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VARIOUS CENTROIDS OF POLYGONS AND SOME CHARACTERIZATIONS OF RHOMBI

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Abstract. For a polygon P, we consider the centroid G_0 of the vertices of P, the centroid G_1 of the edges of P and the centroid G_2 of the interior of P. When P is a triangle, (1) we always have $G_0 = G_2$ and (2) P satisfies $G_1 = G_2$ if and only if it is equilateral. For a quadrangle P, one of $G_0 = G_2$ and $G_0 = G_1$ implies that P is a parallelogram.

In this paper, we investigate the relationships between centroids of quadrangles. As a result, we establish some characterizations for rhombi and show that among convex quadrangles whose two diagonals are perpendicular to each other, rhombi and kites are the only ones satisfying $G_1 = G_2$. Furthermore, we completely classify such quadrangles.

1. Introduction

Let us denote by P a polygon in the plane $\mathbb{R}^2$ and we consider the centroid (or center of mass, or center of gravity, or barycenter) G_2 of the interior of P, the centroid G_1 of the edges of P and the centroid G_0 of the vertices of P. The centroid G_1 of the edges of P is also called the perimeter centroid of P ([3]).

When P is a triangle, then the centroid G_1 coincides with the center of the Spieker circle, which is the incircle of the triangle formed by connecting midpoint of each side of the original triangle P ([2, p. 249]). In this case, the centroid G_0 always coincides with the centroid $G_2(= G)$, where $G = (A + B + C)/3$. Furthermore, the perimeter centroid G_1 of P satisfies $G_1 = G_2$ if and only if the triangle P is equilateral ([14, Theorem 2]).

For a quadrangle, we have the following ([11]).

Proposition 1.1. *Let P denote a quadrangle. Then the following are equivalent.*

(1) *P satisfies $G_0 = G_1$.*
(2) *P satisfies $G_0 = G_2$.*

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(3) *P is a parallelogram.*

In order to study the relationships between the centroid G_1 and the centroid G_2 of a convex quadrangle, for the intersection point M of the two diagonals AC and BD we define as follows:

$$\triangle ABM = m_1, \quad \triangle BCM = m_2, \quad \triangle CDM = m_3, \quad \triangle DAM = m_4. \tag{1.1}$$

The perimeter l and the area m of the convex quadrangle $ABCD$ are respectively given by

$$l = l_1 + l_2 + l_3 + l_4 \tag{1.2}$$

and

$$m = m_1 + m_2 + m_3 + m_4, \tag{1.3}$$

where we put as follows:

$$AB = l_1, BC = l_2, CD = l_3, DA = l_4. \tag{1.4}$$

In [11], for convex quadrangles satisfying $G_1 = G_2$ a characterization theorem was established as follows.

Proposition 1.2. *Let us denote by P a convex quadrangle $ABCD$. Then the following are equivalent.*

(1) *P satisfies $G_1 = G_2$.*

(2) *P satisfies both*

$$l(m_3 + m_4) = m\{3(l_3 + l_4) - l\} \tag{1.5}$$

and

$$l(m_1 + m_4) = m\{3(l_1 + l_4) - l\}. \tag{1.6}$$

Furthermore, there exist some examples of quadrangles which are not parallelograms but satisfy $G_1 = G_2$ as follows ([11]).

Example 1.3. We consider the four points in the plane $\mathbb{R}^2$ defined by

$$A(x, 0), B(0, 1), C(-1, 0), D(0, -1). \tag{1.7}$$

We denote by $P(x)$ the quadrangle $ABCD$. Then there exist two real numbers $a_1 \in (0, 1)$ and $a_2 \in (-\infty, -2)$ such that $P(a_1)$ and $P(a_2)$ satisfy $G_1 = G_2$.

Hence, it is quite natural to ask the following (Question D of [11]):

Which quadrangles satisfy the condition $G_1 = G_2$?

The convex quadrangle $P(a_1)$ in Example 1.3 is a kite, which is a convex quadrangle whose four sides can be grouped into two pairs of equal-length sides that are adjacent to each other. Note that a convex quadrangle is a kite if and only if one diagonal is the perpendicular bisector of the other diagonal.

