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STARLIKENESS AND SCHWARZIAN DERIVATIVES OF HIGHER ORDER OF ANALYTIC FUNCTIONS

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Abstract. In this paper we apply the third-order differential subordination results to normalized analytic functions in the open unit disk. We obtain appropriate classes of admissible functions and find some sufficient conditions of functions to be starlike associated with Tamanoi's Schwarzian derivative of third order. Several interesting examples are also discussed.

1. Introduction

Let $\mathcal{H}=\mathcal{H}(\mathbb{U})$ denote the class of functions analytic in $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\right\}.$$

We define the class $\mathcal{A}_n$ of normalized analytic functions by

$$\mathcal{A}_n = \left\{f \in \mathcal{H} : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots\right\},$$

with $\mathcal{A}_1 = \mathcal{A}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{S}$ is said to be starlike, if $\Re\{zf'(z)/f(z)\} > 0$ in $\mathbb{U}$. Similarly, a function $f \in \mathcal{S}$ is said to be convex, if $\Re\{1 + zf''(z)/f'(z)\} > 0$ in $\mathbb{U}$.

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, if there exists a Schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In this case, we write $f \prec F$ (or $f(z) \prec F(z)$) $(z \in \mathbb{U})$. If the function F is univalent in $\mathbb{U}$, we have

$$f \prec F \Longleftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

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We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus \mathbf{E}(q)$, where

$$\mathbf{E}(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$. Furthermore, we denote by $\mathcal{Q}(a)$ the subclass of $\mathcal{Q}$ for which $q(0) = a$ and put $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Let $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (third order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1).

For $f \in \mathcal{A}$, the Schwarzian derivative S_f of f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f^{(3)}}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{1.2}$$

There are several necessary and several sufficient conditions relating the Schwarzian derivative to the univalency of f (see [9] and [10]). Especially, in [11], Owa and Obradović found a sufficient condition for $f \in \mathcal{A}$ to be convex. And Miller and Mocanu [7] (see also [8, p. 244]) investigated another various conditions for function in $\mathcal{A}$ to be convex. Indeed, they also found admissibility conditions on $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that

$$\Re\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2S_f(z); z\right)\right\} > 0 \quad (z \in \mathbb{U})$$

implies that f is starlike in $\mathbb{U}$. Moreover, Ali *et al.* [2] investigated sufficient conditions relating the Schwarzian derivative to the starlikeness or convexity of $f \in \mathcal{A}$ using the methods of second order differential subordination and superordination.

On the other hand, several authors [1, 4, 5, 6, 12, 13] tried to generalize the Schwarzian derivatives and investigated their properties. Especially, in [13], Tamanoi defined a Schwarzian derivative of higher order $S_n[f]$ $(n \in \mathbb{N})$. And Kim and Sugawa [6] gave a recursive formula for the Tamanoi's Schwarzian derivatives as follows:

$$S_n[f] = S_{n-1}[f]' + \frac{1}{2}S_2[f]\sum_{k=1}^{n-1}\binom{n}{k}S_{k-1}[f]S_{n-k-1}[f] \quad (n \geq 3). \tag{1.3}$$

Here, $S_0[f] = 1$, $S_1[f] = 0$ and $S_2[f]$ is the classical Schwarzian derivative S_f defined by (1.2). Especially, the formula given by (1.3) gives us an equality

$$S_3[f] = S_2[f]'.$$

In this work, we find sufficient conditions involving the Tamanoi's Schwarzian derivatives for functions to be starlike, by making use of the third-order differential subordination results of Antonino and Miller [3]. In Section 2, we modify slightly the result due to Antonino and Miller [3, Lemma D] and consider certain classes of admissible functions. In Section 3, we apply the result investigated in Section 2 to obtain a sufficient conditions for functions to be starlike with considering two special cases. Furthermore, several interesting examples are also discussed.

2. Main results

First of all, we present the following results due to Antonino and Miller [3].

Definition 1 ([3, Definition 3]). Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \geq 2$. The class $\Psi_n[\Omega, q]$ of admissible operators consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition

$$\psi(r, s, t, u; z) \notin \Omega,$$

when $z \in \mathbb{U}$, $r = q(\xi)$, $s = n\xi q'(\xi)$,

$$\Re\left\{\frac{t}{s} + 1\right\} \geq n\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}$$

and

$$\Re\left\{\frac{u}{s}\right\} \geq n^2\Re\left\{\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}\right\}$$

for $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$.

