Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 75–84
https://doi.org/10.4134/CKMS.c160043
pISSN: 1225-1763 / eISSN: 2234-3024

GENERATING FUNCTIONS FOR THE EXTENDED WRIGHT TYPE HYPERGEOMETRIC FUNCTION

Ranjan Kumar Jana, Bhumika Maheshwari, and Ajay Kumar Shukla

Abstract. In recent years, several interesting families of generating functions for various classes of hypergeometric functions were investigated systematically. In the present paper, we introduce a new family of extended Wright type hypergeometric function and obtain several classes of generating relations for this extended Wright type hypergeometric function.

1. Introduction

The Gauss hypergeometric function [7] is defined as

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \qquad (|z|<1,\ c\neq 0,-1,-2,\ldots) \tag{1}$$

where $(\lambda)_v,\ (\lambda, v\in\mathbb{C})$ denotes the Pochhammer symbol defined by

$$(\lambda)_v:=\frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}=\begin{cases}1 & (v=0;\ \lambda\in\mathbb{C}\backslash\{0\}),\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (v\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases} \tag{2}$$

it being assumed conventionally that $(0)_0:=1$ and understood tacitly that the Γ-quotient exists (see, for details, [11]).

The classical generalized hypergeometric function [4] is defined by

$$ {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p;\\ b_1,\ldots,b_q;\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \qquad (p=q+1,\ |z|<1) \tag{3}$$

where no denominator parameter is zero or negative integer.

Wright [16] further extended the generalized hypergeometric function in the following form

$$ {}_p\Psi_q(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+\beta_1 n)\cdots\Gamma(\alpha_p+\beta_p n)}{\Gamma(\rho_1+\mu_1 n)\cdots\Gamma(\rho_q+\mu_q n)}\frac{z^n}{n!}, \tag{4}$$

Received February 29, 2016; Revised June 13, 2016.
2010 *Mathematics Subject Classification.* 33B15, 33B99, 33C05, 33C20, 33C99.
Key words and phrases. generating functions, gamma and beta functions, hypergeometric functions, Wright type hypergeometric function, linear, bilinear and bilateral (or mixed multilateral) generating functions.

where β_r and μ_t are real positive numbers such that

$$1+\sum_{t=1}^{q}\mu_t-\sum_{r=1}^{p}\beta_r>0.$$

When β_r and μ_t are equal to 1 then equation (4) differs from the generalized hypergeometric function ${}_pF_q(z)$ only by a constant multiplier.

This generalized form of the hypergeometric function has been investigated by Dotsenko [3], Malovichko [5] and others. One of the special cases of this generalized form of the hypergeometric function considered in [3] has the form

$$(5)\quad {}_2R_1^{\omega,\mu}(z)={}_2R_1(a,b;c;\omega,\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!}.$$

In 2001, Virchenko et al. [15] defined the said Wright type hypergeometric function by taking $\frac{\omega}{\mu}=\tau>0$ in (5) as

$$(6)\qquad {}_2R_1^{\tau}(z)={}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!},$$

whose various properties have been studied by Rao et al. [8, 9].

Recently, Parmar [6] introduced the following extension of the τ-hypergeometric function (6) as

$$(7)\qquad {}_2R_1^{\tau}((a,p),b;c;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!},$$

$$(\Re(p)>0;\ \tau>0,\ |z|<1;\ \Re(c)>\Re(b)>0 \text{ when } p=0)$$

in terms of the generalized Pochhammer symbol [10]

$$(8)\qquad (\lambda;p)_v=\begin{cases}\dfrac{\Gamma_p(\lambda+v)}{\Gamma(\lambda)} & (\Re(p)>0;\ \lambda,v\in\mathbb{C}),\\ (\lambda)_v & (p=0;\lambda,v\in\mathbb{C}).\end{cases}$$

The generalized gamma function $\Gamma_p(z)$ involved in the definition (8) is introduced by Chaudhry and Zubair [1] (see also [2]) as

$$(9)\qquad \Gamma_p(z)=\begin{cases}\displaystyle\int_0^{\infty}t^{z-1}\exp\left(-t-\frac{p}{t}\right)dt & (\Re(p)>0;\ z\in\mathbb{C}),\\ \Gamma(z) & (p=0;\ \Re(z)>0).\end{cases}$$

