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A COMMON FIXED POINT THEOREM FOR *T*-CONTRACTIONS ON GENERALIZED CONE b-METRIC SPACES

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Abstract. In this paper, we establish a unique common fixed point theorem for T-contraction of two self maps on generalized cone b-metric spaces with solid cone. The result of this paper improves and generalizes several well-known results in the literature. Two examples are also given to support the result.

1. Introduction and preliminaries

Let X be a non-empty set. A mapping $S : X \to X$ is called a self-map of X. If there is an element $x \in X$ such that $S(x) = x$, then x is called a fixed point of the self-map S of X. A result giving a set of conditions on S and X under which S has a fixed point is known as a fixed point theorem. In recent times fixed point theorems have gained importance because of their numerous applications. It is well known that the classical Banach contraction principle [3] is the first ever fixed point theorem. Many authors established the Banach contraction principle on certain spaces (see; [5], [6], [7], [8], [9]). In 1989, Bakhtin [2] introduced b-metric spaces as a generalization of metric spaces. In 2000, Branciari [5] introduced the notion of generalized (rectangular) metric, where the triangle inequality of a metric space was replaced by another inequality, the so called rectangular inequality which involves four or more points instead of three points. In 2007, L. G. Huang and X. Zhang [8] introduced the concept of cone metric spaces. They have replaced real number system by an ordered Banach space. In 2009, A. Azam, M. Arshad and I. Beg [1] introduced the concept of cone rectangular metric space. In 2011, Hussain and Shah [9] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Recently, R. George et al. [7] have introduced the concept of rectangular b-metric space, which is not necessarily Hausdorff

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and which generalizes the concept of metric space, rectangular metric space and b-metric space.

Very recently, R. George, et al. [6] have introduced the concept of generalized cone b-metric space, which generalizes the concepts of cone metric space, cone rectangular metric space and cone b-metric space. They have proved Banach fixed point theorem and Kannan fixed point theorem in generalized cone b-metric space with solid cone. A generalization of contraction mapping has been introduced and called T-contraction mapping on metric spaces which is depending on another function by Beiranvand [4].

In this paper, we obtain a unique common fixed point theorem for two self mappings which satisfy T-contraction mapping on generalized cone b-metric spaces. The main result of this paper extends and generalizes result of R. George, et al. [6] on generalized cone b-metric spaces.

The following definitions and results will be needed in the sequel.

Definition 1.1 ([8])**.** A subset P of a real Banach space E is called a *cone* if it has following properties:

(1) P is non-empty, closed and $P \neq \{\theta\}$, where, θ is a zero vector in E;

(2) $a, b \in \mathbb{R}$, a, $b \geq 0$, x, $y \in P \implies ax + by \in P$;

(3) $x \in P$ and $-x \in P \implies x = \theta$, *i.e.*, $P \cap (-P) = \{\theta\}$.

For a given cone $P \subset E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$ for $x, y \in E$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stands for $y - x \in int\,(P)$, where $int\,(P)$ denotes the interior of P. A cone P is called *solid* if $int\,(P) \neq \varnothing$.

Remark 1.2 ([10]). Let P be a cone in a real Banach space E and Let $a, b, c, x, y \in P$. The following properties hold:

(1) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(2) If $\theta \preceq x \ll c$ for each $c \in int\,(P)$, then $x = \theta$.

(3) If $a \preceq b + c$ for each $c \in int\,(P)$, then $a \preceq b$.

(4) If $\theta \preceq x \preceq y$ and $a \geq 0$, then $\theta \preceq ax \preceq ay$.

(5) If $\theta \preceq x_n \preceq y_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $\theta \preceq x \preceq y$.

(6) If $\theta \preceq d\,(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $d\,(x_n, x) \ll c$, where $\{x_n\}$ and x are respectively, a sequence and a given point in X.

(7) If $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = \theta$.

(8) If $c \in int\,(P)$, $\theta \preceq x_n$, and $x_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $x_n \ll c$.

