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INTEGRAL CURVES OF THE CHARACTERISTIC VECTOR FIELD ON CR-SUBMANIFOLDS OF MAXIMAL CR-DIMENSION

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Abstract. In this paper we study CR-submanifolds of maximal CR-dimension by investigating extrinsic behaviors of integral curves of characteristic vector field on them. Also we consider the notion of ruled CR-submanifold of maximal CR-dimension which is a generalization of that of ruled real hypersurface and find some characterizations of ruled CR-submanifold of maximal CR-dimension concerning extrinsic shapes of integral curves of the characteristic vector field and those of CR-Frenet curves.

1. Introduction

Let M be an $n(>1)$-dimensional CR-submanifold of maximal CR-dimension, that is, of $(n-1)$ CR-dimension isometrically immersed in a complex space form $M^{(n+p)/2}(c)$. Denoting by $(J, \overline{g})$ the Kähler structure of $M^{(n+p)/2}(c)$, we find by definition (cf. [4, 6, 11, 12, 14]) that the maximal J-invariant subspace

$$\mathcal{D}_x := T_xM \cap JT_xM$$

of the tangent space T_xM of M is of $(n-1)$ dimensional at any point $x \in M$. So there exists a unit vector field U_1 tangent to M such that

$$\mathcal{D}_x^{\perp} = Span\{U_1\}, \quad {}^{\forall}x \in M,$$

where $\mathcal{D}_x^{\perp}$ denotes the subspace of T_xM complementary orthogonal to $\mathcal{D}_x$. Moreover, the vector field ξ defined by

$$\xi := JU_1 \tag{1.1}$$

is normal to M and satisfies

$$JTM \subset TM \oplus Span\{\xi\}.$$

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Hence we have, for any tangent vector field X and for a local orthonormal basis $\{\xi_\alpha\ ;\ \alpha = 1, \dots, p\}$ $(\xi_1 := \xi)$ of normal vectors to M, the following decomposition in tangential and normal components:

$$JX = FX + u^1(X)\xi_1, \tag{1.2}$$

$$J\xi_\alpha = -U_\alpha + P\xi_\alpha, \quad \alpha = 1, \dots, p. \tag{1.3}$$

Since the structure $(J, \overline{g})$ is hermitian and $J^2 = -I$, we can easily see from (1.2) and (1.3) that F and P are skew-symmetric linear endomorphisms acting on T_xM and $T_x^\perp M$, respectively, and that

$$g(FU_\alpha, X) = -u^1(X)\overline{g}(\xi_1, P\xi_\alpha), \tag{1.4}$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - \overline{g}(P\xi_\alpha, P\xi_\beta), \tag{1.5}$$

where $T_x^\perp M$ denotes the normal space of M at x and g the metric on M induced from $\overline{g}$. Furthermore, we also have

$$g(U_\alpha, X) = u^1(X)\delta_{1\alpha} \tag{1.6}$$

and consequently

$$g(U_1, X) = u^1(X), \quad U_\alpha = 0, \quad \alpha = 2, \dots, p. \tag{1.7}$$

Next, applying J to (1.2) and using (1.3) and (1.7), we have

$$F^2X = -X + u^1(X)U_1, \quad u^1(X)P\xi_1 = -u^1(FX)\xi_1, \tag{1.8}$$

from which, taking account of the skew-symmetry of P and (1.4),

$$u^1(FX) = 0, \quad FU_1 = 0, \quad u^1(U_1) = 1, \quad P\xi_1 = 0. \tag{1.9}$$

Thus (1.3) may be written in the form

$$J\xi_1 = -U_1, \quad J\xi_\alpha = P\xi_\alpha, \quad \alpha = 2, \dots, p. \tag{1.10}$$

Those equations tell us that (F, g, U_1, u^1) defines an almost contact metric structure on M (cf. [4, 6, 10, 14]). In this sense the vector field U_1 is called the *characteristic vector field* of M in $M^{(n+p)/2}(c)$.