In 2013, Srivastava [13] defined a family of the generalized and extended hypergeometric function ${}_u\mathcal{F}_v$ as

$$(10)\qquad {}_u\mathcal{F}_v\left[\begin{array}{r}\left(a_0;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),a_2,\ldots,a_u;\\ b_1,\ldots,b_v;\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\left(a_0;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right)_n(a_2)_n\cdots(a_u)_n}{(b_1)_n\cdots(b_v)_n}\frac{z^n}{n!}$$

in terms of the generalized and extended Pochhammer symbol [13] defined by

$$(\lambda; p, \{\kappa_l\}_{l\in\mathbb{N}_0})_v = \frac{\Gamma_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda+v)}{\Gamma_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(\lambda)} \quad (\lambda, v \in \mathbb{C}), \tag{11}$$

provided that the series on the right hand side of (10) converges.

The function $\Gamma_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(z)$ is the generalized and extended gamma function [12] defined by

$$\Gamma_p^{(\{\kappa_l\}_{l\in\mathbb{N}_0})}(z) = \int_0^\infty t^{z-1}\,\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0}; -t-\frac{p}{t}\right)dt, \ (\Re(z)>0; \Re(p)\geq 0) \tag{12}$$

where, for an appropriately bounded sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$ of essentially arbitrary (real or complex) number, the function $\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0}; z\right)$ is due to Srivastava [13]
(13)

$$\Theta\left(\{\kappa_l\}_{l\in\mathbb{N}_0}; z\right) = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\kappa_l\frac{z^l}{l!} & (|z|<R;\ R>0;\ \kappa_0 := 1), \\ \mathfrak{M}_0\, z^{\omega}\, \exp(z)\left[1+O\left(\frac{1}{|z|}\right)\right] & (|z|\to\infty;\ \mathfrak{M}_0>0;\ \omega\in\mathbb{C}) \end{cases}$$

for some suitable constant $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$.

Motivated by the aforecited investigations of hypergeometric function, we introduce an extension of the τ-Wright type hypergeometric function (6) by using the generalized and extended Pochhammer symbol defined by (11). We then derive several classes of generating relations involving this extended τ-hypergeometric function. We also consider some interesting special cases from our main results.

2. Extension of τ-Wright type hypergeometric function

In terms of the generalized and extended Pochhammer symbol

$$(\lambda; p, \{\kappa_l\}_{l\in\mathbb{N}_0})_v$$

defined by (11), we introduce the extension of the τ-Wright type hypergeometric function (6) as follows

$$ {}_2R_1^{\tau}\left((a; p, \{\kappa_l\}_{l\in\mathbb{N}_0}), b; c; z\right) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a; p, \{\kappa_l\}_{l\in\mathbb{N}_0})_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}, \tag{14}$$

$$(\Re(p)\geq 0,\ \tau>0,\ |z|<1)$$

where c is never zero or negative integer.

Remark 2.1. In a very special case, when $\kappa_l \equiv 1$ ($l \in \mathbb{N}_0$) and $p = 0$ in the denominator in definition (11), the generalized and extended Pochhammer symbol $(\lambda; p, \{\kappa_l\}_{l\in\mathbb{N}_0})_v$ reduces to the generalized Pochhammer symbol (8), while (14) reduces to the extended τ-hypergeometric function (7). If in (11), we have $p = 0$ also in the numerator, we get the τ-Wright type hypergeometric function (6) as a special case of (14).

Also, (10) becomes the extended τ-Wright type hypergeometric function (14) on setting $u = 2$, $v = 1$ in (10) and $\tau = 1$ in (14).

The extended τ-Wright type hypergeometric function (14) has the following integral representation

$$
\begin{aligned}
&{}_2R_1^{\tau}\left(\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),b;c;z\right)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1\mathcal{F}_0\left[\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right);-;zt^{\tau}\right]dt,\\
&(\Re(c) > \Re(b) > 0).
\end{aligned}
\tag{15}
$$

We can easily derive (15) by using $\frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} = \frac{B(b+\tau n,c-b)}{\Gamma(c-b)}$, $(\Re(c) > \Re(b) > 0)$ and applying the definition of beta function [7]

$$
B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad (\Re(\alpha) > 0;\ \Re(\beta) > 0). \tag{16}
$$

3. Results on generating functions

In this section, we derive generating functions for the extended τ-Wright type hypergeometric function defined by (14). Our main results are asserted by Theorem 3.1 and Theorem 3.3 given below.