Lemma 2.1 ([3, Lemma D]). *Let* $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *be analytic in* $\mathbb{U}$ *with* $p(z) \not\equiv a$ *and* $n \geq 2$, *and let* $q \in \mathcal{Q}(a)$. *If there exist a point* $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U} \setminus \mathbf{E}(q)$ *such that* $p(z_0) = q(\zeta_0)$, $p(\overline{\mathbb{U}}_{r_0}) \subset q(\mathbb{U})$,

$$\Re\left\{\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\} \geq 0 \tag{2.1}$$

and

$$|zp'(z)| \leq n|q'(\xi)|,$$

when $z \in \overline{\mathbb{U}}_{r_0}$ *and* $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$, *then the following relations are satisfied*:

$$z_0 p'(z_0) = n\zeta_0 q'(\zeta_0), \tag{2.2}$$

$$\Re\left\{\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right\} \geq n\Re\left\{\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right\} \tag{2.3}$$

and

$$\Re\left\{\frac{z_0^2 p^{(3)}(z_0)}{p'(z_0)}\right\} \geq n^2\Re\left\{\frac{\zeta_0^2 q^{(3)}(\zeta_0)}{q'(\zeta_0)}\right\}.$$

Theorem 2.2 ([3, Theorem 1]). *Let $p \in \mathcal{H}[a,n]$ with $n \geq 2$ and let $q \in \mathcal{Q}(a)$ satisfy*

$$\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n,$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z) \in \Omega,$$

then $p(z) \prec q(z) \quad (z \in \mathbb{U})$.

Now, we modify Lemma 2.1 as follows.

Lemma 2.3. *Let $p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $n \geq 2$. And let $q \in \mathcal{Q}(a)$. If there exist a point $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus \mathbf{E}(q)$ such that $p(z_0) = q(\zeta_0)$, $p(\overline{\mathbb{U}}_{r_0}) \subset q(\mathbb{U})$ and*

$$|zp'(z)| \leq n|q'(\xi)|,$$

when $z \in \overline{\mathbb{U}}_{r_0}$ and $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$, then the equations (2.2), (2.3) *hold and*

(2.4)
$$\Re\left\{\frac{z_0^2p^{(3)}(z_0)}{p'(z_0)} + \frac{3z_0p''(z_0)}{p'(z_0)} + 1\right\} \geq n^2\Re\left\{\frac{\zeta_0^2q^{(3)}(\zeta_0)}{q'(\zeta_0)} + \frac{3\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right\}.$$

Proof. We note that the relations (2.2) and (2.3) are valid although the condition (2.1) is eliminated from Lemma 2.1. Therefore it is sufficient to show that the inequality (2.4) is satisfied. Let $\zeta = f(z) = q^{-1}(p(z))$ for $z \in p(\overline{\mathbb{U}}_{r_0})$. Then, we have (see the proof of Lemma D in [3])

$$\begin{aligned}
&\frac{z_0^2p^{(3)}(z_0)}{p'(z_0)} + \frac{3z_0p''(z_0)}{p'(z_0)} + 1\\
&= \frac{\zeta_0^2q^{(3)}(\zeta_0)}{q'(\zeta_0)}\left(\frac{z_0f'(z_0)}{f(z_0)}\right)^2 + \frac{3\zeta_0q''(\zeta_0)}{q'(\zeta_0)}\frac{z_0f'(z_0)}{f(z_0)}\left(\frac{z_0f''(z_0)}{f'(z_0)} + 1\right)\\
&\quad + \frac{{z_0}^2f^{(3)}(z_0)}{f'(z_0)} + \frac{3z_0f''(z_0)}{f'(z_0)} + 1,
\end{aligned} \tag{2.5}$$

where $\zeta_0 = f(z_0)$. Following the proof of Lemma D in [3], we can take $k \in \mathbb{R}$ with $k \geq n \geq 2$ such that

$$\frac{z_0f'(z_0)}{f(z_0)} = n, \quad \frac{z_0f''(z_0)}{f'(z_0)} = k - 1 \tag{2.6}$$

and

$$\Re\left\{\frac{z_0^2f^{(3)}(z_0)}{f'(z_0)} + \frac{3z_0f''(z_0)}{f'(z_0)} + 1\right\} \geq k^2. \tag{2.7}$$

