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EXPONENTIAL FORM OF BIQUATERNIONIC VARIABLES IN CLIFFORD ANALYSIS

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Abstract. We give expressions of a biquaternion and research operations and calculations of each form of a biquaternion. Also, we investigate representations and properties of exponential and trigonometric forms of a biquaternion.

1. Introduction

An usual complex number $x + iy$ can be represented by the modulus ρ is $\sqrt{x^2 + y^2}$ and by the polar angle $\theta = \arctan(y/x)$. Also, the quaternions, which have been introduced by Hamilton, are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, while Kaledin [5] introduced many other hypercomplex systems whose multiplications are commutative. So, hypercomplex systems were applied by the development of the theory of functions of a complex variable (see [1, 16]). Leo and Grant et al. [2, 3] described complex numbers in n dimension by using hypercomplex bases and the variables in real numbers. Olariu [13, 14] presented properties of the usual hypercomplex numbers, polar and planar hypercomplex numbers in two to six dimensions which were played between the algebraic and the geometric relations. Ornea et al. [15] described a diagram containing the zero sets of the moment maps associated with the diagonal and actions on the quaternionic projective space. Whittaker et al. [17] gave the theory of functions of a complex variable and presented the study of such functions and their expansions as special functions and their related differential equations. Also, Kajiwara et al. [4] gave a basic estimate for inhomogeneous Cauchy-Riemann system and applied the theory to a closed densely defined operator in a Hilbert space. Kim et al. [6, 7] obtained the regularity of functions on the reduced quaternion field in Clifford analysis, and for the regularity of functions on the

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form of dual split quaternions in Clifford analysis. Kim et al. [8, 9] gave the definition and properties of hyperholomorphic functions in special quaternions and investigated their derivative and integral operations. Also, Kim et al. [10, 11] researched corresponding Cauchy-Riemann systems and properties of functions with values in special quaternions such as reduced quaternions and split quaternions by using a regular function with values in dual split quaternions. Kim et al. [12] researched properties and calculations of functions of bicomplex variables in bicomplex numbers which are commutative for multiplication rule.

In this paper, the system of hypercomplex numbers in four dimensions will be described for which the multiplication is associative and commutative, and some properties such that exponential and trigonometric forms exist can be defined and researched.

2. Preliminaries

A biquaternion is determined as the ordered pair by its four real components (x_0, x_1, x_2, x_3). For two biquaternions (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3), we define a sum and a product of the biquaternions as follows:

$$(x_0, x_1, x_2, x_3) + (y_0, y_1, y_2, y_3) = (x_0 + y_0, x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

and

$$\begin{aligned}&(x_0, x_1, x_2, x_3)(y_0, y_1, y_2, y_3)\\ &= (x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3, x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2,\\ &\quad x_0y_2 + x_2y_0 + x_1y_3 + x_3y_1, x_0y_3 + x_3y_0 - x_1y_2 - x_2y_1).\end{aligned}$$

Any biquaternion can be written by

$$p = (x_0, 0, 0, 0) + (0, x_1, 0, 0) + (0, 0, x_2, 0) + (0, 0, 0, x_3),$$

and it is seen that

$$\begin{aligned}&(0, 1, 0, 0)(x_1, 0, 0, 0) = (0, x_1, 0, 0),\ (0, 0, 1, 0)(x_2, 0, 0, 0) = (0, 0, x_2, 0),\\ &(0, 0, 0, 1)(x_3, 0, 0, 0) = (0, 0, 0, x_3).\end{aligned}$$

Hence, we have

$$\begin{aligned}p = &(x_0, 0, 0, 0) + (0, 1, 0, 0)(x_1, 0, 0, 0)\\ &+ (0, 0, 1, 0)(x_2, 0, 0, 0) + (0, 0, 0, 1)(x_3, 0, 0, 0),\end{aligned}$$

and let i, j and k denote the pure imaginary number $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, respectively. Then we have

$$p = x_0 + ix_1 + jx_2 + kx_3,$$

where i, j and k are called basis elements whose products satisfy the following rules:

$$i^2 = j^2 = -1,\ k^2 = 1,\ ij = ji = -k,\ ik = ki = j,\ jk = kj = i.$$

Then we give a set of biquaternions:

$$\mathbb{BQ} := \{p = x_0 + ix_1 + jx_2 + kx_3 \mid x_t \in \mathbb{R}\ (t = 0, 1, 2, 3)\}.$$