Definition 1.3 ([8])**.** Let X be a non-empty set, E be a real Banach space and P be a solid cone in E and $\preceq$ is a partial ordering with respect to P. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $\theta \prec d\,(x, y)$ for all $x, y \in X$ and $d\,(x, y) = \theta$ if and only if $x = y$;

(2) $d\,(x, y) = d\,(y, x)$ for all $x, y \in X$;

(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x,y,z \in X$ [triangular inequality].
Then d is called a *cone metric* on X and the pair (X,d) is called a *cone metric space.*

Definition 1.4 ([9]). Let X be a non-empty set, E be a real Banach space, P be a solid cone in E, $\preceq$ be a partial ordering with respect to P and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $\theta \prec d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$, $x,y \in X$;

(3) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$ [b-triangular inequality].
Then d is called a *cone b-metric* on X and the pair (X,d) is called a *cone b-metric space.*

Definition 1.5 ([1]). Let X be a non-empty set, E be a real Banach space, P be a solid cone in E and $\preceq$ is a partial ordering with respect to P. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $\theta \prec d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $d(x,y) \preceq d(x,w) + d(w,z) + d(z,y)$ for all $x,y \in X$ and for all distinct points $w,z \in X - \{x,y\}$ [rectangular inequality].

Then d is called a *cone rectangular metric* on X and (X,d) is called a *cone rectangular metric space.*

Definition 1.6 ([6]). Let X be a non-empty set, E be a real Banach space, P be a solid cone in E and $\preceq$ be a partial ordering with respect to P. Suppose that the mapping $d : X \times X \to E$ satisfies:

(1) $\theta \prec d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) there exists a real number $s \geq 1$ such that $d(x,y) \preceq s[d(x,w) + d(w,z) + d(z,y)]$ for all $x,y \in X$ and for all distinct points $w,z \in X - \{x,y\}$ [b-rectangular inequality].

Then d is called a *generalized cone b-metric* on X and (X,d) is called a *generalized cone b-metric space* with coefficient s.

Definition 1.7 ([6]). Let (X,d) be a generalized cone b-metric space with coefficient $s \geq 1$. The sequence $\{x_n\}$ in X is said to be:

(i) a *convergent sequence* if for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$ for some $x \in X$. We say that the sequence $\{x_n\}$ converges to x and we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to +\infty$.

(ii) a *Cauchy sequence* if for every $c \in E$, with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $m,n > n_0$, $d(x_n, x_m) \ll c$.

(iii) The generalized cone b-rectangular metric space (X,d) is said to be *complete* if every Cauchy sequence is convergent in X.

First we give the definition of T-contraction mapping on generalized cone b-metric spaces which is based on the idea of A. Beiranvand et al. [4].

Definition 1.8. Let (X, d) be a generalized cone b-metric space with coefficient $s \geq 1$ and $T, f : X \to X$ be two self maps. A mapping f of X is said to be a T-*contraction* if there exists a real number $0 \leq \lambda < \frac{1}{s}$ such that

$$d\,(Tfx, Tfy) \preceq \lambda \mathrm{d}\,(Tx, Ty)$$

for all $x, y \in X$.

2. Main results

Theorem 2.1. *Let (X, d) be a generalized cone b-metric space with coefficient $s > 1$, P be a solid cone and let the mappings f and $T : X \to X$ satisfy the inequality:*

$$d\,(Tfx, Tfy) \preceq \lambda d\,(Tx, Ty) \tag{2.1}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{s}\right)$. Suppose T is one to one and $T(X)$ is a complete subspace of X, then the mapping f has a unique fixed point in X. Moreover, if f and T are commuting at the fixed point of f, then f and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$ for all $n = 0, 1, 2, \ldots$. If $x_m = x_{m+1}$ for some $m \in \mathbb{N}$, then $x_m = fx_m$. That is, f has a fixed point x_m in X.

Assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (2.1) it follows that,

$$\begin{aligned} d\,(Tx_n,\, Tx_{n+1}) &= d\,(Tfx_{n-1},\, Tfx_n) \\ &\preceq \lambda d\,(Tx_{n-1}, Tx_n) \\ &\vdots \\ &\preceq \lambda^n d\,(Tx_0, Tx_1) \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$, where $0 \leq \lambda < \frac{1}{s}$.

From (2.1), (2.2), b-rectangular inequality and using the fact that $0 \leq \lambda < \frac{1}{s}$, we get,

$$\begin{aligned} d\,(Tx_n,\, Tx_{n+2}) &= d\,(Tfx_{n-1}, Tfx_{n+1}) \\ &\preceq \lambda d\,(Tx_{n-1}, Tx_{n+1}) \\ &\preceq \lambda s\,[d\,(Tx_{n-1}, Tx_n) + d\,(Tx_n, Tx_{n+2}) + d\,(Tx_{n+2}, Tx_{n+1})] \end{aligned}$$

which implies that,

$$\begin{aligned} d\,(Tx_n,\, Tx_{n+2}) &\preceq \frac{\lambda s}{1 - \lambda s}\,[d\,(Tx_{n-1}, Tx_n) + d\,(Tx_{n+2}, Tx_{n+1})] \\ &\preceq \frac{\lambda s}{1 - \lambda s}\left[\lambda^{n-1} d\,(Tx_0, Tx_1) + \lambda^{n+1} d\,(Tx_0, Tx_1)\right] \\ &\preceq \frac{\lambda s}{1 - \lambda s}\left[1 + \lambda^2\right] \lambda^{n-1} d\,(Tx_0, Tx_1) \\ &= \frac{s}{1 - \lambda s}\left[1 + \lambda^2\right] \lambda^n d\,(Tx_0, Tx_1) \end{aligned}$$

$$= \alpha s\lambda^n d(Tx_0, Tx_1), \tag{2.3}$$

where $\alpha = \frac{1+\lambda^2}{1-\lambda s} \geq 0$ for all $n \geq 0$.

For the sequence $\{Tx_n\}$, we consider $d(Tx_n, Tx_{n+p})$ in two cases.

If p is odd say $2m+1$ for $m \geq 1$, then by using b-rectangular inequality and (2.2) we get,

$$\begin{aligned}
& d(Tx_n, Tx_{n+2m+1}) \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+2}, Tx_{n+2m+1})\right] \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})\right] + s^2\left[d(Tx_{n+2}, Tx_{n+3})\right. \\
& \left. + d(Tx_{n+3}, Tx_{n+4}) + d(Tx_{n+4}, Tx_{n+2m-1})\right] \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})\right] \\
& + s^2\left[d(Tx_{n+2}, Tx_{n+3}) + d(Tx_{n+3}, Tx_{n+4})\right] + \cdots \\
& + s^m d(Tx_{n+2m}, Tx_{n+2m+1}) \\
\preceq\ & s\left[\lambda^n d(Tx_0, Tx_1) + \lambda^{n+1} d(Tx_0, Tx_1)\right] \\
& + s^2\left[\lambda^{n+2} d(Tx_0, Tx_1) + \lambda^{n+3} d(Tx_0, Tx_1)\right] + \cdots \\
& + s^m \lambda^{n+2m} d(Tx_0, Tx_1)] \\
\preceq\ & s\lambda^n\left[1 + s\lambda^2 + \cdots\right] d(Tx_0, Tx_1) \\
& + s\lambda^{n+1}\left[1 + s\lambda^2 + \cdots\right] d(Tx_0, Tx_1) \\
=\ & (1+\lambda) s\lambda^n\left[1 + s\lambda^2 + \cdots\right] d(Tx_0, Tx_1).
\end{aligned}$$

Hence, $d(Tx_n, Tx_{n+2m+1}) \preceq \left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d(Tx_0, Tx_1)$ for all $n, m \in \mathbb{N}$.