By taking the real parts in the both sides of (2.5) and using the relations (2.6) and (2.7), we obtain

$$\Re\left\{\frac{z_0^2p^{(3)}(z_0)}{p'(z_0)} + \frac{3z_0p''(z_0)}{p'(z_0)} + 1\right\}$$

$$= n^2\Re\left\{\frac{\zeta_0^2 q^{(3)}(\zeta_0)}{q'(\zeta_0)}\right\}+nk\Re\left\{\frac{3\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\}+\Re\left\{\frac{{z_0}^2 f^{(3)}(z_0)}{f'(z_0)}+\frac{3z_0 f''(z_0)}{f'(z_0)}+1\right\}$$
$$\geq n^2\Re\left\{\frac{\zeta_0^2 q^{(3)}(\zeta_0)}{q'(\zeta_0)}+\frac{3\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}+1\right\}.$$

This shows that the inequality (2.4) holds and the proof of Lemma 2.3 is completed. □

Now, we introduce a class of admissible functions.

Definition 2. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \geq 2$. The class $\Psi'_n[\Omega, q]$ of admissible operators consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition

$$\psi(r, s, t, u; z) \notin \Omega,$$

when $z \in \mathbb{U}$, $r = q(\xi)$, $s = n\xi q'(\xi)$,

$$\Re\left\{\frac{t}{s}+1\right\} \geq n\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}+1\right\}$$

and

$$\Re\left\{\frac{u}{s}+\frac{3t}{s}+1\right\} \geq n^2\Re\left\{\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}+\frac{3\xi q''(\xi)}{q'(\xi)}+1\right\}$$

for $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$.

By applying Lemma 2.3, we can obtain the following lemma.

Lemma 2.4. *Let $p \in \mathcal{H}[a, n]$ with $n \geq 2$ and let $q \in \mathcal{Q}(a)$ satisfy*

$$\left|\frac{zp'(z)}{q'(\xi)}\right| \leq n,$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi'_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z) \in \Omega,$$

then $p(z) \prec q(z)$ $(z \in \mathbb{U})$.

Proof. The proof of this lemma is very similar to [3, Theorem 1] and we omit it. □

We define the following class of admissible functions that are required for our results.

Definition 3. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \geq 2$. The class $\Phi'_n[\Omega, q]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(v, w, x, y; z) \notin \Omega,$$

whenever

$$v = q(\xi), \quad w = \frac{n\xi q'(\xi)}{q(\xi)} + q(\xi) \quad (q(\xi) \neq 0),$$

$$\Re\left\{\frac{2x+v^2-1+3(w-v)^2}{2(w-v)}\right\} \geq n\Re\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}\right\}$$

and

$$\Re\left\{\frac{2x+y}{w-v}+3(w-v)^2+3v^2+4x-2\right\} \geq n^2\Re\left\{\frac{\xi^2 q^{(3)}(\xi)}{q'(\xi)}+\frac{3\xi q''(\xi)}{q'(\xi)}+1\right\}$$

for $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)$.

Theorem 2.5. *Let* $q \in \mathcal{Q}_1$ *and* $\phi \in \Phi'_n[\Omega, q]$ *with* $\Omega \subset \mathbb{C}$ *and* $n \geq 2$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)| \quad (z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus \mathbf{E}(q))$$

and

$$\left\{\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2 S_2[f](z), z^3 S_3[f](z); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.8}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Proof. Define a function p in $\mathbb{U}$ by

$$p(z) = \frac{zf'(z)}{f(z)}. \tag{2.9}$$

We note that $p \in \mathcal{H}[1, n]$ since $f \in \mathcal{A}_n$. And we have

$$1+\frac{zf''(z)}{f'(z)} = p(z)+\frac{zp'(z)}{p(z)}, \tag{2.10}$$

$$z^2 S_2[f](z) = \frac{z^2 p''(z)}{p(z)}+\frac{zp'(z)}{p(z)}-\frac{3}{2}\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{1-p^2(z)}{2} \tag{2.11}$$

and

$$\begin{aligned} z^3 S_3[f](z) &= z^3 (S_2[f](z))' \\ &= \frac{z^2 p''(z)}{p(z)}-\frac{zp'(z)}{p(z)}-\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^3 p^{(3)}(z)}{p(z)} \\ &\quad -\frac{4z^3 p'(z)p''(z)}{p^2(z)}+3\left(\frac{zp'(z)}{p(z)}\right)^3-zp'(z)p(z)-1+p^2(z). \end{aligned} \tag{2.12}$$