In this paper we study CR-submanifolds of maximal CR-dimension by investigating extrinsic behaviors of integral curves of characteristic vector field on them. Also we consider the notion of ruled CR-submanifold of maximal CR-dimension which is a generalization of that of ruled real hypersurface and find some characterizations of ruled CR-submanifold of maximal CR-dimension concerning extrinsic shapes of integral curves of the characteristic vector field and those of CR-Frenet curves.

2. Preliminaries

Let M be as in Section 1 and let us use the same notations as shown in that section. We denote by $\overline{\nabla}$ and ∇ the Levi-Civita connection on $M^{(n+p)/2}(c)$ and M, respectively. Then the Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\overline{\nabla}_X \xi_\alpha = -A_\alpha X + \nabla^\perp_X \xi_\alpha, \quad \alpha = 1, \ldots, p \tag{2.2}$$

for any tangent vector fields X, Y to M. Here $\nabla^\perp$ denotes the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^\perp$ of M, and h and A_α the second fundamental form and the shape operator corresponding to ξ_α, respectively. It is clear that h and A_α are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)\xi_\alpha.$$

Especially, if we put

$$\nabla^\perp_X \xi_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)\xi_\beta, \tag{2.3}$$

then $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Now, by using (2.1)-(2.3) and taking account of the Kähler condition $\overline{\nabla} J = 0$, we differentiate (1.2) and (1.3) covariantly and compare the tangential and the normal parts. Then we can easily find that

$$(\nabla_X F)Y = u^1(Y)A_1 X - g(A_1 Y, X)U_1, \tag{2.4}$$

$$(\nabla_X u^1)Y = g(FA_1 X, Y), \tag{2.5}$$

$$\nabla_X U_1 = FA_1 X, \tag{2.6}$$

$$g(A_\alpha U_1, X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P_{\beta\alpha}, \quad \alpha = 2, \ldots, p \tag{2.7}$$

for any X, Y tangent to M, where we have put $P\xi_\alpha = \sum_{\beta=2}^{p} P_{\alpha\beta}\xi_\beta$ for $2 \le \alpha \le p$.

In what follows, *we assume that the distinguished normal vector field* $\xi_1 := \xi$ *is parallel with respect to the normal connection* $\nabla^\perp$. Then (2.3) gives

$$s_{1\alpha} = 0, \quad \alpha = 2, \ldots, p, \tag{2.8}$$

which together with (2.7) yields

$$A_\alpha U_1 = 0, \quad \alpha = 2, \ldots, p. \tag{2.9}$$

On the other hand, as the ambient manifold $M^{(n+p)/2}(c)$ is of constant holomorphic sectional curvature c, its Riemannian curvature tensor $\overline{R}$ satisfies

$$\begin{aligned}\overline{R}(\overline{X}, \overline{Y})\overline{Z} = \frac{c}{4}\{&\overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y} + \overline{g}(J\overline{Y}, \overline{Z})J\overline{X} \\ &- \overline{g}(J\overline{X}, \overline{Z})J\overline{Y} - 2\overline{g}(J\overline{X}, \overline{Y})J\overline{Z}\}\end{aligned}$$

for any $\overline{X}, \overline{Y}, \overline{Z}$ tangent to $M^{(n+p)/2}(c)$ (cf. [2, 15]). So, the equations of Gauss, Codazzi and Ricci imply

$$\begin{aligned}R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(FY,Z)FX \\ &- g(FX,Z)FY - 2g(FX,Y)FZ\}\end{aligned} \tag{2.10}$$

$$+\sum_{\alpha}\{g(A_\alpha Y,Z)A_\alpha X - g(A_\alpha X,Z)A_\alpha Y\},$$

$(2.11)_{(a)}$
$$(\nabla_X A_1)Y - (\nabla_Y A_1)X = \frac{c}{4}\{g(X,U_1)FY - g(Y,U_1)FX - 2g(FX,Y)U_1\},$$

$(2.11)_{(b)}$
$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = \sum_{\beta=2}^{p}\{s_{\beta\alpha}(Y)A_\beta X - s_{\beta\alpha}(X)A_\beta Y\}, \quad \alpha = 2,\dots,p,$$

$$[A_1, A_\alpha] = 0, \quad \alpha = 2,\dots,p \tag{2.12}$$

for any X, Y, Z tangent to M with the aid of (2.8), where R denotes the Riemannian curvature tensor of M with respect to g.

