Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 119–133
https://doi.org/10.4134/CKMS.c160009
pISSN: 1225-1763 / eISSN: 2234-3024

HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study half lightlike submanifolds of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. First, we characterize the geometry of two types of half lightlike submanifolds of such an indefinite Kaehler manifold. Next, we investigate the geometry of half lightlike submanifolds of an indefinite complex space form with a semi-symmetric non-metric connection.

1. Introduction

Ageshe-Chafle introduced the notion of semi-symmetric non-metric connection on a semi-Riemannian manifold in their papers [1, 2] as follow:

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *semi-symmetric non-metric connection* if $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

$$(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) - \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y}), \tag{1.1}$$

$$\bar{T}(\bar{X}, \bar{Y}) = \theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}, \tag{1.2}$$

where θ is a 1-form associated with a smooth unit vector field ζ, called the *characteristic vector field* on $\bar{M}$, defined by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. From now and in the sequel, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

Denote by $\widetilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. It is known [12] that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is a semi-symmetric non-metric connection if and only if $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\bar{X}. \tag{1.3}$$

A submanifold M of a semi-Riemannian manifold, of codimension 2, is called a lightlike submanifold if $Rad(TM) \neq \{0\}$, where $Rad(TM)$ denotes the radical distribution of M defined by $Rad(TM) = TM \cap TM^{\perp}$ with TM and $TM^{\perp}$ the tangent and normal bundle of M, respectively. In this case, we say that M is

Received January 13, 2016.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. semi-symmetric non-metric connection, recurrent, Lie recurrent, half lightlike submanifold.

(1) *half lightlike submanifold* if $rank\{Rad(TM)\} = 1$,
(2) *coisotropic submanifold* if $rank\{Rad(TM)\} = 2$.

Half lightlike submanifold [5] is a special case of r-lightlike submanifolds [4]. Its geometry is more general than that of lightlike hypersurfaces or coisotropic submanifolds. Recently several authors studied lightlike submanifolds of a manifold with a semi-symmetric non-metric connection ([7]∼[13], [15]).

In this paper, we study half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection $\bar{\nabla}$ given by (1.3). First, we characterize the geometry of two types of half lightlike submanifolds, named by *recurrent* and *Lie recurrent*, of such an indefinite Kaehler manifold. Next, we investigate the geometry of half lightlike submanifolds of an indefinite complex space form with a semi-symmetric non-metric connection.

2. Half lightlike submanifolds

Let $\bar{M} = (\bar{M}, \bar{g}, J)$ be an indefinite Kaeler manifold, where $\bar{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure satisfying

$$J^2 = -I, \quad \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}), \quad (\widetilde{\nabla}_{\bar{X}} J)\bar{Y} = 0, \tag{2.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to the metric $\bar{g}$.

Let (M, g) be a half lightlike submanifold of $\bar{M}$. As $rank(Rad(TM)) = 1$, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^\perp)$ of $Rad(TM)$ in TM and $TM^\perp$, respectively, which are called *screen distribution* and *co-screen distribution* of M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \;\; TM^\perp = Rad(TM) \oplus_{orth} S(TM^\perp),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. In the following, we choose $L \in \Gamma(S(TM^\perp))$ as a unit spacelike vector field without loss of generality. Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in $T\bar{M}$. Certainly, $Rad(TM)$ and $S(TM^\perp)$ are vector subbundles of $S(TM)^\perp$. As the co-screen distribution $S(TM^\perp)$ is non-degenerate, we have

$$S(TM)^\perp = S(TM^\perp) \oplus_{orth} S(TM^\perp)^\perp,$$

where $S(TM^\perp)^\perp$ is the orthogonal complementary to $S(TM^\perp)$ in $S(TM)^\perp$. For any null section ξ of $Rad(TM)$, there exist a uniquely defined lightlike vector bundle $ltr(TM)$ and a null vector field N of $ltr(TM)$ satisfying

$$\bar{g}(\xi, N) = 1, \;\; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \;\; \forall\, X \in \Gamma(S(TM)).$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^\perp) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector* *bundle* of M with respect to $S(TM)$ respectively [5]. $T\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM)$$
$$= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}).$$

In the sequel, let X, Y, Z and W be the smooth vector fields on M unless otherwise specified. Let $\bar{\nabla}$ be a semi-symmetric non-metric connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.2}$$

$$\bar{\nabla}_X N = -A_{_N} X + \tau(X)N + \rho(X)L, \tag{2.3}$$

$$\bar{\nabla}_X L = -A_{_L} X + \lambda(X)N; \tag{2.4}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.5}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.6}$$

where ∇ and ∇^* are linear connections on M and $S(TM)$ respectively, B and D are the *local second fundamental forms* of M, C is the *local second fundamental form* on $S(TM)$. $A_{_N}$ and $A_{_L}$ are the *shape operators* on M, A^*_ξ is the *shape operator* of $S(TM)$ and τ, ρ, λ and σ are 1-forms on M.

