsymbol{t}(s) - t\sin\phi\tau_h(s)\boldsymbol{n}(s) + t\cos\phi\tau_h(s)\boldsymbol{e}(s),$$

$$\frac{\partial \mathbb{LH}_C}{\partial \phi} = t(-\sin\phi\boldsymbol{n}(s) + \cos\phi\boldsymbol{e}(s)),$$

$$\frac{\partial \mathbb{LH}_C}{\partial t} = \boldsymbol{\gamma}(s) + \cos\phi\boldsymbol{n}(s) + \sin\phi\boldsymbol{e}(s).$$

Under the condition that $\kappa_h\tau_h \neq 0$, $\{\partial\mathbb{LH}/\partial s, \partial\mathbb{LH}/\partial\phi, \partial\mathbb{LH}/\partial t\}$ is linearly dependent if and only if $t = 0$ or $1 + t(1 - \kappa_h(s)\cos\phi) = 0$. Therefore, (s,ϕ,t) is a singular point of $\mathbb{LH}_C$ if and only if $t = 0$ or $t = 1/(\kappa_h(s)\cos\phi - 1)$. If $t = 0$, then we have $\mathbb{LH}_C(s,\phi,0) = \gamma(s)$, which is the trivial singular locus of $\mathbb{LH}_C$. We call the non-trivial singular locus of $\mathbb{LH}_C$ a *lightlike focal set* of C. Thus the lightlike focal set of C is defined by

$$\mathbb{LF}_C = \left\{ \boldsymbol{\gamma}(s) + \tfrac{1}{\kappa_h(s)\cos\phi - 1}\left(\boldsymbol{\gamma}(s) + \cos\phi\boldsymbol{n}(s) + \sin\phi\boldsymbol{e}(s)\right) \;\middle|\; (s,\phi) \in I \times [0,2\pi) \right\}.$$

We remark that

$$\begin{aligned}&\boldsymbol{\gamma}(s) + \frac{1}{\kappa_h(s)\cos\phi - 1}\left(\boldsymbol{\gamma}(s) + \cos\phi\boldsymbol{n}(s) + \sin\phi\boldsymbol{e}(s)\right)\\ = &\frac{1}{\kappa_h(s)\cos\phi - 1}\left(\cos\phi(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sin\phi\boldsymbol{e}(s)\right).\end{aligned}$$

We define a mapping $\mathbb{LF}_{\kappa_h} : I \times [0, 2\pi) \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$ by

$$\mathbb{LF}_{\kappa_h}(s,\phi) = \frac{1}{\kappa_h(s)\cos\phi - 1}\left(\cos\phi(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sin\phi\boldsymbol{e}(s)\right).$$

Then we have $\mathbb{LF}_C = \mathbb{LF}_{\kappa_h}(I \times [0,2\pi) \times \mathbb{R})$. Since $\langle \mathbb{LF}_{\kappa_h}(s,\phi), \mathbb{LF}_{\kappa_h}(s,\phi)\rangle = 1-\cos^2\phi\kappa_h^2(s)$, we have For $\mathbb{LF}_{\kappa_h}(s,\phi)$, we can show that

$$\mathbb{LF}_{\kappa_h}(s,\phi) \text{ is } \begin{cases} \text{timelike if} & \kappa_h^2(s)\cos^2\phi > 1, \\ \text{lightlike if} & \kappa_h^2(s)\cos^2\phi = 1, \\ \text{spacelike if} & \kappa_h^2(s)\cos^2\phi < 1. \end{cases}$$

We define a mapping

$$\Phi : \mathbb{R}^4_1 \setminus LC_0 \longrightarrow H^3(-1) \cup S^3_1$$

by $\Phi(\boldsymbol{x}) = \boldsymbol{x}/\|\boldsymbol{x}\|$. We have $\mathbb{R}^4_1 \setminus LC_0 = S \cup T$, where $S = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x},\boldsymbol{x}\rangle > 0\}$ and $T = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x},\boldsymbol{x}\rangle < 0\}$. Therefore, we have projections $\Phi^S = \Phi|S : S \longrightarrow S^3_1$ and $\Phi^T = \Phi|T : T \longrightarrow H^3(-1)$. Suppose that $\kappa_h^2(s)\cos^2\phi < 1$. Then we have

$$\Phi^S \circ \mathbb{LF}_{\kappa_h}(s,\phi) = \frac{1}{\sqrt{1-\kappa_h^2(s)\cos^2\phi}}\left(\cos\phi(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sin\phi\boldsymbol{e}(s)\right).$$

Since $|(1-\kappa_h^2(s))/(1-\kappa_h^2(s)\cos^2\phi)| \leq 1$, there exists $\theta \in [0,\pi)$ such that

$$\cos\theta = \frac{\sqrt{1-\kappa_h^2(s)}\cos\phi}{\sqrt{1-\kappa_h^2(s)\cos^2\phi}},$$

so that we have

$$\sin\theta = \frac{\pm\sin\phi}{\sqrt{1-\kappa_h^2(s)\cos^2\phi}}.$$

If

$$\sin\theta = \frac{-\sin\phi}{\sqrt{1-\kappa_h^2(s)\cos^2\phi}},$$

then we choose $-\theta$ instead of θ. It follows that there exists $\theta \in [0,2\pi)$ such that

$$\Phi^S \circ \mathbb{LF}_{\kappa_h}(s,\phi) = \frac{\cos\theta}{\sqrt{1-\kappa_h^2(s)}}(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sin\theta\boldsymbol{e}(s) = DF_\gamma(s,\theta).$$