If $x_0 = y_0$, $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$, then two biquaternions $p = x_0 + ix_1 + jx_2 + kx_3$ and $q = y_0 + iy_1 + jy_2 + ky_3$ are said to be equal. For two biquaternions p and q, the sum $p+q$ and the product pq which are defined above can be expressed as follows:

$$p + q = (x_0 + y_0) + i(x_1 + y_1) + j(x_2 + y_2) + k(x_3 + y_3)$$

and

$$\begin{aligned} pq = {}& (x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3) + i(x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2) \\ &+ j(x_0y_2 + x_2y_0 + x_1y_3 + x_3y_1) + k(x_0y_3 + x_3y_0 - x_1y_2 - x_2y_1). \end{aligned}$$

Proposition 2.1. *The product satisfies the following properties: For any biquaternions p, p_1 and p_2,*

1. *Associative: $(pp_1)p_2 = p(p_1p_2)$,*
2. *Commutative: $pp_1 = p_1p$,*
3. *Existence of an identity: there is a unique identity 1 such that $p1 = 1p = p$,*
4. *Existence of an inverse: there is an inverse element p^{-1} such that $pp^{-1} = p^{-1}p = 1$.*

According to the above properties, which defines the product of two biquaternions, p and q satisfy the following equations:

$$x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3 = 1,\ x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2 = 0,$$

$$x_0y_2 + x_2y_0 + x_1y_3 + x_3y_1 = 0,\ x_0y_3 + x_3y_0 - x_1y_2 - x_2y_1 = 0.$$

The linear simultaneous equations have the solution:

$$\begin{aligned} y_0 &= \frac{x_0(x_0^2 + x_1^2 + x_2^2 - x_3^2) - 2x_1x_2x_3}{r^4}, \\ y_1 &= \frac{x_1(-x_0^2 - x_1^2 + x_2^2 - x_3^2) - 2x_0x_2x_3}{r^4}, \\ y_2 &= \frac{x_2(-x_0^2 + x_1^2 - x_2^2 - x_3^2) - 2x_0x_1x_3}{r^4}, \\ y_3 &= \frac{x_3(-x_0^2 + x_1^2 + x_2^2 + x_3^2) - 2x_0x_1x_2}{r^4}, \end{aligned}$$

where a nonzero real number r is

$$\begin{aligned} r^4 = {}& x_0^4 + x_1^4 + x_2^4 + x_3^4 + 2(x_0^2x_1^2 + x_0^2x_2^2 - x_0^2x_3^2 - x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) \\ &- 8x_0x_1x_2x_3. \end{aligned}$$

We set

$$r_1^2 := (x_0 + x_3)^2 + (x_1 + x_2)^2,\ r_2^2 := (x_0 - x_3)^2 + (x_1 - x_2)^2.$$

Since

$$(1) \qquad r^4 = r_1^2 r_2^2 = \{(x_0 + x_3)^2 + (x_1 + x_2)^2\}\{(x_0 - x_3)^2 + (x_1 - x_2)^2\},$$

a biquaternion p has an inverse, except for $x_0 + x_3 = 0$, $x_1 + x_2 = 0$ or $x_0 - x_3 = 0$, $x_1 - x_2 = 0$. The biquaternion p can be written by the point P of coordinates (x_0, x_1, x_2, x_3), and then the distance $|p|$ from P to the origin O can be represented as

$$|p|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

called modulus of the biquaternion p.

Theorem 2.2. *Let an orientation in the four dimensional space of the line OP be expressed with three angles θ, σ and ϕ defined with respect to the rotated system of*

$$\alpha = \frac{x_0 + x_3}{\sqrt{2}}, \ \beta = \frac{x_1 + x_2}{\sqrt{2}}, \ \gamma = \frac{x_0 - x_3}{\sqrt{2}}, \ \delta = \frac{x_1 - x_2}{\sqrt{2}}.$$

Then we have

$$\begin{aligned} p &= x_0 + ix_1 + jx_2 + kx_3 \\ &= \frac{|p|}{\sqrt{2}}\{\cos\phi(\cos\sigma + i\sin\sigma - j\sin\sigma - k\cos\sigma) \\ &\quad + \sin\phi(\cos\theta + i\sin\theta + j\sin\theta + k\cos\theta)\}. \end{aligned}$$

Proof. For convenience for the representation of the biquaternions in exponential and trigonometric form, the variables α, β, γ and δ are used for the definition of the angles θ, σ and ϕ. That is, we give angles θ, σ and ϕ by using α, β, γ and δ as axes (see Figure 1). Hence, we obtain θ is the angle between the projection of P in the plane α, β and the $O\alpha$ axis, $0 \le \theta < 2\pi$, σ is the angle between the projection of P in the plane γ, δ and the $O\gamma$ axis, $0 \le \sigma < 2\pi$ and ϕ is the angle between the line OP and the plane $\gamma O\delta$, $0 \le \phi \le \pi/2$, where the inequality $0 \le \phi \le \pi/2$ means that ϕ has the same sign on both faces of the two dimensional hyperplane $\beta O\delta$.