For a half lightlike submanifold M of an indefinite almost Hermitian manifold $\bar{M}$, it is known [6] that $J(Rad(TM)), J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$ with mutually trivial intersections, of rank 1. Therefore,

$$J(Rad(TM)) \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp}))$$

is a vector subbundle of $S(TM)$, of rank 3. Then there exist two non-degenerate almost complex distribution H_o and H on M with respect to J such that

$$S(TM) = J(Rad(TM)) \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp}) \oplus_{orth} H_o,$$
$$H = \{Rad(TM) \oplus_{orth} J(Rad(TM))\} \oplus_{orth} H_o.$$

In this case, the decomposition form of TM is reduced to

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{2.7}$$

Consider two null and one spacelike vector fields $\{U, V\}$ and W such that

$$U = -JN, \quad V = -J\xi, \qquad W = -JL. \tag{2.8}$$

Denote by S the projection morphism of TM on H with respect to the decomposition (2.7). Any vector field X on M is expressed as follows

$$X = SX + u(X)U + w(X)W,$$

where u, v and w are 1-forms locally defined on M by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = g(X, W). \tag{2.9}$$

Using (2.8), the action JX of X by J is expressed as follow:

$$JX = FX + u(X)N + w(X)L, \tag{2.10}$$

where F is a tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Applying J to (2.10) and using (2.1) and (2.8), we have

$$F^2 X = -X + u(X)U + w(X)W. \tag{2.11}$$

As $u(U) = w(W) = 1$, $FU = FW = 0$ and $u \circ F = w \circ F = 0$, the set $(F,\, u,\, w,\, U,\, W)$ defines a so-called (f, g, u, v, λ)-structure such that $\lambda = 0$ on M. In this case, F is called the *structure tensor field* of M.

Denote by $\bar{R}$, R and R^* the curvature tensors of the semi-symmetric non-metric connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$ respectively. Using the Gauss-Weingarten formulas, we obtain two Gauss equations for M and $S(TM)$ respectively:

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_{N} Y - B(Y,Z)A_{N} X \\ &+ D(X,Z)A_{L} Y - D(Y,Z)A_{L} X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &+ \lambda(X)D(Y,Z) - \lambda(Y)D(X,Z) + B(T(X,Y),Z)\}N, \\ &+ \{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) \\ &+ \rho(X)B(Y,Z) - \rho(Y)B(X,Z) + D(T(X,Y),Z)\}L, \end{aligned} \tag{2.12}$$

$$\begin{aligned} R(X,Y)PZ = R^*(X,Y)PZ &+ C(X,PZ)A^*_{\xi} Y - C(Y,PZ)A^*_{\xi} X \\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \sigma(X)C(Y,PZ) \\ &+ \sigma(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi, \end{aligned} \tag{2.13}$$

where T is the torsion tensor with respect to the induced connection ∇ on M.

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}. \tag{2.14}$$

In general, $R^{(0,2)}$ is not symmetric. $R^{(0,2)}$ is called the *induced Ricci tensor* of M and denote it by Ric if it is symmetric. It is known that, for any half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with a semi-symmetric non-metric connection, $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, *i.e.*, $d\tau = 0$ on M [10, 13].

3. Semi-symmetric non-metric connections

Let $(\bar{M}, \bar{g})$ be an indefinite Kaehler manifold with a semi-symmetric non-metric connection $\bar{\nabla}$ given by (1.3). Replacing the Levi-Civita connection $\tilde{\nabla}$ by the semi-symmetric non-metric connection $\bar{\nabla}$, the equation $(2.1)_3$ reduces

$$(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \theta(J\bar{Y})\bar{X} - \theta(\bar{Y})J\bar{X}. \tag{3.1}$$

Also, using (1.1), (1.2) and (2.2), we see that

$$\begin{aligned} (\nabla_X g)(Y,Z) = &\, B(X,Y)\eta(Z) + B(X,Z)\eta(Y) \\ &- \theta(Y)g(X,Z) - \theta(Z)g(X,Y). \end{aligned} \tag{3.2}$$

$$T(X,Y) = \theta(Y)X - \theta(X)Y, \tag{3.3}$$

and B and D are symmetric on TM, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N).$$

From the facts that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \bar{g}(\bar{\nabla}_X Y, L)$, we show that B and D are independent of the choice of $S(TM)$ and satisfy

$$B(X,\xi) = 0, \qquad D(X,\xi) = -\lambda(X). \tag{3.4}$$

From (2.2), (2.6) and (3.4), we obtain

$$\bar{\nabla}_X \xi \ = \ -A^*_\xi X - \sigma(X)\xi - \lambda(X)L. \tag{3.5}$$

Definition. A half lightlike submanifold M of a semi-Riemannian manifold is called *irrotational* [14] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$, *i.e.*, $\lambda = 0$.