Let us define parameters v, w, x and y by

$$\begin{aligned} v &= r, \\ w &= r+\frac{s}{r}, \\ x &= \frac{t+s}{r}-\frac{3}{2}\left(\frac{s}{r}\right)^2+\frac{1-r^2}{2}, \\ y &= \frac{t}{r}-\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{u}{r}-\frac{4st}{r^2}+3\left(\frac{s}{r}\right)^3-rs-1+r^2. \end{aligned}$$

Now, we define a transformation $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ by

$$\psi(r, s, t, u; z) = \phi(v, w, x, y; z).$$

By using the relations (2.9), (2.10), (2.11) and (2.12), we have

$$\begin{aligned}&\psi\left(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z\right)\\ &= \phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right) \quad (z \in \mathbb{U}).\end{aligned}$$

Therefore, we can rewrite (2.8) as

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p^{(3)}(z); z\right) \in \Omega \quad (z \in \mathbb{U}).$$

Then, the proof is completed by showing that the admissibility condition $\phi \in \Phi'_n[\Omega, q]$ in Definition 3 is equivalent to the admissibility condition for ψ as given in Definition 2. Since

$$\frac{t}{s} + 1 = \frac{2x + v^2 - 1 + 3(w - v)^2}{2(w - v)}$$

and

$$\frac{u}{s} + \frac{3t}{s} + 1 = \frac{2x + y}{w - v} + 3(w - v)^2 + 3v^2 + 4x - 2,$$

we have $\psi \in \Psi'_n[\Omega, q]$. Therefore, by Lemma 2.4, we have $p(z) \prec q(z)$ $(z \in \mathbb{U})$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'_n[h(\mathbb{U}), q]$ is written as $\Phi'_n[h, q]$. The following result is an immediate consequence of Theorem 2.5.

Theorem 2.6. *Let h be a conformal mapping in $\mathbb{U}$ and let $q \in \mathcal{Q}_1$. Let $\phi \in \Phi'_n[h, q]$ with $n \geq 2$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)| \quad (z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus \mathbf{E}(q))$$

and

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right)$$

is analytic in $\mathbb{U}$, then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Following similar arguments as in [8, Theorem 2.3d], Theorem 2.6 can be extended to the following corollary where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2.7. *Let $q \in \mathcal{Q}_1$ and for $\rho \in (0,1)$ set $q_\rho(z) \equiv q(\rho z)$. Let $f \in \mathcal{A}_n$ with $n \geq 2$ satisfy*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq n|q'_\rho(\xi)| \quad (z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus \mathbf{E}(q_\rho)).$$

If $\phi \in \Phi'_n[h, q_\rho]$, where h is conformal in $\mathbb{U}$, and

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right)$$

is analytic in $\mathbb{U}$, then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right) \prec h(z) \quad (z \in \mathbb{U}) \tag{2.13}$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Next theorem yields the best dominant of the differential subordination (2.13).

Theorem 2.8. *Let h be univalent in $\mathbb{U}$ and let $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi(q(z), Q_1(z), Q_2(z), Q_3(z); z) = h(z), \tag{2.14}$$

where

$$Q_1(z) = q(z) + \frac{zq'(z)}{q(z)},$$

$$Q_2(z) = \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \frac{3}{2}\left(\frac{zq'(z)}{q(z)}\right)^2 + \frac{1-q^2(z)}{2}$$

and

$$\begin{aligned} Q_3(z) = &\frac{z^2q''(z)}{q(z)} - \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2 + \frac{z^3q^{(3)}(z)}{q(z)} \\ &- \frac{4z^3q'(z)q''(z)}{q^2(z)} + 3\left(\frac{zq'(z)}{q(z)}\right)^3 - zq'(z)q(z) - 1 + q^2(z), \end{aligned}$$

has a solution $q \in \mathcal{Q}_1$ such that $\phi \in \Phi'_n[h, q]$ with $n \geq 2$. If $f \in \mathcal{A}_n$ satisfies

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right| \leq n|q'(\xi)| \quad (z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus \mathbf{E}(q)),$$

then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1+\frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$\frac{zf'(z)}{f(z)} \prec q(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

Proof. Applying the same arguments as in [8, Theorem 2.3e], we note that q is dominant from Theorem 2.6 and Corollary 2.7. Since q satisfies (2.14), it is also a solution of (2.13) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

3. Two special cases

In this section we will apply Theorem 2.5 to two special cases. For the first case, let $q(z) = 1 + Mz,\ M > 0$. In view of Definition 3, the class of admissible functions $\Phi'_n[\Omega, q]$, denoted by $\Phi'_n[\Omega, M]$, is described below.