First of all, the generalized binomial coefficient $\binom{\lambda}{\mu}$ is defined (for real or complex parameters λ and μ) as (see [14]):

$$
\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \ (\lambda,\mu\in\mathbb{C}), \tag{17}
$$

so that, in the special case when $\mu = n$ $(n \in \mathbb{N}_0)$, we have

$$
\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \ (n\in\mathbb{N}_0). \tag{18}
$$

Theorem 3.1. *The following generating relations hold true for the extended τ-Wright type hypergeometric function ${}_2R_1^{\tau}$ defined by* (14)

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\,{}_2R_1^{\tau}\left[\begin{array}{c}\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),\lambda+n;\\ c;\end{array} z\right]t^n\\
&= (1-t)^{-\lambda}\,{}_2R_1^{\tau}\left[\begin{array}{c}\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),\lambda;\\ c;\end{array} \frac{z}{(1-t)^{\tau}}\right]\\
&\quad(|t| < 1;\ \lambda\in\mathbb{C};\ \tau > 0),
\end{aligned}
\tag{19}
$$

and

$$
\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\,{}_2R_1^{\tau}\left[\begin{array}{c}\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),b;\\ 1-\lambda-n;\end{array} z\right]t^n
$$

$$= (1-t)^{-\lambda}\, {}_2R_1^{\tau}\left[\begin{array}{cc} \left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),b; & z(1-t)^{\tau} \\ 1-\lambda; & \end{array}\right] \tag{20}$$

$$(|t|<1;\ \lambda\in\mathbb{C};\ \tau>0)$$

provided that both members of each of the generating functions (19) *and* (20) *exist.*

Proof. By using the definitions of the generalized bionomial coefficient (18) and the extended τ-Wright type hypergeometric function (14), we have the left hand side of the relation (19) as

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}\lambda+n-1\\ n\end{array}\right){}_2R_1^{\tau}\left[\begin{array}{cc} \left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),\lambda+n; & z \\ c; & \end{array}\right]t^n$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(-\lambda-n+1)_n}{n!}\left(\frac{\Gamma(c)}{\Gamma(\lambda+n)}\sum_{k=0}^{\infty}\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right)_k\frac{\Gamma(\lambda+n+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}\right)t^n$$

$$= \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{\Gamma(c)}{\Gamma(\lambda+n)}\sum_{k=0}^{\infty}\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right)_k\frac{\Gamma(\lambda+n+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}\right)t^n$$

$$= \frac{\Gamma(c)}{\Gamma(\lambda)}\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}(\lambda+\tau k)_n\frac{t^n}{n!}\right)\frac{\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right)_k\Gamma(\lambda+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!},$$

where in the last step, we used $\frac{(\lambda)_n\Gamma(\lambda+n+\tau k)}{\Gamma(\lambda+n)} = (\lambda+\tau k)_n\frac{\Gamma(\lambda+\tau k)}{\Gamma(\lambda)}$ and inverted the order of the summation.

Now, by applying the generalized bionomial expansion [7], $\sum_{n=0}^{\infty}(a)_n\frac{t^n}{n!} = (1-t)^{-a}$, we are led to the assertion (19).

In order to derive assertion (20), we use $\frac{(\lambda)_n\Gamma(1-\lambda-n)}{\Gamma(1-\lambda-n+\tau k)} = (\lambda-\tau k)_n\frac{\Gamma(1-\lambda)}{\Gamma(1-\lambda+\tau k)}$ and apply similar steps of the proof of assertion (19). □

Remark 3.2. We can get the same generating relations (19) and (20) of Theorem 3.1 for the extended τ-hypergeometric function (7) due to Parmar [6].

Next, we introduce the sequences $\left\{\varphi_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ and $\left\{\theta_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ of the extended τ-Wright type hypergeometric function as

$$\varphi_n^{(\lambda,\tau)}(z) = {}_2R_1^{\tau}\left(\begin{array}{cc} \left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),\lambda+n; & z \\ c; & \end{array}\right), \tag{21}$$

and

$$\theta_n^{(\lambda,\tau)}(z) = {}_2R_1^{\tau}\left(\begin{array}{cc} \left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),b; & z \\ 1-\lambda-n; & \end{array}\right). \tag{22}$$

Theorem 3.3. *The following generating relations hold true for the sequences* $\left\{\varphi_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ *and* $\left\{\theta_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ *respectively defined by* (21) *and* (22)