Now we set $a = \theta(N)$, $b = \theta(\xi)$ and $e = \theta(L)$. Then the three local second fundamental forms B, D and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y) + bg(X,Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{3.6}$$

$$\begin{aligned} &D(X,Y) = g(A_L X, Y) + eg(X,Y) - \lambda(X)\eta(Y), \\ &\bar{g}(A_L X, N) = \rho(X) - e\eta(X), \end{aligned} \tag{3.7}$$

$$\begin{aligned} &C(X,PY) = g(A_N X, PY) + ag(X,PY) + \eta(X)\theta(PY), \\ &\bar{g}(A_N X, N) = -a\eta(X), \qquad\qquad \sigma(X) = \tau(X) - b\eta(X). \end{aligned} \tag{3.8}$$

Replacing X by ξ to $(3.6)_1$ and using $(3.4)_1$, we have

$$A^*_\xi \xi = 0. \tag{3.9}$$

Applying $\bar{\nabla}_X$ to (2.8) and (2.10) and using (2.2)$\sim$(2.4), (2.8), (2.10), (3.1) and (3.5), we have

$$\begin{aligned} &B(X,U) = u(A_N X) + au(X) = C(X,V) - \theta(V)\eta(X), \\ &D(X,U) = w(A_N X) + aw(X) = C(X,W) - \theta(W)\eta(X), \\ &D(X,V) = B(X,W), \end{aligned} \tag{3.10}$$

$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W + aFX + \theta(U)X, \tag{3.11}$$

$$\nabla_X V = F(A^*_\xi X) - \sigma(X)V - \lambda(X)W + bFX + \theta(V)X, \tag{3.12}$$

$$\nabla_X W = F(A_L X) + \lambda(X)U + eFX + \theta(W)X, \tag{3.13}$$

$$\begin{aligned} (\nabla_X F)Y = {} & u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ & + \theta(JY)X - \theta(Y)FX. \end{aligned} \tag{3.14}$$

Theorem 3.1. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. If V is parallel with respect to ∇, then M is irrotational, $\tau = 0$ and $R^{(0,\,2)}$ is symmetric.*

Proof. If V is parallel with respect ∇, then, from (3.12), we have

$$F(A^*_\xi X) - \sigma(X)V - \lambda(X)W + bFX + \theta(V)X = 0.$$

Taking the scalar product with V, U, W and N to this by turns, we have

$$\theta(V) = 0, \qquad \tau = 0, \qquad \lambda = 0, \qquad B(X, U) = 0. \tag{3.15}$$

As $\lambda = 0$, M is irrotational. As $\tau = 0$, $d\tau = 0$ and $R^{(0,2)}$ is symmetric. □

Theorem 3.2. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. If W is parallel with respect to ∇, then M is irrotational, i.e., $\lambda = 0$, and $\rho = 0$.*

Proof. If W is parallel with respect to ∇, then, from (3.13), we have

$$F(A_L X) + \lambda(X)U + eFX + \theta(W)X = 0. \tag{3.16}$$

Taking the scalar product with W, U, V and N to this by turns, we have

$$\theta(W) = 0, \qquad \rho = 0, \qquad \lambda = 0, \qquad D(X, U) = 0. \tag{3.17}$$

As $\lambda = 0$, M is irrotational. And $\rho = 0$. □

Theorem 3.3. *Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. If U is parallel with respect to ∇, then $\rho = 0$, $\tau = 0$ and $R^{(0,2)}$ is symmetric.*

Proof. If U is parallel with respect to ∇, then, from (3.11), we have

$$F(A_N X) + \tau(X)U + \rho(X)W + aFX + \theta(U)X = 0. \tag{3.18}$$

Taking the scalar product with U, V, W and N to this by turns, we have

$$\theta(U) = 0, \qquad \tau = 0, \qquad \rho = 0, \qquad C(X, U) = 0. \tag{3.19}$$

As $\tau = 0$, $d\tau = 0$ and $R^{(0,2)}$ is a symmetric induced Ricci tensor of M. □

4. Recurrent and Lie recurrent half lightlike submanifolds

Definition. The structure tensor field F of M is said to be *recurrent* [11] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

A half lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called *recurrent* if it admits a recurrent structure tensor field F.