Definition 4. Let Ω be a set in $\mathbb{C}$, $M > 0$ and $n \geq 2$. The class of admissible functions $\Phi'_n[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ such that

$$\phi\left(q_1, q_2, q_3, q_4; z\right) \notin \Omega, \tag{3.1}$$

where

$$q_1 = 1 + Me^{i\theta},$$

$$q_2 = 1 + Me^{i\theta} + \frac{nM}{M + e^{-i\theta}},$$

$$q_3 = \frac{Le^{-i\theta} + nM}{M + e^{-i\theta}} - \frac{3}{2}\left(\frac{nM}{M + e^{-i\theta}}\right)^2 - \frac{1}{2}Me^{i\theta}(2 + Me^{i\theta})$$

and

$$q_4 = \frac{Le^{-i\theta} - nM}{M + e^{-i\theta}} - \left(\frac{nM}{M + e^{-i\theta}}\right)^2 + \frac{Ne^{-i\theta}}{M + e^{-i\theta}} - \frac{4nLM}{e^{-i\theta} + 2M + M^2e^{i\theta}}$$
$$+ 3\left(\frac{nM}{M + e^{-i\theta}}\right)^3 - Me^{i\theta}\left(n - 2 + (n-1)Me^{i\theta}\right),$$

whenever $\Re\left\{Le^{-i\theta}\right\} \geq n(n-1)M$ and $\Re\left\{(N + 3L)e^{-i\theta}\right\} \geq (n^2 - 1)M$ for all $\theta \in [0, 2\pi]$.

Throughout this section, for given $f \in \mathcal{A}$, we will define $\hat{f}$ by

$$\hat{f}(z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}$$

for the sake of our convenient notation.

We can obtain the following result from Theorem 2.5, immediately.

Corollary 3.1. *Let $\phi \in \Phi'_n[\Omega, M]$ with $\Omega \subset \mathbb{C}$, $M > 0$ and $n \geq 2$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|zf'(z)\hat{f}(z)\right| \leq nM|f(z)| \quad (z \in \mathbb{U})$$

and

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2S_2[f](z), z^3S_3[f](z); z\right) \in \Omega \quad (z \in \mathbb{U}),$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (z \in \mathbb{U}).$$

Example 3.2. Define a transformation $\phi_1 : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ by

$$\phi_1(v, w, x, y; z) = v\left\{3(w-v)^3 + (3v^2 + 4x - 3)(w - v) + y + 2x + 1\right\}$$

and let $\Omega = \{w \in \mathbb{C} : |w - 1| < 4M\}$. We will show that $\phi_1 \in \Phi'_n[\Omega, M]$, for $n \geq 2$, by showing that (3.1) is satisfied. For this transformation ϕ_1, we have

$$\begin{aligned}|\phi_1(q_1, q_2, q_3, q_4; \theta) - 1| &= \left|M + 3Le^{-i\theta} + Ne^{-i\theta}\right| \\ &\geq M + \Re\left\{(3L + N)e^{-i\theta}\right\} \\ &> M + (n^2 - 1)M \\ &\geq 4M,\end{aligned}$$

when $\Re\left\{Le^{-i\theta}\right\} \geq n(n-1)M$ and $\Re\left\{(N + 3L)e^{-i\theta}\right\} \geq (n^2 - 1)M$ for $n \geq 2$ and $\theta \in [0, 2\pi]$. Hence from Corollary 3.1, we obtain:
Let $M > 0$. If $f \in \mathcal{A}_n$ with $n \geq 2$ satisfies

$$\left|zf'(z)\hat{f}(z)\right| \leq nM|f(z)| \quad (z \in \mathbb{U}),$$

then

$$\begin{aligned}&\left|\frac{zf'(z)}{f(z)}\left\{3(\hat{f}(z))^3 + \left(3\left(\frac{zf'(z)}{f(z)}\right)^2 + 4z^2S_2[f](z) - 3\right)\hat{f}(z)\right.\right. \\ &\qquad \left.\left. + z^3S_3[f](z) + 2z^2S_2[f](z) + 1\right\} - 1\right| < 4M \quad (z \in \mathbb{U})\end{aligned}$$

implies

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < M \quad (z \in \mathbb{U}).$$