In this paper, we investigate the various centroids of the convex quadrangles whose two diagonals are perpendicular to each other and completely answer the above question for such quadrangles.

First of all, in Section 2 we establish the characterization theorem for rhombi as follows. Note that a rhombus is a simple (non-self-intersecting) quadrangle all of whose four sides have the same length.

Theorem 1.4. *Suppose that P denotes a convex quadrangle whose two diagonals are perpendicular to each other. We denote by O the intersection point of diagonals of P. Then the following are equivalent.*

(1) *P satisfies $G_0 = O$.*
(2) *P satisfies $G_1 = O$.*
(3) *P satisfies $G_2 = O$.*
(4) *P is a rhombus.*

Finally, in Section 3, using a series of lemmas we prove the following.

Theorem 1.5. *Suppose that P denotes a convex quadrangle whose two diagonals are perpendicular to each other. We denote by O the intersection point of diagonals of P. Then we have the following.*

(1) *P satisfies $G_1 = G_2(= O)$ if and only if P is a rhombus.*
(2) *If P satisfies $G_1 = G_2(\neq O)$, then P is a kite.*

In Section 4, conversely, we completely classify the kites satisfying $G_1 = G_2$.

In order to find the centroid G_2 of polygons, see [4]. In [13], mathematical definitions of centroid G_2 of planar bounded domains were given. For higher dimensions, it was shown that the centroid G_0 of the vertices of a simplex in an n-dimensional space always coincides with the centroid G_n of the simplex ([1, 14]).

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([15]). Some characterizations of parabolas using these properties were given in [6, 9, 10]. Furthermore, Archimedes also proved the volume properties of the region surrounded by a paraboloid of rotation and a plane ([15]). For characterizations of ellipsoids, elliptic paraboloid or elliptic hyperboloids with respect to these volume properties, we refer [5, 7, 8, 12].

2. Preliminaries and proof of Theorem 1.4

In this section, first of all we recall the centroids of a quadrangle. For centroids of a quadrangle $ABCD$, we have the following, where we use the notations given in Section 1.

Proposition 2.1. *Let us denote by P the convex quadrangle $ABCD$. Then we have the following.*

(1) *The centroid G_0 of P is given by*

$$G_0 = \frac{A+B+C+D}{4}. \tag{2.1}$$

(2) *The centroid G_1 of P is given by*

$$G_1 = \frac{(l_4+l_1)A + (l_1+l_2)B + (l_2+l_3)C + (l_3+l_4)D}{2l}. \tag{2.2}$$

(3) *If $m = \delta + \beta$, where $\delta = \triangle ABC$ and $\beta = \triangle ACD$, then the centroid G_2 of P is given by*

$$G_2 = \frac{mA + \delta B + mC + \beta D}{3m}. \tag{2.3}$$

Proof. It is straightforward to prove (1), (2) and (3) or see [4, 11]. □

Now, we prove Theorem 1.4 stated in Section 1.

Suppose that P denotes a convex quadrangle whose two diagonals are perpendicular to each other. We denote by O the intersection point of diagonals of P. Let us put by A the vertex which is closest to the point O. By a similarity transformation if necessary, we may introduce a coordinate system so that the point O is the origin and the vertices of P are given by

$$A(1,0),\ B(0,s),\ C(-t,0),\ D(0,-u), \tag{2.4}$$

where s, t and u are positive real numbers with $s \geq u \geq 1, t \geq 1$.

The centroids of P are given by

$$\begin{aligned} G_0 &= \frac{1}{4}(1-t, s-u),\ G_2 = \frac{1}{3}(1-t, s-u), \\ G_1 &= \frac{1}{2l}(l_1 + l_4 - tl_2 - tl_3, sl_1 + sl_2 - ul_3 - ul_4), \end{aligned} \tag{2.5}$$

where we put

$$l_1 = \sqrt{s^2+1},\ l_2 = \sqrt{s^2+t^2},\ l_3 = \sqrt{t^2+u^2},\ l_4 = \sqrt{u^2+1}. \tag{2.6}$$

It follows from (2.5) that each of (1) and (3) in Theorem 1.4 implies (4).