Theorem 4.1. *Let M be a recurrent half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with a semi-symmetric non-metric connection. Then:*

(1) *F is parallel with respect to the induced connection ∇ on M,*
(2) *M is irrotational, i.e., $\lambda = 0$, and $\rho = 0$,*
(3) *H, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M,*
(4) *M is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^{\sharp}$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^{\perp}))$, and $M^{\sharp}$ is a leaf of the distributions H.*

Proof. (1) From the above definition and (3.14), we get

$$\varpi(X)FY = u(Y)A_{N}X + w(Y)A_{L}X - B(X,Y)U - D(X,Y)W + \theta(JY)X - \theta(Y)FX. \tag{4.1}$$

Replacing Y by ξ and using (2.8), (3.4) and the fact that $F\xi = -V$, we get

$$\varpi(X)V = -\lambda(X)W + \theta(V)X + bFX. \tag{4.2}$$

Taking the scalar product with N to this equation, we obtain

$$\theta(V)\eta(X) + bv(X) = 0.$$

Taking $X = V$ and then $X = \xi$ to this equation, we have

$$b = 0, \qquad \theta(V) = 0. \tag{4.3}$$

Taking the scalar product with U to (4.2), we get $\varpi = 0$. It follows that $\nabla_X F = 0$. Therefore, F is parallel with respect to the connection ∇.

(2) Taking the scalar product with W to (4.2) such that $b = \theta(V) = \varpi = 0$, we obtain $\lambda = 0$. Therefore, M is irrotational.

Taking $Y = U$ and $Y = W$ to (4.1) such that $\varpi = 0$ by turns, we have

$$A_{N}X = B(X,U)U + D(X,U)W - aX + \theta(U)FX, \tag{4.4}$$

$$A_{L}X = B(X,W)U + D(X,W)W - eX + \theta(W)FX. \tag{4.5}$$

Taking the scalar product with N and U to (4.4) by turns, we get

$$\theta(U) = 0, \qquad C(X,U) = 0, \tag{4.6}$$

due to (3.8). Taking the scalar product with N and U to (4.5) by turns and using (3.7) and the fact that $\lambda = 0$, we obtain

$$\rho(X) = \theta(W)v(X), \qquad D(X,U) = -\theta(W)\eta(X). \tag{4.7}$$

Replacing X by ξ to $(4.7)_2$ and using $(3.4)_2$ and the fact that $\lambda = 0$, we obtain

$$\theta(W) = 0.$$

From this result and $(4.7)_1$, we see that $\rho = 0$.

(3) In general, by using (3.2), (3.4), (3.6), (3.7), (3.12) and (3.13), we derive

$$\begin{aligned} &g(\nabla_X\xi, V) = -B(X,V) + bu(X),\\ &g(\nabla_X\xi, W) = -B(X,W) + bw(X),\\ &g(\nabla_X V, V) = \theta(V)u(X),\\ &g(\nabla_X V, W) = -\lambda(X) + \theta(V)w(X),\\ &g(\nabla_X Z, V) = B(X,FZ) + \theta(Z)u(X),\\ &g(\nabla_X Z, W) = D(X,FZ) + \theta(Z)w(X), \end{aligned} \tag{4.8}$$

for any $X \in \Gamma(TM)$ and $Z \in \Gamma(H_o)$.

Taking the scalar product with V and W to (4.1) by turns, we have

$$B(X,Y) = u(Y)u(A_{N}X) + w(Y)u(A_{L}X) + \theta(JY)u(X),$$
$$D(X,Y) = u(Y)w(A_{N}X) + w(Y)w(A_{L}X) + \theta(JY)w(X).$$

Taking $Y = V$ and $Y = FZ$, $Z \in \Gamma(D_o)$ to these equations by turns and using the facts that $b = 0$, $u(FZ) = w(FZ) = 0$ and $FZ = JZ$, we have

$$B(X, V) = bu(X) = 0, \qquad B(X, FZ) = -\theta(Z)u(X), \tag{4.9}$$

$$D(X, V) = B(X, W) = bw(X) = 0, \quad D(X, FZ) = -\theta(Z)w(X). \tag{4.10}$$

Using (4.3), (4.9), (4.10) and the fact that $\lambda = 0$, (4.8) is equivalent to

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

It follows that H is a parallel distribution on M.

Applying F to (4.4) and (4.5) and using the fact: $\theta(U) = \theta(W) = 0$, we get

$$F(A_N X) = -aFX, \qquad F(A_L X) = -eFX.$$

Using these results and the facts that $\rho = \lambda = 0$ and $\theta(U) = \theta(W) = 0$, the equations (3.11) and (3.13) reduce to

$$\nabla_X U = \tau(X)U, \qquad \nabla_X U \in \Gamma(J(ltr(TM))), \tag{4.11}$$

$$\nabla_X W = 0, \qquad \nabla_X W \in \Gamma(J(S(TM^\perp))). \tag{4.12}$$

Thus $J(ltr(TM))$ and $J(S(TM^\perp))$ are also parallel distributions on M.

(4) As H, $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions and satisfied (2.7), by the decomposition theorem of de Rham [3], M is locally a product manifold $\mathcal{C}_U \times \mathcal{C}_W \times M^\sharp$, where $\mathcal{C}_U$ is a null curve tangent to $J(ltr(TM))$, $\mathcal{C}_W$ is a spacelike curve tangent to $J(S(TM^\perp))$, and $M^\sharp$ is a leaf of H. □

Definition. The structure tensor field F of M is said to be *Lie recurrent* [11] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$. A half lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ is called *Lie recurrent* if it admits a Lie recurrent structure tensor field F.