Example 3.3. Define a transformation $\phi_2 : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ by

$$\phi_2(v, w, x, y; z) = \frac{2x + y}{w - v} + 3(w - v)^2 + 3v^2 + 4x - 3$$

and $\Omega = \{w \in \mathbb{C} : \Re\{w\} < 3/2\}$. We will show that $\phi_2 \in \Phi'_n[\Omega, M]$, for $n \geq 2$, by showing that (3.1) is satisfied. Since

$$\Re\{\phi_2(q_1, q_2, q_3, q_4; z)\} = \Re\left\{\frac{(3L + N)e^{-i\theta}}{nM}\right\} \geq \frac{3}{2},$$

where $\Re\left\{Le^{-i\theta}\right\} \geq n(n-1)M$ and $\Re\left\{(N + 3L)e^{-i\theta}\right\} \geq (n^2 - 1)M$ for $n \geq 2$ and $\theta \in [0, 2\pi]$. Hence from Corollary 3.1, we obtain:
Let $M > 0$. If $f \in \mathcal{A}_n$ with $n \geq 2$ satisfies

$$\left|zf'(z)\hat{f}(z)\right| \leq nM|f(z)| \quad (z \in \mathbb{U}),$$

then

$$\Re\Bigg\{\Big\{2z^2S_2[f](z)+z^3S_3[f](z)\Big\}(\hat{f}(z))^{-1}$$
$$+3(\hat{f}(z))^2+3\left(\frac{zf'(z)}{f(z)}\right)^2+4z^2S_2[f](z)-3\Bigg\}<\frac{3}{2}\quad(z\in\mathbb{U})$$

implies

$$\left|\frac{zf'(z)}{f(z)}-1\right|<M\quad(z\in\mathbb{U}).$$

Now, we specialize the class of admissible functions and corresponding theorems for the second case of $q(\mathbb{U})$ being the half-plane $\Delta:=\{w:\Re\{w\}>0\}$. The function

$$q(z)=\frac{1+z}{1-z}\quad(z\in\mathbb{U}) \tag{3.2}$$

is convex univalent in $\mathbb{U}$ and satisfies $q(\mathbb{U})=\Delta$, $q(0)=1$ and $q\in\mathcal{Q}_1$. Straightforward calculation lead to

$$\xi q'(\xi)=\frac{2\xi}{(1-\xi)^2}=-\frac{1}{2}(1+|q(\xi)|^2),$$
$$\Re\left\{1+\frac{\xi q''(\xi)}{q'(\xi)}\right\}=\Re\left\{\frac{1+\xi}{1-\xi}\right\}=0$$

and

$$\Re\left\{\frac{\xi^2q^{(3)}(\xi)}{q'(\xi)}\right\}=\Re\left\{\frac{6\xi^2}{(1-\xi)^2}\right\}=\frac{3}{2}(1-|q(\xi)|^2),$$

for $\xi\in\partial\mathbb{U}\setminus\{1\}$.

Using these equalities and Lemma 2.3, we can obtain the following lemma.

Lemma 3.4. *Let the function q be given by (3.2). Let $p(z)=1+a_nz^n+a_{n+1}z^{n+1}+\cdots$ be analytic in $\mathbb{U}$ with $p(z)\not\equiv 1$ and $n\geq 2$. If there exists a point $z_0=r_0\mathrm{e}^{\mathrm{i}\theta_0}\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}\setminus\{1\}$ such that $p(z_0)=q(\zeta_0)$, $p(\overline{\mathbb{U}}_{r_0})\subset q(\mathbb{U})$ and*

$$|zp'(z)||1-\xi|^2\leq 2n,$$

when $z\in\overline{\mathbb{U}}_{r_0}$ and $\xi\in\partial\mathbb{U}\setminus\{1\}$, then

$$z_0p'(z_0)=-\frac{n}{2}(1+|q(\zeta_0)|^2),$$
$$\Re\left\{1+\frac{z_0p''(z_0)}{p'(z_0)}\right\}\geq 0$$

and

$$\Re\left\{\frac{z_0^2p^{(3)}(z_0)}{p'(z_0)}+\frac{3z_0p''(z_0)}{p'(z_0)}+1\right\}\geq n^2\left(1-3|q(\zeta_0)|^2\right).$$

We will use Lemma 3.4 and Definition 3 to define the class of admissible functions for the special function q defined by (3.2). We denote by $\Phi_n'[\Omega,q]$ by $\Phi_n'[\Omega]$, when q is given by (3.2).