Now, suppose that P satisfies $G_1 = O$. Then from (2.5) we get

$$l_1 + l_4 = t(l_2 + l_3), s(l_1 + l_2) = u(l_3 + l_4). \tag{2.7}$$

Since the two diagonals are perpendicular, we have $l_1^2 + l_3^2 = l_2^2 + l_4^2$, and hence we obtain the following.

$$\begin{aligned} l_1^2 - l_2^2 &= l_4^2 - l_3^2, \\ l_1^2 - l_4^2 &= l_2^2 - l_3^2. \end{aligned} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$t(l_1 - l_4) = l_2 - l_3,\ \frac{s}{u}(l_4 - l_3) = l_1 - l_2. \tag{2.9}$$

Combining (2.7) and (2.9), we get

$$\begin{aligned} (t + \frac{1}{t})l_1 - (t - \frac{1}{t})l_4 &= 2l_2, \\ (\frac{s}{u} - \frac{u}{s})l_2 + (\frac{s}{u} + \frac{u}{s})l_1 &= 2l_4. \end{aligned} \tag{2.10}$$

By eliminating l_4 in (2.10), we find

(2.11)
$$\{(s-u)t+(s+u)\}[\{(u-s)t+(u+s)\}l_1-\{(u+s)t+(u-s)\}l_2]=0.$$

Since $(s-u)t+(s+u)>0$, (2.11) yields

$$\{(u-s)t+(u+s)\}l_1=\{(u+s)t+(u-s)\}l_2. \tag{2.12}$$

By replacing l_1 and l_2 in (2.12) with those in (2.6), we get

$$(t^2-1)\{(u+s)^2t^2+2(u^2-s^2)t+4us^3+(u+s)^2\}=0. \tag{2.13}$$

The discriminant D of the quadratic polynomial in the parenthesis in (2.13) is given by

$$D/4=-4(u+s)^2us(1+s^2)<0. \tag{2.14}$$

Hence, (2.13) shows that $t=1$. Thus, we have from (2.6)

$$l_1=l_2=\sqrt{s^2+1}, l_3=l_4=\sqrt{u^2+1}. \tag{2.15}$$

Therefore, the second equation of (2.7) implies

$$(s^2-u^2)(s^2+u^2+1)=0, \tag{2.16}$$

which leads $s=u$. Hence we see that the quadrangle P is a rhombus. This completes the proof of (2) ⇒ (4) in Theorem 1.4.

Conversely, it is obvious that a rhombus P satisfies $G_0=G_1=G_2=O$, where O denotes the intersection point of diagonals of P. This completes the proof of Theorem 1.4.

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5 stated in Section 1.

Suppose that P denotes a convex quadrangle whose two diagonals are perpendicular to each other. We denote by O the intersection point of diagonals of P. Let us put by A the vertex which is closest to the point O. As in Section 2, we may introduce a coordinate system so that the point O is the origin and the vertices of P are given by

$$A(1,0),\ B(0,s),\ C(-t,0),\ D(0,-u), \tag{3.1}$$

where s,t and u are positive real numbers with $s\ge u\ge 1, t\ge 1$. Note that P is a rhombus if $t=1$ and $s=u$, and P is a kite if $t=1$ or $s=u$.

The centroids of P are given by

$$\begin{aligned} G_2&=\frac{1}{3}(1-t,s-u),\\ G_1&=\frac{1}{2l}(l_1+l_4-tl_2-tl_3, sl_1+sl_2-ul_3-ul_4), \end{aligned} \tag{3.2}$$

where we put

$$l_1=\sqrt{s^2+1},\ l_2=\sqrt{s^2+t^2},\ l_3=\sqrt{t^2+u^2},\ l_4=\sqrt{u^2+1}. \tag{3.3}$$

We also have $l_1^2 + l_3^2 = l_2^2 + l_4^2$, and hence we obtain the following:

$$(l_1 - l_2)(l_1 + l_2) = (l_4 - l_3)(l_4 + l_3) \tag{3.4}$$

and

$$(l_1 - l_4)(l_1 + l_4) = (l_2 - l_3)(l_2 + l_3). \tag{3.5}$$

Now, we suppose that P satisfies $G_1 = G_2$ with $t \neq 1$ and $s \neq u$.