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}\lambda+m+n-1\\ n\end{array}\right)\varphi_{m+n}^{(\lambda,\tau)}(z)t^n=(1-t)^{-\lambda-m}\,\varphi_m^{(\lambda,\tau)}\left(\frac{z}{(1-t)^{\tau}}\right) \tag{23}$$

$$(|t|<1;\ m\in\mathbb{N}_0;\ \lambda\in\mathbb{C};\ \tau>0),$$

and

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}\lambda+m+n-1\\ n\end{array}\right)\theta_{m+n}^{(\lambda,\tau)}(z)t^n=(1-t)^{-\lambda-m}\,\theta_m^{(\lambda,\tau)}(z(1-t)^{\tau}) \tag{24}$$

$$(|t|<1;\ m\in\mathbb{N}_0;\ \lambda\in\mathbb{C};\ \tau>0).$$

Proof. By substituting for the sequence $\left\{\varphi_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ and using the definitions of the generalized bionomial coefficient (18) and the extended τ-Wright type hypergeometric function (14), and inverting the order of the summation, we have the left hand side of the relation (23) as

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}\lambda+m+n-1\\ n\end{array}\right)\varphi_{m+n}^{(\lambda,\tau)}(z)t^n$$
$$=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}(\lambda+m+\tau k)_n\frac{t^n}{n!}\right)\frac{\Gamma(c)}{\Gamma(\lambda+m)}\frac{\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right)_n\Gamma(\lambda+m+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}.$$

Further employing the generalized bionomial expansion, this leads to the assertion (23).

The same method of the proof of assertion (23) can be applied for the sequence $\left\{\theta_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ in order to derive the second assertion (24) of the theorem. □

4. Applications

In this section, we use Theorem 3.3 in order to derive various interesting classes of linear, bilinear and bilateral (or mixed multilateral) generating functions for the sequences $\left\{\varphi_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ and $\left\{\theta_n^{(\lambda,\tau)}(z)\right\}_{n\in\mathbb{N}_0}$ respectively defined by (21) and (22) in terms of the extended τ-Wright type hypergeometric function ${}_2R_1^{\tau}\left(\left(a;p,\{\kappa_l\}_{l\in\mathbb{N}_0}\right),b;c;z\right)$ defined by (14).

Theorem 4.1. *Corresponding to a non-vanishing function* $\Pi_\mu(\xi_1,\ldots,\xi_s)$ *of* s *complex variables* $\xi_1,\ldots,\xi_s$ $(s\in\mathbb{N})$ *and involving a complex parameter* μ, *called the order, let*

$$\Omega_{m,p,q}^{(1)}[z;\xi_1,\ldots,\xi_s;t]:=\sum_{n=0}^{\infty}a_n\varphi_{m+nq}^{(\lambda+\sigma nq,\tau)}(z)\cdot\Pi_{\mu+np}(\xi_1,\ldots,\xi_s)t^n \tag{25}$$

$$(a_n\neq 0;m\in\mathbb{N}_0;p,q\in\mathbb{N};\tau>0,\sigma\in\mathbb{C}),$$

and

$$\Omega^{(2)}_{m,p,q}\left[z;\xi_1,\ldots,\xi_s;t\right] := \sum_{n=0}^{\infty} a_n \theta^{(\lambda+\sigma nq,\tau)}_{m+nq}(z)\Pi_{\mu+np}\left(\xi_1,\ldots,\xi_s\right)t^n \tag{26}$$

$$\left(a_n \neq 0;\ m \in \mathbb{N}_0;\ p,q \in \mathbb{N};\ \tau > 0; \sigma \in \mathbb{C}\right).$$

Suppose also that

$$\begin{aligned}(27)\quad &\Theta^{\lambda,p,\mu,\sigma}_{m,n,q}(z;\xi_1,\ldots,\xi_s;\eta)\\ &:= \sum_{l=0}^{[n/q]}\binom{\lambda+m+\sigma ql+n-1}{n-ql} a_l\,\varphi^{(\lambda+\sigma ql,\tau)}_{m+n}(z)\cdot\Pi_{\mu+pl}(\xi_1,\ldots,\xi_s)\eta^l,\end{aligned}$$