Finally we prepare the following lemma for later use. By the definition of CR-submanifold M of maximal CR-dimension, we can easily see that M is a CR-submanifold in the sense of Bejancu (cf. [1, 2]).

Thus we have:

Lemma 2.1 ([1, 3])**.** *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a non-flat complex space form $M^{(n+p)/2}(c)$ (more general, in a Kähler manifold). Then the maximal distribution $\mathcal{D}$ is integrable if and only if*

$$h(JX, Y) = h(X, JY)$$

for any sections X, Y in $\mathcal{D}$. Moreover, each of its integral manifolds is totally geodesic in $M^{(n+p)/2}(c)$ if and only if

$$h(X, Y) = 0$$

for any sections X, Y in $\mathcal{D}$.

3. Integral curves of the characteristic vector field as an eigenvector of the shape operator A_1

We first introduce the following lemma included the proof for later use which is provided in [6].

Lemma 3.1. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a non-flat complex space form $M^{(n+p)/2}(c)$. Suppose the distinguished normal vector field ξ be parallel with respect to the normal connection. If the characteristic vector U_1 is an eigenvector of the shape operator A_1, then the corresponding eigenvalue is locally constant.*

Proof. Let λ be the eigenvalue of A_1 corresponding to the eigenvector U_1, that is,

$$A_1 U_1 = \lambda U_1. \tag{3.1}$$

Differentiating (3.1) covariantly along M and using (1.9), (2.6) and $(2.11)_{(a)}$, we have

$$-\frac{c}{2}g(FX,Y)+2g(FA_1X,A_1Y) \tag{3.2}$$
$$=(X\lambda)u^1(Y)-(Y\lambda)u^1(X)+\lambda g((FA_1+A_1F)X,Y),$$

from which, putting $X=U_1$ and using (1.9) and (3.1),

$$X\lambda=\mu u^1(X) \quad (\mu:=(U_1\lambda)). \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$-\frac{c}{2}g(FX,Y)+2g(FA_1X,A_1Y)=\lambda g((FA_1+A_1F)X,Y), \tag{3.4}$$

or equivalently,

$$-\frac{c}{2}FX+2A_1FA_1X=\lambda(FA_1+A_1F)X. \tag{3.4$'$}$$

On the other hand, differentiating (3.3) covariantly and using (2.5), we have

$$YX\lambda=(Y\mu)u^1(X)+\mu g(FA_1Y,X)+\mu u^1(\nabla_Y X),$$

from which, taking the skew-symmetric part,

$$(Y\mu)u^1(X)-(X\mu)u^1(Y)-\mu g((FA_1+A_1F)X,Y)=0. \tag{3.5}$$

Putting $X=U_1$ into (3.5) and using (1.9) and (3.1), we have

$$Y\mu=(U_1\mu)u^1(Y), \tag{3.6}$$

which and (3.5) yield

$$\mu(FA_1+A_1F)X=0. \tag{3.7}$$

As was already shown in (3.3), it suffices to prove $\mu=0$ in order that λ is locally constant. From now on we assume that $\mu\neq 0$. Then (3.7) gives

$$(FA_1+A_1F)X=0, \tag{3.8}$$

which together with (3.4) implies

$$-\frac{c}{4}FX+A_1FA_1X=0. \tag{3.9}$$

Substituting FX into (3.9) instead of X and using (1.8), (1.9), (3.1) and (3.8), we can easily see that

$$A_1^2X=-\frac{c}{4}X+(\frac{c}{4}+\lambda^2)u^1(X)U_1. \tag{3.10}$$

Differentiating (2.6) covariantly and using (2.4), (2.6) itself and (3.1), we have