We prove, first of all, a series of lemmas as follows, and then we will show that the assumption $t \neq 1$ and $s \neq u$ leads a contradiction.

First, we prove the following relationship between l_1 and l_2.

Lemma 3.1. *Suppose that P satisfies $G_1 = G_2$. Then we have the following.*

(1) *The relationship between l_1 and l_2 is given by*

$$\phi l_2 = \psi l_1, \tag{3.6}$$

where we put

$$\begin{aligned} \phi &= 4(2+t)(1+2t)(2s+u)(s+2u) + 9(t^2-1)(s^2-u^2), \\ \psi &= 2(2s+u)(s+2u)(5t^2+8t+5) + 3(t^2-1)(5s^2+8su+5u^2). \end{aligned} \tag{3.7}$$

(2) *If $t \neq 1$, then we get*

$$t = \frac{-(7s+5u) + \sqrt{12(s^2+1)(2s+u)(s+2u)}}{5s+u}. \tag{3.8}$$

Proof. It follows from (3.2) that

$$(1+2t)(l_1 + l_4) = (2+t)(l_2 + l_3) \tag{3.9}$$

and

$$(s+2u)(l_1 + l_2) = (2s+u)(l_3 + l_4). \tag{3.10}$$

Together with (3.4), (3.10) shows that

$$(2s+u)(l_1 - l_2) = (s+2u)(l_4 - l_3). \tag{3.11}$$

Combining (3.5) and (3.9) also gives

$$(2+t)(l_1 - l_4) = (1+2t)(l_2 - l_3). \tag{3.12}$$

It follows from (3.9) and (3.12) that

$$(a + \frac{1}{a})l_1 + (a - \frac{1}{a})l_4 = 2l_2 \tag{3.13}$$

and from (3.10) and (3.11) we also obtain

$$(b + \frac{1}{b})l_1 + (b - \frac{1}{b})l_2 = 2l_4, \tag{3.14}$$

where we use

$$a = \frac{1+2t}{2+t},\ b = \frac{s+2u}{2s+u}. \tag{3.15}$$

In the equations (3.13) and (3.14), we may eliminate l_4 and then replace l_1 and l_2 with those in (3.3). Then we get (3.6). This shows that (1) of Lemma 3.1 holds.

In order to prove (2), first note that ϕ and ψ can be written, respectively, as

$$\begin{aligned}\phi &= \{(5s+7u)t+s+5u\}\{(5s+u)t+7s+5u\},\\ \psi &= \{(5s+7u)t+s+5u\}\{(7s+5u)t+5s+u\}.\end{aligned} \tag{3.16}$$

Since s, t and u are positive, it follows from (3.3), (3.6) and (3.16) that

$$\begin{aligned}&(t^2-1)\{(5s+u)^2t^2+2(7s+5u)(5s+u)t\\ &\qquad +(s^2+1)(5s+u)^2-s^2(7s+5u)^2\}=0.\end{aligned} \tag{3.17}$$

This completes the proof of Lemma 3.1. □

Second, just the similar argument as in the proof of Lemma 3.1 gives the relationship between l_3 and l_4, from which we get the following. Note that it can be also obtained by interchanging s and u in (3.8).

Lemma 3.2. *Suppose that P satisfies $G_1 = G_2$. If $t \neq 1$, then we have*

$$t = \frac{-(7u+5s)+\sqrt{12(u^2+1)(2u+s)(u+2s)}}{5u+s}. \tag{3.18}$$

Using Lemma 3.1 and Lemma 3.2, we may obtain the following.