Definition 5. Let Ω be a set in $\mathbb{C}$ and let $n \geq 2$. The class $\Phi'_n[\Omega]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(\mathrm{i}\rho, \mathrm{i}\sigma, \mu + \mathrm{i}\nu, \alpha + \mathrm{i}\beta; z) \notin \Omega, \tag{3.3}$$

when $\rho, \sigma, \mu, \nu, \alpha, \beta \in \mathbb{R}$ such that

$$\sigma = \rho + \frac{n(1+\rho^2)}{2\rho}, \tag{3.4}$$

$$\nu(\sigma - \rho) \leq 0, \tag{3.5}$$

$$\frac{2\nu + \beta}{\sigma - \rho} + 3\rho^2 + 3(\sigma - \rho)^2 - 4\mu + 2 \leq n^2\left(3\rho^2 - 1\right) \tag{3.6}$$

and $z \in \mathbb{U}$.

The above definition and Theorem 2.6 lead us to the following theorem.

Theorem 3.5. *Let $f \in \mathcal{A}_n$ with $n \geq 2$ satisfy*

$$|zf'(z)\hat{f}(z)||1-\xi|^2 \leq 2n|f(z)| \quad (z \in \mathbb{U},\ \xi \in \partial\mathbb{U} \setminus \{1\}).$$

If Ω is a set in $\mathbb{C}$ and $\phi \in \Phi'_n[\Omega]$, then

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, z^2 S_2[f](z), z^3 S_3[f](z); z\right) \in \Omega \quad (z \in \mathbb{U})$$

implies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Finally, as an example, we suggest a sufficient condition for functions to be starlike.

Example 3.6. Fix $n \geq 2$ and define a transformation ϕ_3 by

$$\phi_3(v, w, x, y; z) = 3(n^2 - 1)v^2 - \frac{2x + y}{w - v} - 3(w - v)^2 - 4x + n^2 + 2.$$

And let Ω be the right half plane Δ. We will show that $\phi_3 \in \Phi'_n[\Omega]$, by showing that (3.3) is satisfied. For this particular ϕ_3, we have

$$\begin{aligned} &\Re\left\{\phi_3(\mathrm{i}\rho, \mathrm{i}\sigma, \mu + \mathrm{i}\nu, \alpha + \mathrm{i}\beta; z)\right\} \\ &= n^2(1 - 3\rho^2) + \frac{2\nu + \beta}{\sigma - \rho} + 3(\sigma - \rho)^2 + 3\rho^2 - 4\mu + 2 \\ &\leq 0, \end{aligned}$$

that is,

$$\phi_3(\mathrm{i}\rho, \mathrm{i}\sigma, \mu + \mathrm{i}\nu, \alpha + \mathrm{i}\beta; z) \notin \Omega,$$

when $\rho, \sigma, \mu, \nu, \alpha, \beta \in \mathbb{R}$ satisfy (3.4), (3.5) and (3.6) and $z \in \mathbb{U}$. Hence, from Theorem 3.5, we obtain:
If $f(z) = z + a_{n+1}z^{n+1} + \cdots \in \mathcal{A}_n$ with $n \geq 2$ and $a_{n+1} \neq 0$ satisfies

$$|zf'(z)\hat{f}(z)||1-\xi|^2 \leq 2n|f(z)| \quad (z \in \mathbb{U},\ \xi \in \partial\mathbb{U} \setminus \{1\}),$$

then

$$\Re\Big\{3(n^2-1)\left\{(zf'(z))/(f(z))\right\}^2 - z^2(2S_2[f](z) + zS_3[f](z))(\hat{f}(z))^{-1}$$
$$-4z^2S_2[f](z) - 3(\hat{f}(z))^2 + n^2 + 2\Big\} > 0 \quad (z \in \mathbb{U})$$

implies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

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