Theorem 4.2. *Let M be a Lie recurrent half lightlike submanifold of an indefinite Kaehler manifold with a semi-symmetric non-metric connection. Then:*

(1) *F is Lie parallel,*
(2) *the 1-form τ is satisfied $\tau = 0$,*
(3) *$R^{(0,2)}$ is a symmetric induced Ricci tensor of M.*

Proof. (1) Using the above definition, (2.10), (3.3) and (3.14), we get

$$\begin{aligned}\vartheta(X)FY &= u(Y)A_N X + w(Y)A_L X - B(X,Y)U - D(X,Y)W \\ &\quad + \{au(Y) + ew(Y)\}X - \nabla_{FY}X + F\nabla_Y X.\end{aligned} \tag{4.13}$$

Taking $Y = \xi$ to (4.13) and using (3.4) and the fact that $F\xi = -V$, we have

$$-\vartheta(X)V = \lambda(X)W + \nabla_V X + F\nabla_\xi X. \tag{4.14}$$

Taking the scalar product with V and W to (4.14) by turns, we have

$$u(\nabla_V X) = 0, \qquad w(\nabla_V X) = -\lambda(X). \tag{4.15}$$

Replacing Y by V to (4.13) and using the fact that $FV = \xi$, we have

$$\vartheta(X)\xi = -B(X,V)U - D(X,V)W - \nabla_\xi X + F\nabla_V X. \tag{4.16}$$

Applying F to this equation and using (2.11) and (4.15), we obtain

$$\vartheta(X)V = \lambda(X)W + \nabla_V X + F\nabla_\xi X.$$

Comparing this equation with (4.14), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N to (4.13), we obtain

$$w(Y)\rho(X) - \bar{g}(\nabla_{FY}X, N) + \bar{g}(F\nabla_Y X, N) = 0. \tag{4.17}$$

Replacing X by ξ to (4.17) and using (2.6), (2.8) and (2.10), we have

$$g(A^*_\xi X, U) = \sigma(FX) + w(X)\rho(\xi).$$

From this equation, (3.6), $(3.8)_3$ and the fact that $v(X) = \eta(FX)$, we have

$$B(X,U) - w(X)\rho(\xi) = \tau(FX).$$

Taking $X = U$ and $X = W$ to this by turns and using $(3.10)_{1,3}$, we get

$$C(U,V) = B(U,U) = 0, \quad D(U,V) = B(U,W) = \rho(\xi). \tag{4.18}$$

Taking the scalar product with W to (4.16), we have

$$D(X,V) = -g(\nabla_\xi X, W).$$

Replacing X by U to this equation and using (3.11), we get

$$D(U,V) = -\rho(\xi).$$

From this result and $(4.18)_2$, we obtain

$$\rho(\xi) = 0, \qquad D(U,V) = 0, \tag{4.19}$$

$$B(X,U) = \tau(FX). \tag{4.20}$$

Replacing X by W to (4.13) and using $(3.10)_3$ and (3.13), we have

$$\begin{aligned} &u(Y)A_{N}W + w(Y)A_{L}W - A_{L}Y \\ &- F(A_{L}FY) - \lambda(FY)U + u(Y)\{aW - eU\} = 0. \end{aligned}$$

Taking the scalar product with N to this and using (3.7) and $(3.8)_2$, we have

$$D(FY,U) = w(Y)\rho(W) - \rho(Y).$$

Replacing Y by V to this equation and using $(3.4)_2$, we obtain

$$-\lambda(U) = D(\xi,U) = -\rho(V).$$

On the other hand, replacing X by U to $(4.15)_2$ and using (3.11), we get

$$\rho(V) = -\lambda(U).$$

Comparing the last two equations, we obtain $\rho(V) = 0$ and $\lambda(U) = 0$.