Lemma 3.3. *Suppose that P satisfies $G_1 = G_2$. If $t \neq 1$ and $s \neq u$, then we have*

$$\begin{aligned}&3\sqrt{3}\{(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}\}\\ &= -(s^2+10su+u^2-24)\sqrt{(2u+s)(2s+u)}.\end{aligned} \tag{3.19}$$

Proof. Since $t \neq 1$, it follows from (3.8) and (3.18) that

$$\frac{3\sqrt{3}(s^2-u^2)}{\sqrt{(2u+s)(2s+u)}} = (5s+u)\sqrt{u^2+1}-(5u+s)\sqrt{s^2+1}. \tag{3.20}$$

The right hand side of (3.20) can be rewritten as

$$(5s+u)\sqrt{u^2+1}-(5u+s)\sqrt{s^2+1} = \frac{(u^2-s^2)(s^2+10su+u^2-24)}{(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}}. \tag{3.21}$$

Hence, combining (3.20) and (3.21) completes the proof of Lemma 3.3. □

Third, we prove the following relationship between l_1 and l_4.

Lemma 3.4. *Suppose that P satisfies $G_1 = G_2$. Then we have the following.*

(1) *The relationship between l_1 and l_4 is given by*

$$\{(c^2-1)(d^2+1)+2d(c^2+1)\}l_4 = \{4cd-(c^2-1)(d^2-1)\}l_1, \tag{3.22}$$

where we put

$$c = \frac{2s+u}{s+2u}, d = \frac{1+2t}{2+t}. \tag{3.23}$$

(2) *If $t \neq 1$ and $s \neq u$, then we get*

$$t = \frac{-(5s^2+32su+5u^2-30)+18\epsilon\sqrt{(s^2+1)(u^2+1)}}{s^2+10su+u^2-24}, \tag{3.24}$$

where $\epsilon = \pm 1$.

Proof. As in the proof of Lemma 3.1, it follows from (3.9)-(3.12) that

$$\begin{aligned} 2l_1 &= (c - \frac{1}{c})l_3 + (c + \frac{1}{c})l_4, \\ 2l_3 &= (d - \frac{1}{d})l_1 + (d + \frac{1}{d})l_4, \end{aligned} \tag{3.25}$$

where c and d are defined in (3.23). Let us eliminate l_3 in (3.25). Then we get (3.22).

In order to prove (2) of Lemma 3.4, first note that (3.22) can be written as

$$(cd+c+d-1)(c-d+cd+1)l_4 = (cd+c+d-1)(c+d-cd+1)l_1. \tag{3.26}$$

The assumption $s \geq u$ shows that $c - 1 \geq 0$. Hence we see that $cd + c + d - 1$ is positive. Together with (3.26), this implies

$$(c-d+cd+1)l_4 = (c+d-cd+1)l_1. \tag{3.27}$$

By replacing c, d, l_1 and l_4 in (3.27) with those in (3.3) and (3.23), we get

$$(s^2-u^2)(\alpha t^2 + 2\beta t + \gamma) = 0, \tag{3.28}$$

where we put

$$\begin{aligned} \alpha &= s^2+10su+u^2-24, \\ \beta &= 5s^2+32su+5u^2-30, \\ \gamma &= 5(5s^2+14su+5u^2)-24. \end{aligned} \tag{3.29}$$

Note that (3.19) yields $\alpha < 0$. Since $s \neq u$, from (3.28) we obtain (3.24). This completes the proof of Lemma 3.4. □

Now, it follows from (3.8) and (3.24) that

$$\begin{aligned} &-(s^2+10su+u^2-24)\sqrt{(2s+u)(s+2u)} \\ &= 3\sqrt{3}\{(5u+s)\sqrt{s^2+1} - \epsilon(5s+u)\sqrt{u^2+1}\}. \end{aligned} \tag{3.30}$$

From (3.18) and (3.24) we also obtain

$$\begin{aligned} &-(s^2+10su+u^2-24)\sqrt{(2s+u)(s+2u)} \\ &= 3\sqrt{3}\{(5s+u)\sqrt{u^2+1} - \epsilon(5u+s)\sqrt{s^2+1}\}. \end{aligned} \tag{3.31}$$

Together with (3.19), (3.30) shows that

$$(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}=(5u+s)\sqrt{s^2+1}-\epsilon(5s+u)\sqrt{u^2+1}. \tag{3.32}$$

Finally, according to $\epsilon=\pm1$ in Lemma 3.4 we divide by two cases as follows.