$$\nabla_Y\nabla_X U_1-\nabla_{\nabla_Y X}U_1=\lambda u^1(X)A_1Y-g(A_1X,A_1Y)U_1+F(\nabla_Y A_1)X,$$

from which, taking the skew symmetric part and using (1.8), (1.9) and $(2.11)_{(a)}$

$$R(Y,X)U=\lambda\{u^1(X)A_1Y-u^1(Y)A_1X\}+\frac{c}{4}\{u^1(X)Y-u^1(Y)X\}.$$

On the other hand, it is clear from (1.9), (2.9) and (2.10) that

$$R(Y,X)U = \frac{c}{4}\{u^1(X)Y - u^1(Y)X\},$$

from which, comparing with the above equation, we have

$$\lambda\{u^1(X)A_1Y - u^1(Y)A_1X\} = 0. \tag{3.11}$$

Substituting U_1 and A_1Y into (3.11) instead of X and Y, respectively, and using (1.9), (3.1) and (3.10), we can obtain

$$c\lambda\{u^1(Y)U_1 - Y\} = 0. \tag{3.12}$$

Since $c \neq 0$ and $n > 1$, (3.12) gives $\lambda = 0$ on the set $\{x \in M \mid \mu(x) \neq 0\}$, which is a contradiction because of $\mu := U_1\lambda$. Hence $\mu = 0$ identically on M, which and (3.3) imply our assertion. □

Proposition 3.2. *Let M be as in Lemma 2.1. Suppose the distinguished normal vector field ξ be parallel with respect to the normal connection. If the characteristic vector U_1 is an eigenvector of the shape operator A_1, then every integral curve for U_1 is a circle lying on a complex line as a curve in $M^{(n+p)/2}(c)$.*

Proof. By means of Lemma 3.1, if the characteristic vector U_1 is an eigenvector of the shape operator A_1, then $A_1U_1 = \lambda U_1$ for some locally constant function λ, which together with (1.9) and (2.6) yields $\nabla_{U_1}U_1 = 0$. Hence, from (2.1), (2.2) and (2.9), it is clear that

$$\overline{\nabla}_{U_1}U_1 = \lambda\xi, \quad \overline{\nabla}_{U_1}\xi = -\lambda U_1.$$

If γ is an integral curve of U_1, those equations yield that it is a circle of curvature $|\lambda|$ in $M^{(n+p)/2}(c)$. Furthermore, if we take a totally geodesic complex line of $M^{(n+p)/2}(c)$ whose tangent space at $\gamma(0)$ is spanned by $\dot\gamma(0) = U_{1\gamma(0)}$ and $J\dot\gamma(0) = \xi_{\gamma(0)}$, those equations guarantee that γ lies on this complex line. □

The next theorem gives an affirmative answer of the converse problem of Proposition 3.2.

Theorem 3.3. *Let M be as in Lemma 2.1. Suppose the distinguished normal vector field ξ be parallel with respect to the normal connection. Then the characteristic vector U_1 is an eigenvector of the shape operator A_1 if and only if every integral curve for U_1 lies on a totally geodesic complex line of $M^{(n+p)/2}(c)$.*

Proof. By means of Proposition 3.2 it suffices to show the "if" part. Let γ be an integral curve of the characteristic vector U_1. Then there is a smooth function κ which satisfies $\overline{\nabla}_{\dot\gamma}\dot\gamma(s) = \kappa(s)J\dot\gamma(s)$ as a curve on $M^{(n+p)/2}(c)$. Since $\dot\gamma(s) = U_{1\gamma(s)}$, it follows from (1.1), (2.1), (2.6) and (2.9) that

$$\begin{aligned}\kappa(s)\xi_{\gamma(s)} &= \kappa(s)J\dot\gamma(s) = \nabla_{\dot\gamma}\dot\gamma(s) + g(A_1\dot\gamma(s), \dot\gamma(s))\xi_{\gamma(s)} \\ &= FA_1U_{1\gamma(s)} + g(A_1U_{1\gamma(s)}, U_{1\gamma(s)})\xi_{\gamma(s)},\end{aligned}$$

from which, taking the tangential components for M, we have $FA_1U_{1\gamma(s)} = 0$. Hence $U_{1\gamma(s)}$ is an eigenvector of the shape operator A_1 and consequently so is U_1 because γ is an arbitrary integral curve of U_1. □