On the other hand, suppose that $\kappa_h^2(s)\cos^2\phi > 1$. Then we have

$$\Phi^T \circ \mathbb{LF}_{\kappa_h}(s,\phi) = \frac{1}{\sqrt{\kappa_h^2(s)\cos^2\phi - 1}}\left(\cos\phi(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sin\phi\boldsymbol{e}(s)\right).$$

Since $(\kappa_h^2(s)-1)/(\kappa_h^2(s)\cos^2\phi - 1) \geq 1$, there exists $\theta \in \mathbb{R}$ such that

$$\cosh\theta = \frac{\sqrt{\kappa_h^2(s)-1}\cos\phi}{\sqrt{\kappa_h^2(s)\cos^2\phi - 1}},$$

so that we have

$$\sinh\theta = \frac{\pm\sin\phi}{\sqrt{\kappa_h^2(s)\cos^2\phi - 1}}.$$

By the same reason as the above case, there exists $\theta \in \mathbb{R}$ such that

$$\Phi^T \circ \mathbb{LF}_{\kappa_h}(s,\phi) = \frac{\cosh\theta}{\sqrt{\kappa_h^2(s)-1}}(\kappa_h(s)\boldsymbol{\gamma}(s) + \boldsymbol{n}(s)) + \sinh\theta\boldsymbol{e}(s) = HF_\gamma(s,\theta).$$

We now define

$$\mathbb{LF}_C^S = \left\{ \mathbb{LF}_{\kappa_h}(s,\phi) \;\middle|\; (s,\phi) \in I \times [0,2\pi), \kappa_h^2(s)\cos^2\phi < 1 \right\},$$
$$\mathbb{LF}_C^T = \left\{ \mathbb{LF}_{\kappa_h}(s,\phi) \;\middle|\; (s,\phi) \in I \times [0,2\pi), \kappa_h^2(s)\cos^2\phi > 1 \right\}.$$

We call $\mathbb{LF}_C^S$ the *spacelike part of the lightlike focal set* of C and $\mathbb{LF}_C^T$ the *timelike part of the lightlike focal set* of C, respectively. We have the following theorem.

Theorem 5.1. *Let $\gamma : I \longrightarrow H_+^3(-1)$ be a unit speed hyperbolic space curve with $\kappa_h \tau_h \neq 0$. Then we have*

$$\Phi^S(\mathbb{LF}_C^S) = DF_\gamma(I_{<1} \times J) \text{ and } \Phi^T(\mathbb{LF}_C^T) = HF_\gamma(I_{>1} \times J),$$

where $I_{<1} = \{s \in I \mid \kappa_h^2(s) < 1\}$ and $I_{>1} = \{s \in I \mid \kappa_h^2(s) > 1\}$

Proof. We have shown that

$$\Phi^S(\mathbb{LF}_C^S) \subset DF_\gamma(I_{<1} \times J) \text{ and } \Phi^T(\mathbb{LF}_C^T) \subset HF_\gamma(I_{>1} \times J).$$

Suppose that $\kappa_h^2(s) < 1$. Then we have $1 - \kappa_h^2(s)\sin^2\theta > 1 - \sin^2\theta = \cos^2\theta$. It follows that there exists $\phi \in [0, 2\pi)$ such that

$$\cos^2\phi = \frac{\cos^2\theta}{1 - \kappa_h^2(s)\sin^2\theta}.$$

This means that $DF_\gamma(s,\theta) \in \Phi^S(\mathbb{LF}_C^S)$. By the similar arguments to the above case, we can show that $HF_\gamma(s,\theta) \in \Phi^T(\mathbb{LF}_C^T)$ for $s \in I_{>1}$. This completes the proof. $\square$

References

[1] J. W. Bruce and P. J. Giblin, *Curves and Singularities*, Second Edition, Cambridge University press, 1992.

[2] S. Izumiya, D.-H. Pei, and T. Sano, *Horospherical surfaces of curves in hyperbolic space*, Publ. Math. Debrecen **64** (2004), no. 1-2, 1–13.

[3] S. Izumiya and T. Sato, *Lightlike hypersurfaces along spacelike submanifolds in Minkowski space-time*, J. Geom. Phys. **71** (2013), 30–52.

Ryota Hayashi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail address: dooh67@gmail.com

Shyuichi Izumiya
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail address: izumiya@math.sci.hokudai.ac.jp

Takami Sato
Shiseikan Elementary School
Chuoku Minami 3 Nishi 7
Sapporo 060-0063, Japan
E-mail address: takami.s1218@gmail.com