and

$$\begin{aligned}(28)\quad &\Phi^{\lambda,p,\mu,\sigma}_{m,n,q}\left(z;\xi_1,\ldots,\xi_s;\eta\right)\\ &:= \sum_{l=0}^{[n/q]}\binom{\lambda+m+\sigma ql+n-1}{n-ql} a_l\,\theta^{(\lambda+\sigma ql,\tau)}_{m+n}(z)t^n\cdot\Pi_{\mu+pl}\left(\xi_1,\ldots,\xi_s\right)\eta^l,\end{aligned}$$

where the sequences $\left\{\varphi^{(\lambda,\tau)}_n(z)\right\}_{n\in\mathbb{N}_0}$ *and* $\left\{\theta^{(\lambda,\tau)}_n(z)\right\}_{n\in\mathbb{N}_0}$ *are defined by* (21) *and* (22)*, respectively. Then*

$$\begin{aligned}(29)\quad &\sum_{n=0}^{\infty}\Theta^{\lambda,p,\mu,\sigma}_{m,n,q}(z;\xi_1,\ldots,\xi_s;\eta)t^n\\ &= (1-t)^{-\lambda-m}\Omega^{(1)}_{m,p,q}\left[\frac{z}{(1-t)^{\tau}};\xi_1,\ldots,\xi_s;\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right]\quad(|t|<1),\end{aligned}$$

and

$$\begin{aligned}(30)\quad &\sum_{n=0}^{\infty}\Phi^{\lambda,p,\mu,\sigma}_{m,n,q}\left(z;\xi_1,\ldots,\xi_s;\eta\right)t^n\\ &= (1-t)^{-\lambda-m}\Omega^{(2)}_{m,p,q}\left[z(1-t)^{\tau};\xi_1,\ldots,\xi_s;\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right]\quad(|t|<1).\end{aligned}$$

Proof. By using the definition of $\Theta^{\lambda,p,\mu,\sigma}_{m,n,q}(z;\xi_1,\ldots,\xi_s;\eta)$ from (27) in (29), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\Theta^{\lambda,p,\mu,\sigma}_{m,n,q}(z;\xi_1,\ldots,\xi_s;\eta)t^n\\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{[n/q]}\binom{\lambda+m+\sigma ql+n-1}{n-ql}a_l\\ &\qquad\varphi^{(\lambda+\sigma ql,\tau)}_{m+n}(z)\cdot\Pi_{\mu+pl}(\xi_1,\ldots,\xi_s)\ \eta^l t^n\end{aligned}$$

$$= \sum_{l=0}^{\infty} a_l \, \Pi_{\mu+pl}(\xi_1, \ldots, \xi_s) \, (\eta t^q)^l$$

$$\sum_{n=0}^{\infty} \binom{\lambda + m + (\sigma+1)ql + n - 1}{n} \varphi_{m+n+ql}^{(\lambda+\sigma ql,\tau)}(z) \; t^n, \tag{31}$$

where we have inverted the order of the double summation involved.

By applying Theorem 3.3 on (31), we find

$$\sum_{n=0}^{\infty} \Theta_{m,n,q}^{\lambda,p,\mu,\sigma}(z; \xi_1, \ldots, \xi_s; \eta) t^n$$
$$= \sum_{l=0}^{\infty} a_l \, \Pi_{\mu+pl}(\xi_1, \ldots, \xi_s)(\eta t^q)^l \left((1-t)^{-\lambda-m-(\sigma+1)ql} \, \varphi_{m+ql}^{(\lambda+\sigma ql,\tau)} \left(\frac{z}{(1-t)^{\tau}} \right) \right)$$

which, in view of the definition (25), is precisely the right side of the assertion (29).

We have thus completed the proof of the assertion (29) under the assumption that the double series involved in the first two steps of our proof are absolutely (and uniformly) convergent. Thus, in general, the assertion (29) holds true at least as a relation between formal power series.