Remark. As already mentioned in §1, a real hypersurface M of $M^{(n+p)/2}(c)$ is a typical example of CR-submanifold of maximal CR-dimension. Moreover, when the characteristic vector U_1 is a principal curvature vector of M, M is called a *Hopf hypersurface*(cf. [5, 7, 8, 9]). By means of Proposition 3.2 and Theorem 3.3, we can see the features of integral curves for the characteristic vector of a Hopf hypersurface in a non-flat complex space form $M^{(n+p)/2}(c)$. For more details, see [5, 8].

4. Extrinsic shapes of the characteristic vector field on CR-submanifolds with integrable maximal holomorphic distribution $\mathcal{D}$

Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a non-flat complex space form $M^{(n+p)/2}(c)$ with parallel distinguished normal vector field ξ with respect to the normal connection. In particular, we consider those CR-submanifolds M such that the maximal holomorphic distribution $\mathcal{D}$ is integrable and each of its integral manifolds is a totally geodesic complex submanifold of $M^{(n+p)/2}(c)$ which is locally congruent to $M^{(n-1)/2}(c)$. In what follows, such a submanifold M is called a *ruled CR-submanifold of maximal CR-dimension*. The notion of ruled CR-submanifold of maximal CR-dimension is a natural extension of that of ruled real hypersurface.

On the other hand, in their paper [8], Maeda and Adachi proved the following lemma.

Lemma 4.1 ([8]). *A real hypersurface M of a non-flat complex space form $M^{(n+1)/2}(c)$ is a ruled real hypersurface if and only if every geodesic on M whose initial vector is orthogonal to the characteristic vector U_1 is a geodesic as a curve in $M^{(n+1)/2}(c)$.*

In this section we want to extend Lemma 4.1 to the case of ruled CR-submanifold of maximal CR-dimension and prove the following theorem.

Theorem 4.2. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a non-flat complex space form $M^{(n+p)/2}(c)$ with parallel distinguished normal vector field ξ with respect to the normal connection. Then M is a ruled CR-submanifold of maximal CR-dimension if and only if every geodesic on M whose initial vector is orthogonal to the characteristic vector U_1 is a geodesic as a curve in $M^{(n+p)/2}(c)$.*

Proof. Assume that M is a ruled CR-submanifold of maximal CR-dimension. By means of Lemma 2.1 we can see that

$$h(X, Y) = 0 \tag{4.1}$$

for any sections X, Y in $\mathcal{D}$. Since

$$h(X,Y) = A_1(X,Y)\xi + \sum_{\alpha=2}^{p} g(A_\alpha X, Y)\xi_\alpha$$

for any tangent vectors X, Y to M, it follows from (2.9) and (4.1) that

$$A_\alpha = 0, \quad \alpha = 2, \dots, p. \tag{4.2}$$

Thus we can apply the codimension reduction theorems due to Kawamoto ([4]) and Okumura ([13]) since the distinguished normal vector field ξ is parallel with respect to the normal connection. In fact we can see from (4.2) that there exists a totally geodesic non-flat complex space form $M^{(n+1)/2}(c)$ in $M^{(n+p)/2}(c)$ such that $M \subset M^{(n+1)/2}(c)$ and hence that M can be regarded as a ruled real hypersurface of $M^{(n+1)/2}(c)$. Hence Lemma 4.1 implies that every geodesic on M whose initial vector is orthogonal to the characteristic vector U_1 is a geodesic as a curve in $M^{(n+1)/2}(c)$. But $M^{(n+1)/2}(c)$ is totally geodesic in $M^{(n+p)/2}(c)$ and consequently every geodesic in $M^{(n+1)/2}(c)$ is also geodesic in $M^{(n+p)/2}(c)$ because the distinguished normal vector field is parallel with respect to the normal connection.