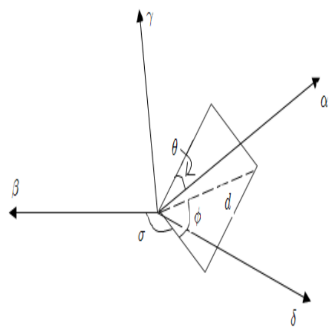

FIGURE 1. Hyperplane

Thus, we have the following components of the point P:

$$\frac{x_0 + x_3}{\sqrt{2}} = |p| \cos\theta \sin\phi, \ \frac{x_1 + x_2}{\sqrt{2}} = |p| \sin\theta \sin\phi,$$

$$\frac{x_0 - x_3}{\sqrt{2}} = |p| \cos\sigma \cos\phi, \ \frac{x_1 - x_2}{\sqrt{2}} = |p| \sin\sigma \cos\phi$$

and we obtain

$$x_0 = \frac{|p|}{\sqrt{2}}(\cos\phi\cos\sigma + \sin\phi\cos\theta), \ x_1 = \frac{|p|}{\sqrt{2}}(\cos\phi\sin\sigma + \sin\phi\sin\theta),$$

$$x_2 = \frac{|p|}{\sqrt{2}}(-\cos\phi\sin\sigma + \sin\phi\sin\theta), \ x_3 = \frac{|p|}{\sqrt{2}}(-\cos\phi\cos\sigma + \sin\phi\cos\theta). \quad \square$$

Let $r_+ = \sqrt{2}|p|\sin\phi$ and $r_- = \sqrt{2}|p|\cos\phi$. Then the angles θ, σ and ϕ can be expressed in terms of the coordinates as

$$\sin\theta = \frac{x_1 + x_2}{\sqrt{2}|p|\sin\phi}, \ \cos\theta = \frac{x_0 + x_3}{\sqrt{2}|p|\sin\phi},$$

$$\sin\sigma = \frac{x_1 - x_2}{\sqrt{2}|p|\cos\phi}, \ \cos\theta = \frac{x_0 - x_3}{\sqrt{2}|p|\cos\phi}.$$

Example 2.3. For any point in the hyperplane $\alpha O\gamma$, for which $\beta = 0$ and $\delta = 0$, a planar angle is $\phi = \frac{\pi}{2}$. Also, for the point in the hyperplane $\beta O\delta$, satisfying $\alpha = 0$ and $\gamma = 0$, a planar angle is $\phi = 0$.

From the notations of moduli and angles, if p and q are biquaternions, then we also write the product for biquaternions as follows:

$$r = r_1 r_2, \ \theta = \theta_1\theta_2, \ \sigma = \sigma_1\sigma_2, \ \tan\phi = \tan\phi_1 \tan\phi_2.$$

2.1. The exponential and trigonometric forms of biquaternions

An exponential function of a biquaternion variable p and the addition theorem for the exponential function can be written as

$$\exp(p) = \exp(x_0)\exp(ix_1)\exp(jx_2)\exp(kx_3)$$

for $p = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{BQ}$. Then we have the following forms:

$$\exp(ix_1) = \cos x_1 + i\sin x_1, \ \exp(jx_2) = \cos x_2 + j\sin x_2,$$
$$\exp(kx_3) = \cosh x_3 + k\sinh x_3.$$

Remark 2.4. By the mathematical induction, we have

$$(\cos x_1 + i\sin x_1)^n = \cos(nx_1) + i\sin(nx_1),$$

$$(\cos x_2 + j\sin x_2)^n = \cos(nx_2) + j\sin(nx_2),$$

$$(\cosh x_3 + k\sinh x_3)^n = \cosh(nx_3) + k\sinh(nx_3).$$

Theorem 2.5. *Let a biquaternion p be written in the form*

$$x_0 + ix_1 + jx_2 + kx_3 = \exp(t_0 + it_1 + jt_2 + kt_3),$$

where the representations of $t_0,\ t_1,\ t_2,\ t_3$ *as functions of* $x_0,\ x_1,\ x_2,\ x_3$. *Then we have*

$$t_0 = \log r,\ \ t_1 = \frac{1}{2}\sin^{-1}\frac{2(x_0x_1 - x_2x_3)}{r^2},$$

$$t_2 = \frac{1}{2}\sin^{-1}\frac{2(x_0x_2 - x_1x_3)}{r^2},\ \ t_3 = \frac{1}{2}\sinh^{-1}\frac{2(x_0x_3 + x_1x_2)}{r^2},$$

where r *is in* (1).