The same method of proof of the first assertion (29) can be applied in order to derive the second assertion (30) of Theorem 4.1, which would yield bilateral or mixed multilateral generating relations for the sequence $\left\{ \theta_n^{(\lambda,\tau)}(z) \right\}_{n \in \mathbb{N}_0}$ defined by (22) in terms of the extended τ-Wright type hypergeometric function ${}_2R_1^{\tau}\left(\left(a; p, \{\kappa_l\}_{l \in \mathbb{N}_0}\right), b; c; z \right)$ defined by (14). □

Remark 4.2. If we define the sequences $\left\{ \varphi_n^{(\lambda,\tau)}(z) \right\}_{n \in \mathbb{N}_0}$ and $\left\{ \theta_n^{(\lambda,\tau)}(z) \right\}_{n \in \mathbb{N}_0}$ in terms of Parmar's τ-hypergeometric function (7) as

$$\varphi_n^{(\lambda,\tau)}(z) = {}_2R_1^{\tau}\left((a; p), \lambda + n; c; z \right) \tag{32}$$

and

$$\theta_n^{(\lambda,\tau)}(z) = {}_2R_1^{\tau}\left((a; p), b; 1 - \lambda - n; z \right), \tag{33}$$

then the generating relations (23) and (24) of Theorem 3.3, and (29) and (30) of Theorem 4.1, still hold true for the sequences defined in (32) and (33).

Remark 4.3. We can further generalize the τ-Wright type hypergeometric function (14) as

$$\begin{aligned} &{}_pR_q^{\tau} \left[\begin{matrix} \left(a_1; p, \{\kappa_l\}_{l \in \mathbb{N}_0}\right), a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix} \; z \right] \\ &= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\left(a_1; p, \{\kappa_l\}_{l \in \mathbb{N}_0}\right)_n \Gamma(a_2 + \tau n) (a_3)_n \cdots (a_p)_n}{\Gamma(b_1 + \tau n) (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \end{aligned} \tag{34}$$

provided that the series on the right hand side converges.

On applying the methodology described in the above mentioned theorems, one can derive analogous generating relations involving the function (34).

Acknowledgment. The authors would like to thank the anonymous referee(s) for their valuable suggestions.

References

[1] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math. **55** (1994), no. 1, 99–123.

[2] ———, *On a Class of Incomplete Gamma Functions with Applications*, Chapman and Hall/CRC, Boca Raton, 2002.

[3] M. Dotsenko, *On some applications of Wright's hypergeometric function*, C. R. Acad. Bulgare Sci. **44** (1991), no. 6, 13–16.

[4] A. Erdélyi, W. Mangus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vol I*, McGraw-Hill, New York, 1953.

[5] V. Malovichko, *On a generalized hypergeometric function and some integral operators*, Math. Phys. **19** (1976), 99–103.

[6] R. K. Parmar, *Extended τ-hypergeometric functions and associated properties*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 5, 421–426.

[7] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[8] S. B. Rao, A. D. Patel, J. C. Prajapati, and A. K. Shukla, *Some properties of generalized hypergeometric function*, Commun. Korean Math. Soc. **28** (2013), no. 2, 303–317.

[9] S. B. Rao and A. K. Shukla, *Note on generalized hypergeometric function*, Integral Transforms Spec. Funct. **24** (2013), no. 11, 896–904.

[10] H. M. Srivastava, A. Çetinkaya, and I. Onur Kymaz, *A certain generalized Pochhammer symbol and its applications to hypergeometric functions*, Appl. Math. Comput. **226** (2014), 484–491.

[11] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited), New York, 1984.

[12] H. M. Srivastava, R. K. Parmar, and P. Chopra, *A class of extended fractional derivative oper ators and associated generating relations involving hypergeometric functions*, Axioms **1** (2012), no. 3, 238–258.

[13] R. Srivastava, *Some generalizations of Pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric polynomials*, Appl. Math. Inf. Sci. **7** (2013), no. 6, 2195–2206.

[14] ———, *Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions*, Appl. Math. Comput. **243** (2014), 132–137.

[15] N. Virchenko, S. L. Kalla, and A. Al-Zamel, *Some results on a generalized hypergeometric function*, Integral Transforms Spec. Funct. **12** (2001), no. 1, 89–100.

[16] E. M. Wright, *On the coefficient of power series having exponential singularities*, J. Lond. Math. Soc. **5** (1933), no. 1, 71–79.

Ranjan Kumar Jana
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat-395007, India
E-mail address: rkjana2003@yahoo.com, rkj@ashd.svnit.ac.in

Bhumika Maheshwari
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat-395007, India
E-mail address: bhumi0512@gmail.com, d13ma011@ashd.svnit.ac.in

Ajay Kumar Shukla
Department of Applied Mathematics & Humanities
S.V. National Institute of Technology
Surat-395007, India
E-mail address: ajayshukla2@rediffmail.com, aks@ashd.svnit.ac.in