For the converse we assume that every geodesic γ on M whose initial vector $\dot\gamma(0)$ is orthogonal to the characteristic vector $U_{1\gamma(0)}$ is a geodesic as a curve in $M^{(n+p)/2}(c)$. Then the Gauss equation (2.1) gives that $h(\dot\gamma(0), \dot\gamma(0)) = 0$. Therefore $h(X, X) = 0$ for arbitrary tangent vector X to M orthogonal to U_1 at each point in M, which together with the symmetry of h yields the condition in Lemma 2.1. Hence we find that M is a ruled CR-submanifold of maximal CR-dimension. □

5. Extrinsic shapes of *CR*-Frenet curves on integrable maximal holomorphic distribution $\mathcal{D}$

Let M be an $n(>1)$-dimensional CR-submanifold of CR-dimension $(n-1)$ which is isometrically immersed in a Kähler manifold $M^{(n+p)/2}$.

We assume that the maximal holomorphic distribution $\mathcal{D}$ is integrable and denote by $M_{\mathcal{D}}$ its maximal integral manifold. Then $M_{\mathcal{D}}$ is locally an invariant submanifold of $M^{(n+p)/2}$ and so a Kähler manifold with complex structure $J|_{\mathcal{D}}$ obtained as the restriction of J to $\mathcal{D}$. We shall denote $J|_{\mathcal{D}}$ also by the same symbol J. A smooth curve $\gamma = \gamma(s)$ on $M_{\mathcal{D}}$, s being its arc length, is called a *CR-Frenet curve* if it satisfies the following differential equation

$$'\nabla_{\dot\gamma}\dot\gamma = \kappa(s)J\dot\gamma \quad \text{or} \quad '\nabla_{\dot\gamma}\dot\gamma = -\kappa(s)J\dot\gamma \tag{5.1}$$

for some positive smooth function $\kappa = \kappa(s)$, where $'\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection $'\nabla$ induced on $M_{\mathcal{D}}$ from that of $M^{(n+p)/2}$. Here we note that for an arbitrary point $x \in M_{\mathcal{D}}$, an arbitrary unit tangent vector v to $M_{\mathcal{D}}$ and an arbitrary positive smooth function κ, there exist just two CR-Frenet curves γ_1, γ_2 of curvature κ with

${}'\nabla_{\dot\gamma}\dot\gamma = \kappa(s)J\dot\gamma$, ${}'\nabla_{\dot\gamma}\dot\gamma = -\kappa(s)J\dot\gamma$ satisfying the initial condition $\gamma_i(0) = x$, $\dot\gamma_i(0) = v$ $(i = 1, 2)$.

On the other hand, a smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold parametrized by its arc length s is called a *Frenet curve of proper order 2* if there exist a field of orthonormal frames $\{\dot\gamma(s), Y_s\}$ along γ and a positive function $\kappa(s)$ satisfying the following system of ordinary differential equations

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)Y_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -\kappa(s)\dot\gamma.$$

A curve is called a *Frenet curve of order 2* if it is either a Frenet curve of proper order 2 or a geodesic (cf. [5, 8]).

In this section we provide the following theorem as an extrinsic property of CR-Frenet curve (for Kähler Frenet curves on Kähler manifold, cf. [5, 8, 9]).