Proof. By multiplying these expressions and separating the hypercomplex components, the representations of $t_0,\ t_1,\ t_2,\ t_3$ as functions of $x_0,\ x_1,\ x_2,\ x_3$ are obtained by calculating $\exp(ix_1),\ \exp(jx_2),\ \exp(kx_3)$. Hence, we have

$$x_0 = \exp(t_0)(\cos t_1 \cos t_2 \cosh t_3 - \sin t_1 \sin t_2 \sinh t_3),$$

$$x_1 = \exp(t_0)(\sin t_1 \cos t_2 \cosh t_3 + \cos t_1 \sin t_2 \sinh t_3),$$

$$x_2 = \exp(t_0)(\cos t_1 \sin t_2 \cosh t_3 + \sin t_1 \cos t_2 \sinh t_3),$$

$$x_3 = \exp(t_0)(-\sin t_1 \sin t_2 \cosh t_3 + \cos t_1 \cos t_2 \sinh t_3),$$

by distributing with $\exp(ix_1),\ \exp(jx_2),\ \exp(kx_3)$ and comparing for each basis in biquaternions. From the above equations, we obtain

$$\begin{aligned} d^2 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ &= \exp(2t_0)(\cos^2 t_1 \cos^2 t_2 \cosh^2 t_3 + \sin^2 t_1 \sin^2 t_2 \sinh^2 t_3 \\ &\quad + \sin^2 t_1 \cos^2 t_2 \cosh^2 t_3 + \cos^2 t_1 \sin^2 t_2 \sinh^2 t_3 \\ &\quad + \cos^2 t_1 \sin^2 t_2 \cosh^2 t_3 + \sin^2 t_1 \cos^2 t_2 \sinh^2 t_3 \\ &\quad + \sin^2 t_1 \sin^2 t_2 \cosh^2 t_3 + \cos^2 t_1 \cos^2 t_2 \sinh^2 t_3) \\ &= \exp(2t_0)(\cos^2 t_2 \cosh^2 t_3 + \sin^2 t_2 \sinh^2 t_3 \\ &\quad + \sin^2 t_2 \cosh^2 t_3 + \cos^2 t_2 \sinh^2 t_3) \\ &= \exp(2t_0)(\cosh^2 t_3 + \sinh^2 t_3) = \exp(2t_0)(2\cosh^2 t_3 - 1) \\ &= \exp(2t_0)\cosh(2t_3) \end{aligned}$$

and

$$2(x_0x_3 + x_1x_2) = \exp(2x_1)\sinh(2t_3)$$

so that

$$\exp(4x_1) = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 - 4(x_0x_3 + x_1x_2)^2.$$

From the expression of the equation (1), we have $\exp(x_1) = r$, and then $x_1 = \log r$. By using the equation

$$r^4 = d^4 - 4(x_0x_3 - x_1x_2)^2,$$

we have t_3 as the variable satisfied by

$$\cosh 2t_3 = \frac{d^2}{r^2},\ \sinh 2t_3 = \frac{2(x_0x_3 + x_1x_2)}{r^2}.$$

Hence, the above equations follow that $\frac{d^2}{r^2} \geq 1$ and $t_3 = 0$ for $x_0x_3 + x_1x_2 = 0$. Similarly, by

$$(x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 \leq r^4 \quad \text{and} \quad (x_0^2 + x_1^2 - x_2^2 - x_3^2)^2 \leq r^4,$$

we have

$$\cos 2t_1 = \frac{x_0^2 - x_1^2 + x_2^2 - x_3^2}{r^2},\ \sin 2t_1 = \frac{2(x_0x_1 - x_2x_3)}{r^2},$$

$$\cos 2t_2 = \frac{x_0^2 + x_1^2 - x_2^2 - x_3^2}{r^2},\ \sin 2t_2 = \frac{2(x_0x_2 - x_1x_3)}{r^2}.$$

Therefore, we obtain the results. □

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