Taking the scalar product with V to (4.16), we obtain

$$B(X,V)+g(\nabla_\xi X,V)=0.$$

Replacing X by W to this equation and using (3.13), we have

$$B(V,W)=-\lambda(\xi).$$

Replacing X by ξ to $(4.15)_2$ and using (2.6) and (3.6), we get

$$B(V,W)=\lambda(\xi).$$

Comparing the last two equations, we obtain $\lambda(\xi)=0$. Therefore, we get

$$\rho(V)=0, \qquad \lambda(U)=0, \qquad \lambda(\xi)=0. \tag{4.21}$$

Replacing X by V to (4.17) and using (2.11), (3.12) and $(4.21)_1$, we have

$$g(A^*_\xi FY,U)+\sigma(Y)=0.$$

Using this equation, (3.6) and $(3.8)_3$, we obtain

$$B(FY,U)=-\tau(Y).$$

Taking $Y=U$ and $Y=W$ by turns and using $FU=FW=0$, we obtain

$$\tau(U)=0, \qquad \tau(W)=0. \tag{4.22}$$

Replacing X by U to (4.13) and using (2.11), (3.6), and $(3.10)_{1,\,2}$, we get

$$\begin{aligned}&u(Y)A_{\scriptscriptstyle N}U+w(Y)A_{\scriptscriptstyle L}U-F(A_{\scriptscriptstyle N}FY)-A_{\scriptscriptstyle N}Y\\&-\tau(FY)U-\rho(FY)W+w(Y)\{eU-aW\}=0.\end{aligned}$$

Taking the scalar product with V to this equation and using (3.7), (3.8), (4.18) and (4.19), we get

$$B(X,U)=-\,\tau(FX).$$

Comparing this with (4.20), we obtain $\tau(FX)=0$. Replacing X by FY to this and using (3.6) and (4.22), we have $\tau=0$.

(3) As $\tau=0$, we see that $d\tau=0$ and $R^{(0,\,2)}$ is symmetric. $\square$

5. Indefinite complex space forms

An indefinite complex space form, denoted by $\bar M(c)$, is a connected indefinite Kaehler manifold of a constant holomorphic sectional curvature c such that

$$\begin{aligned}\bar R(\bar X,\bar Y)\bar Z \;=\; &\frac{c}{4}\{\bar g(\bar Y,\bar Z)\bar X-\bar g(\bar X,\bar Z)\bar Y+\bar g(J\bar Y,\bar Z)J\bar X\\&-\bar g(J\bar X,\bar Z)J\bar Y+2\bar g(\bar X,J\bar Y)J\bar Z\}.\end{aligned} \tag{5.1}$$

Comparing the tangential, lightlike transversal and co-screen components of the two equations (2.12) and (5.1), and using (2.10) and (3.3), we get

$$\begin{aligned}R(X,Y)Z \;=\; &B(Y,Z)A_{\scriptscriptstyle N}X-B(X,Z)A_{\scriptscriptstyle N}Y\\&+D(Y,Z)A_{\scriptscriptstyle L}X-D(X,Z)A_{\scriptscriptstyle L}Y\\&+\frac{c}{4}\{g(Y,Z)X-g(X,Z)Y+\bar g(JY,Z)FX\end{aligned} \tag{5.2}$$

$$- \bar{g}(JX, Z)FY + 2\bar{g}(X, JY)FZ\},$$

$$\begin{aligned}(5.3)\qquad &(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)\\ &+ \{\tau(X) - \theta(X)\}B(Y, Z) - \{\tau(Y) - \theta(Y)\}B(X, Z)\\ &+ \lambda(X)D(Y, Z) - \lambda(Y)D(X, Z)\\ &= \frac{c}{4}\{u(X)g(FY, Z) - u(Y)g(FX, Z) + 2u(Z)\bar{g}(X, JY)\},\end{aligned}$$

$$\begin{aligned}(5.4)\qquad &(\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z)\\ &+ \rho(X)B(Y, Z) - \rho(Y)B(X, Z) - \theta(X)D(Y, Z) + \theta(Y)D(X, Z)\\ &= \frac{c}{4}\{w(X)g(FY, Z) - w(Y)g(FX, Z) + 2w(Z)\bar{g}(X, JY)\}.\end{aligned}$$

Taking the scalar product with N to (2.13) and then, substituting (5.2) into the resulting equation and using (3.3), $(3.7)_2$ and $(3.8)_2$, we obtain

$$\begin{aligned}(5.5)\qquad &(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)\\ &- \{\sigma(X) + \theta(X)\}C(Y, PZ) + \{\sigma(Y) + \theta(Y)\}C(X, PZ)\\ &+ a\{\eta(X)B(Y, PZ) - \eta(Y)B(X, PZ)\}\\ &- \{\rho(X) - e\eta(X)\}D(Y, PZ) - \{\rho(Y) - e\eta(Y)\}D(X, PZ)\}\\ &= \frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) + v(X)g(FY, PZ)\\ &- v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.\end{aligned}$$

Theorem 5.1. *Let M be a half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. Suppose that one of the following conditions holds.*

(1) *M is recurrent.*
(2) *M is Lie recurrent.*
(3) *V is parallel with respect to the connection ∇ on M.*
(4) *W is parallel with respect to the connection ∇ on M.*

Then $c = 0$, i.e., $\bar{M}(c)$ is flat. Moreover, in cases (1), (2) *and* (3), *the induced Ricci type tensor $R^{(0,2)}$, defined by* (2.14), *is an induced Ricci tensor of M.*

Proof. (1) As M is recurrent, by Theorem 4.1, we see that

$$(5.6)\qquad D(X, U) = 0.$$

Applying ∇_Y to $(4.6)_2$: $C(Y, U) = 0$ and using (4.11), we have

$$(\nabla_X C)(Y, U) = 0.$$

Replacing PZ by U to (5.5) and using the last equations, we get

$$(5.7)\qquad a\{\eta(X)B(Y, U) - \eta(Y)B(X, U)\} = \frac{c}{2}\{v(Y)\eta(X) - v(X)\eta(Y)\}.$$

Taking $X = \xi$ and $Y = V$ and using $(3.4)_1$ and $(4.9)_1$, we have $c = 0$.