Theorem 5.1. *Let M be an n-dimensional CR-submanifold of $(n-1)$ CR-dimension in a Kähler manifold $M^{(n+p)/2}$. Then M is a ruled CR-submanifold of maximal CR-dimension if and only if*

(i) *the distribution $\mathcal{D}$ is integrable, and*

(ii) *for some positive smooth function $\kappa(s)$ there exists such an orthonormal basis $\{v_1, \dots, v_n, Jv_1, \dots, Jv_n\}$ at each point $x \in M_{\mathcal{D}}$ that CR-Frenet curves γ_{ij} of curvature $\kappa(s)$ on $M_{\mathcal{D}}$ through x satisfying that the initial vector $\dot\gamma_{ij}(0)$ is in the direction $v_i + v_j$ $(1 \le i \le j \le n)$ are mapped to Frenet curves of order 2 in $M^{(n+p)/2}$ by the immersion $M \subset M^{(n+p)/2}$.*

Proof. First of all we define the covariant differentiation $\tilde\nabla$ of the second fundamental form $h_{\mathcal{D}}$ of $M_{\mathcal{D}}$ with respect to the connection in (tangent bundle)$\oplus$ (normal bundle) as follows (cf. [2, 14]):

$$(\tilde\nabla_X h_{\mathcal{D}})(Y, Z) = {}'\nabla^{\perp}_X h_{\mathcal{D}}(Y, Z) - h_{\mathcal{D}}({}'\nabla_X Y, Z) - h_{\mathcal{D}}(Y, {}'\nabla_X Z), \tag{5.2}$$

where ${}'\nabla^{\perp}$ denotes the normal connection induced from $\overline{\nabla}$ in the normal bundle $TM^{\perp}_{\mathcal{D}}$ of $M_{\mathcal{D}}$.

The "only if" part is trivial. The rest of the proof is to verify the "if" part is true. But the proof of "if" part is quite similar as was given in [5] and [9] in the case of Kähler Frenet curve.

As was already mentioned in the above, the maximal integral manifold $M_{\mathcal{D}}$ is Kählerian, and consequently Lemma 2.1 yields that

$$h_{\mathcal{D}}(JX, Y) = h_{\mathcal{D}}(X, JY) = Jh_{\mathcal{D}}(X, Y) \tag{5.3}$$

for any sections X, Y in $\mathcal{D}$.

Let γ_{ij} be a CR-Frenet curve satisfying the hyperthesis of our theorem. In order to deal two equations in (5.1) simultaneously, we set

$${}'\nabla_{\dot\gamma}\dot\gamma = \kappa(s)J\dot\gamma, \tag{5.4}$$

where $\kappa > 0$ or $\kappa < 0$. Note that $\dot\gamma_{ij}(0) = v_i$ (resp. $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$) in the case of $i = j$ (resp. $i \neq j$) and $({}'\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij})(0) = \pm\kappa(0)J\dot\gamma_{ij}(0)$, where $\kappa(0)$

is positive. Then it is clear from the Gauss equation for $M_{\mathcal{D}}$ in $M^{(n+p)/2}$ and (5.2) that

$$\overline{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = \kappa J\dot{\gamma}_{ij} + h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij}). \tag{5.5}$$

Differentiating (5.5) covariantly along $M^{(n+p)/2}$ and using (5.2)-(5.4) and (5.5) itself, we can easily obtain

$$\begin{aligned} \overline{\nabla}_{\dot{\gamma}_{ij}}\overline{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = {}& \kappa' J\dot{\gamma}_{ij} - \kappa^2\dot{\gamma}_{ij} + 3\kappa h_{\mathcal{D}}(\dot{\gamma}_{ij}, J\dot{\gamma}_{ij}) \\ & - A_{h_{\mathcal{D}}(\dot{\gamma}_{ij},\dot{\gamma}_{ij})}\dot{\gamma}_{ij} + (\tilde{\nabla}_X h_{\mathcal{D}})(\dot{\gamma}_{ij}, \dot{\gamma}_{ij}), \end{aligned} \tag{5.6}$$

where $A_{h_{\mathcal{D}}(\dot{\gamma}_{ij},\dot{\gamma}_{ij})}$ denotes the shape operator of $M_{\mathcal{D}}$ in direction of $h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})$. On the other hand, by our assumption we have

$$\overline{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = \overline{\kappa}_{ij}Y_{ij} \quad \text{and} \quad \overline{\nabla}_{\dot{\gamma}_{ij}}Y_{ij} = -\overline{\kappa}_{ij}\dot{\gamma}_{ij}, \tag{5.7}$$

where we set

$$(\overline{\kappa}_{ij})^2 := \kappa^2 + \|h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})\|^2, \quad Y_{ij} := \frac{1}{\overline{\kappa}_{ij}}(\kappa J\dot{\gamma}_{ij} + h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})). \tag{5.8}$$