Case 1. $\epsilon=+1$**.** In this case, from (3.32) we get

$$(5s+u)\sqrt{u^2+1}=0, \tag{3.33}$$

which is a contradiction.

Case 2. $\epsilon=-1$**.** In this case, one of the three equations (3.19), (3.30) and (3.31) is nothing but the same as the other. Since $t>1$, we get from (3.8)

$$s^3+2us^2-5s-u>0. \tag{3.34}$$

From (3.19), we also obtain

$$s^2+10su+u^2-24<0. \tag{3.35}$$

Together with the assumption $s>u\geq 1$, it follows from (3.34) and (3.35) that

$$\sqrt{2}<s<-5+4\sqrt{3}(<2), \quad 1<u<\sqrt{2}. \tag{3.36}$$

We rewrite (3.19) as follows.

$$\frac{(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}}{\sqrt{(2u+s)(2s+u)}}=\frac{24-(s^2+10su+u^2)}{3\sqrt{3}}. \tag{3.37}$$

Since (3.36) holds, we obtain

$$s^2+10su+u^2>3+10\sqrt{2}. \tag{3.38}$$

Hence, the right hand side of (3.37) satisfies

$$\frac{24-(s^2+10su+u^2)}{3\sqrt{3}}<\frac{21-10\sqrt{2}}{3\sqrt{3}}. \tag{3.39}$$

On the other hand, we get from (3.36)

$$\begin{aligned}(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}&>10+\sqrt{2}+(5+\sqrt{2})\sqrt{3}\\&>12+6\sqrt{2},\end{aligned} \tag{3.40}$$

where the second inequality follows from $\sqrt{3}>\sqrt{2}$. We also obtain from (3.36)

$$\begin{aligned}\sqrt{(2u+s)(2s+u)}&<\sqrt{(2+2\sqrt{2})(4+\sqrt{2})}\\&<\sqrt{27}=3\sqrt{3}.\end{aligned} \tag{3.41}$$

Thus, the left hand side of (3.37) satisfies

$$\frac{(5s+u)\sqrt{u^2+1}+(5u+s)\sqrt{s^2+1}}{\sqrt{(2u+s)(2s+u)}}>\frac{12+6\sqrt{2}}{3\sqrt{3}}. \tag{3.42}$$

Note that $12 + 6\sqrt{2} > 21 - 10\sqrt{2}$. Hence, together with (3.39) and (3.42), (3.37) leads a contradiction. This contradiction shows that $t = 1$ or $s = u$, that is, the quadrangle P is a rhombus or a kite. Therefore, Theorem 1.4 completes the proof of Theorem 1.5.

4. Kites satisfying $G_1 = G_2$

In this section, we classify the kites satisfying $G_1 = G_2$ as follows.

Suppose that P denotes a kite. We denote by O the intersection point of diagonals of P. Let us put by A the vertex which is closest to the point O. As in Section 2, we may introduce a coordinate system so that the point O is the origin and P is similar to the kite Q whose vertices are given by

$$A(1,0),\ B(0,s),\ C(-t,0),\ D(0,-s), \tag{4.1}$$

where $s \geq 1$ and $t \geq 1$. Note that when $t = 1$, Q (and hence) P is a rhombus.

Suppose that $s \geq 1$ and $t > 1$. Then, it follows from (3.8) with $u = s$ that

$$t = \sqrt{3}\sqrt{s^2+1} - 2. \tag{4.2}$$

Since $t > 1$, we have $s > \sqrt{2}$.

Now, we prove the following.

Proposition 4.1. *A kite P satisfies $G_1 = G_2$ if and only if P is either a rhombus or similar to the kite Q defined by* (4.1)*, where $s > \sqrt{2}$ and t is given by* (4.2)*.*

Proof. It suffices to prove the if part of Proposition 4.1. It is straightforward to show that

$$G_1 = G_2 = \frac{1}{3}(3 - \sqrt{3}\sqrt{s^2+1}, 0). \tag{4.3}$$

This completes the proof of Proposition 4.1. □

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