As $c = 0$, taking $Y = \xi$ to (5.7) and using $(3.4)_1$, we obtain

$$aB(X,U) = 0. \tag{5.8}$$

By directed calculations from (4.11), we obtain

$$R(X,Y)U = 2d\tau(X,Y)U.$$

Comparing this equation with (5.2) such that $Z = U$ and using (5.6), we have

$$2d\tau(X,Y)U = B(Y,U)A_{_N}X - B(X,U)A_{_N}Y.$$

Substituting (4.4) into the right term of this and using $(4.6)_1$, (5.6) and (5.8), we get $d\tau = 0$. Thus $R^{(0,\,2)}$ is a symmetric induced Ricci tensor of M.

(2) As $\tau = 0$, from (4.20) we obtain

$$B(Y,U) = 0. \tag{5.9}$$

Applying ∇_X to this equation and using (3.11), we have

$$\begin{aligned}(\nabla_X B)(Y,U) = &- B(Y,F(A_{_N}X)) - \rho(X)B(Y,W)\\ &- aB(FX,Y) - \theta(U)B(X,Y).\end{aligned}$$

Substituting the last two equations into (5.3), we have

$$\begin{aligned}&B(X,F(A_{_N}Y) - B(Y,F(A_{_N}X) + a\{B(X,FY) - B(FX,Y)\}\\ &+ \rho(Y)B(X,W) - \rho(X)B(Y,W) + \lambda(X)D(Y,U) - \lambda(Y)D(X,U)\\ = &\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Taking $X = \xi$ and $Y = U$ to this equation and using $(3.4)_{1,\,2}$, $(4.21)_{2,\,3}$ and (5.9), we get $c = 0$. Therefore, $\bar{M}(c)$ is flat.

From Theorem 4.2, we show that $\tau = 0$ and $R^{(0,\,2)}$ is symmetric.

(3) As V is parallel with respect to ∇, we holds (3.15) by Theorem 3.1:

$$\theta(V) = 0, \qquad \tau = 0, \qquad \lambda = 0, \qquad B(X,U) = 0.$$

As $B(X,U) = 0$ and $\theta(V) = 0$, from $(3.10)_1$ we obtain

$$C(X,V) = 0.$$

Applying ∇_X to $C(Y,V) = 0$ and using the fact that $\nabla_X V = 0$, we have

$$(\nabla_X C)(Y,V) = 0.$$

Substituting the last two equations into (5.5) with $PZ = V$, we have

$$\begin{aligned}&a\{\eta(X)B(Y,V) - \eta(Y)B(X,V)\}\\ &- \{\rho(X) - e\eta(X)\}D(Y,V) + \{\rho(Y) - e\eta(Y)\}D(X,V)\}\\ = &\frac{c}{4}\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Taking $X = \xi$ and $Y = U$ to this and using (3.4), $(3.15)_4$ and the facts that $\lambda = 0$ and $D(U,V) = 0$ by $(3.10)_3$ and $(3.15)_4$, we get $c = 0$.

From Theorem 3.1, we show that $\tau = 0$ and $R^{(0,\,2)}$ is symmetric.

(4) As W is parallel with respect to ∇, we holds (3.17) by Theorem 3.2:

$$\theta(W)=0, \qquad \rho=0, \qquad \lambda=0, \qquad D(X,U)=0.$$

As $\lambda=\theta(W)=0$ and $FW=0$, from (3.16), we have

$$F(A_{_L}X)=-eFX, \qquad F(A_{_L}W)=0.$$

Applying ∇_X to $D(Y,U)=0$ and using (3.11) and $\rho=0$, we have

$$(\nabla_X D)(Y,U)=-D(Y,F(A_{_N}X))-aD(FX,Y)-\theta(U)B(X,Y).$$

Substituting this equation into (5.4) such that $Z=U$ and $\lambda=0$, we have

$$D(X,F(A_{_N}Y)-D(Y,F(A_{_N}X)+a\{D(X,FY)-D(FX,Y)\}$$
$$=\frac{c}{4}\{w(Y)\eta(X)-w(X)\eta(Y)\}.$$