Hence we have

$$\overline{\nabla}_{\dot{\gamma}_{ij}}\overline{\nabla}_{\dot{\gamma}_{ij}}\dot{\gamma}_{ij} = \overline{\kappa}_{ij}{}'Y_{ij} - \overline{\kappa}_{ij}{}^2\dot{\gamma}_{ij}, \tag{5.9}$$

from which, comparing the tangential components and the normal components with (5.6) and using (5.8),

$$(\frac{\overline{\kappa}_{ij}{}'}{\overline{\kappa}_{ij}}\kappa - \kappa')J\dot{\gamma}_{ij} - \|h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})\|^2\dot{\gamma}_{ij} + A_{h_{\mathcal{D}}(\dot{\gamma}_{ij},\dot{\gamma}_{ij})}\dot{\gamma}_{ij} = 0, \tag{5.10}$$

$$3\kappa h_{\mathcal{D}}(\dot{\gamma}_{ij}, J\dot{\gamma}_{ij}) + (\tilde{\nabla}_X h_{\mathcal{D}})(\dot{\gamma}_{ij}, \dot{\gamma}_{ij}) = \frac{\overline{\kappa}_{ij}{}'}{\overline{\kappa}_{ij}}h_{\mathcal{D}}(\dot{\gamma}_{ij}, \dot{\gamma}_{ij}). \tag{5.11}$$

Taking the inner product of (5.10) and $J\dot{\gamma}_{ij}$ and using (5.3), we obtain

$$\frac{\overline{\kappa}_{ij}{}'}{\overline{\kappa}_{ij}}(s) = \frac{\kappa'}{\kappa}(s)$$

for each s. Therefore, in the case of $i = j$, the value of (5.11) at $s = 0$ implies

$$\pm 3\kappa(0)h_{\mathcal{D}}(v_i, Jv_i) + (\tilde{\nabla}_{v_i}h_{\mathcal{D}})(v_i, v_i) = \frac{\kappa'}{\kappa}(0)h_{\mathcal{D}}(v_i, v_i),$$

which yields $h_{\mathcal{D}}(v_i, Jv_i) = 0$ and consequently

$$h_{\mathcal{D}}(v_i, v_i) = 0, \quad i = 1, \ldots, n. \tag{5.12}$$

In the case of $i \neq j$, by the same method as above it follows that

$$\begin{aligned} & \pm \frac{3}{2}\kappa(0)h_{\mathcal{D}}(v_i + v_j, Jv_i + Jv_j) + (\tilde{\nabla}_{(v_i+v_j)/\sqrt{2}}h_{\mathcal{D}})(\frac{v_i + v_j}{\sqrt{2}}, \frac{v_i + v_j}{\sqrt{2}}) \\ = {}& \frac{\kappa'}{\kappa}(0)h_{\mathcal{D}}(\frac{v_i + v_j}{\sqrt{2}}, \frac{v_i + v_j}{\sqrt{2}}), \end{aligned}$$

and hence $h_{\mathcal{D}}(\frac{v_i+v_j}{\sqrt{2}}, \frac{v_i+v_j}{\sqrt{2}}) = 0$ for each distinct i, j, from which and (5.12), it is clear that

$$h_{\mathcal{D}}(v_i, v_j) = 0$$

for each distinct i, j. Since x is an arbitrary point in $M_{\mathcal{D}}$, we have

$$h_{\mathcal{D}}(X, Y) = 0$$

for any sections X, Y in $\mathcal{D}$, which together with Lemma 2.1 gives our assertion. □

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