Taking $X=\xi$ and $Y=W$ to this equation and using $(3.4)_2$, we get

$$D(W,F(A_{_N}\xi))-aD(V,W)=-\frac{c}{4}.$$

By directed calculation from (2.10), $(3.7)_{1,2}$, $(3.8)_2$ and (3.17), we see that

$$D(W,F(A_{_N}\xi))=-g(A_{_N}\xi,F(A_{_L}W))+aD(V,W)=aD(V,W).$$

From the last two equations, we see that $c=0$. □

Definition. (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [5] if there exists a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X,PY)=\gamma g(X,PY). \tag{5.10}$$

(2) A half lightlike submanifold M is said to be *screen conformal* [5] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that

$$C(X,PY)=\varphi B(X,Y). \tag{5.11}$$

Theorem 5.2. *Let M be an irrotational half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ with a semi-symmetric non-metric connection. If $S(TM)$ is totally umbilical in M or M is screen conformal, then $c=0$.*

Proof. (1) As $S(TM)$ is totally umbilical, from $(3.10)_{1,2}$ and (5.10), we have

$$B(X,U)=\gamma u(X)-\theta(V)\eta(X), \quad D(X,U)=\gamma w(X)-\theta(W)\eta(X).$$

Replacing X by V to these two equations, we obtain

$$B(V,U)=0, \qquad D(V,U)=0. \tag{5.12}$$

Applying ∇_X to $C(Y,PZ)=\gamma g(Y,PZ)$ and using (3.2), we obtain

$$(\nabla_X C)(Y,PZ)=(X\gamma)g(Y,PZ)$$
$$+\ \gamma\{B(X,PZ)\eta(Y)-\theta(Y)g(X,PZ)-\theta(PZ)g(X,Y)\}.$$

Substituting this equation and (5.10) into (5.5), we have

$$\{X\gamma-\gamma\sigma(X)-\frac{c}{4}\eta(X)\}g(Y,PZ)$$

$$
\begin{aligned}
&- \{Y\gamma - \gamma\sigma(Y) - \frac{c}{4}\eta(Y)\}g(X, PZ) \\
&+ (\gamma - a)\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\} \\
&- \{\rho(X) - e\eta(X)\}D(Y, PZ) + \{\rho(Y) - e\eta(Y)\}D(X, PZ)\} \\
= &\frac{c}{4}\{v(X)g(FY, PZ) - v(Y)g(FX, PZ) + 2v(PZ)\bar{g}(X, JY)\}.
\end{aligned}
$$

Replacing Y by ξ this equation and using (3.4) and $\lambda = 0$, we have

$$
\begin{aligned}
&(\gamma - a)B(X, PY) + \{\rho(\xi) - e\}D(X, PZ) \\
= &\{\xi\gamma - \gamma\sigma(\xi) - \frac{c}{4}\}g(X, PY) - \frac{c}{4}\{v(X)u(PY) + 2u(X)v(PY)\}.
\end{aligned}
$$

Taking $X = U$, $PY = V$ and alternately, taking $X = V$, $PY = U$ to this and using (5.12) and the facts that B and D are symmetric, we have

$$
\xi\gamma - \gamma\sigma(\xi) - \frac{3}{4}c = 0, \qquad \xi\gamma - \gamma\sigma(\xi) - \frac{2}{4}c = 0.
$$

From the two equations of the last relationship, we obtain $c = 0$.

(2) As M is screen conformal, using (3.10) and (5.11), we obtain

$$
B(X, \mu) = -\eta(X)\theta(V), \qquad D(X, \mu) = -\theta(W)\eta(X), \tag{5.13}
$$

where we set $\mu = U - \varphi V$. Replacing X by ξ to (5.13) and using (3.4), we have

$$
\theta(V) = 0, \qquad \theta(W) = \lambda(\mu) = 0. \tag{5.14}
$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$
(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).
$$

Substituting this equation into (5.5) and using (5.3), we have

$$
\begin{aligned}
&\{X\varphi - 2\varphi\tau(X) + [a + \varphi b]\eta(X)\}B(Y, PZ) \\
&- \{Y\varphi - 2\varphi\tau(Y) + [a + \varphi b]\eta(Y)\}B(X, PZ) \\
&- \{\rho(X) - e\eta(X) + \varphi\lambda(X)\}D(Y, PZ) \\
&+ \{\rho(Y) - e\eta(Y) + \varphi\lambda(Y)\}D(X, PZ) \\
= &\frac{c}{4}\{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) + [v(X) - \varphi u(X)]g(FY, PZ) \\
&- [v(Y) - \varphi u(Y)]g(FX, PZ) + 2[v(PZ) - \varphi u(PZ)]\bar{g}(X, JY)\}.
\end{aligned}
$$

Taking $Y = \xi$ and $PZ = \mu$ and using (5.13) and (5.14), we have

$$
\frac{c}{2}\{v(X) - 3\varphi u(X)\} = 0.
$$

Replacing X by V to this equation, we obtain $c = 0$. □

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