Let $\theta \ll c$ be given. Since, $s\lambda^2 < 1$, we note that $\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d(Tx_0, Tx_1) \to \theta$ as $n \to \infty$. By Remark 1.2, for any $c \in int(P)$, we can find $N_1 \in \mathbb{N}$ such that for each $n > N_1$, we have $\left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d(Tx_0, Tx_1) \ll c$.

Thus,

$$d(Tx_n, Tx_{n+2m+1}) \preceq \left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d(Tx_0, Tx_1) \ll c$$

for all $n > N_1$ and $m \geq 1$.

If p is even say $2m$ for $m \geq 1$, then by using b-rectangular inequality, (2.2), (2.3) and the fact that $s\lambda^2 < 1$, we get,

$$\begin{aligned}
& d(Tx_n, Tx_{n+2m}) \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+2}, Tx_{n+2m})\right] \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})\right] \\
& + s^2\left[d(Tx_{n+2}, Tx_{n+3}) + d(Tx_{n+3}, Tx_{n+4}) + d(Tx_{n+4}, Tx_{n+2m})\right] \\
\preceq\ & s\left[d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2})\right]
\end{aligned}$$

$$
\begin{aligned}
&+ s^2\left[d\left(Tx_{n+2}, Tx_{n+3}\right)+d\left(Tx_{n+3}, Tx_{n+4}\right)\right]+\cdots \\
&+ s^{m-1}\left[d\left(Tx_{n+2m-4}, Tx_{n+2m-3}\right)+d\left(Tx_{n+2m-3}, Tx_{n+2m-2}\right)\right. \\
&\left.+d\left(Tx_{n+2m-2}, Tx_{n+2m}\right)\right] \\
\preceq\ & s\left[\lambda^n d\left(Tx_0, Tx_1\right)+\lambda^{n+1} d\left(Tx_0, Tx_1\right)\right] \\
&+ s^2\left[\lambda^{n+2} d\left(Tx_0, Tx_1\right)+\lambda^{n+3} d\left(Tx_0, Tx_1\right)\right]+\cdots \\
&+ s^{m-1}\left[\lambda^{n+2m-4} d\left(Tx_0, Tx_1\right)+\lambda^{n+2m-3} d\left(Tx_0, Tx_1\right)\right] \\
&+ s^{m-1}\alpha\lambda^{n+2m-2} d\left(Tx_0, Tx_1\right) \\
\preceq\ & s\lambda^n\left[1+s\lambda^2+\cdots\right] d\left(Tx_0, Tx_1\right) \\
&+ s\lambda^{n+1}\left[1+s\lambda^2+\cdots\right] d\left(Tx_0, Tx_1\right) \\
&+ s^{m-1}\alpha\lambda^{n+2m-2} d\left(Tx_0, Tx_1\right) \\
=\ & (1+\lambda) s\lambda^n\left[1+s\lambda^2+\cdots\right] d\left(Tx_0, Tx_1\right) \\
&+ s^{m-1}\alpha\lambda^{n+2m-2} d\left(Tx_0, Tx_1\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d\left(Tx_n, Tx_{n+2m}\right) &\preceq \left(\frac{1+\lambda}{1-s\lambda^2}\right) s\lambda^n d\left(Tx_0, Tx_1\right)+s^{m-1}\alpha\lambda^{n+2m-2} d\left(Tx_0, Tx_1\right) \\
&\preceq \left(\frac{1+\lambda}{1-s\lambda^2}+s^{m-2}\alpha\lambda^{2m-2}\right) s\lambda^n d\left(Tx_0, Tx_1\right)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and $\alpha \geq 0$.

Let $\theta \ll c$ be given. Since, $s\lambda^2 < 1$, we have

$$
\left(\frac{1+\lambda}{1-s\lambda^2}+s^{m-2}\alpha\lambda^{2m-2}\right) s\lambda^n d\left(Tx_0, Tx_1\right) \rightarrow \theta
$$

as $n \rightarrow \infty$. By Remark 1.2 for any $c \in int(P)$, we can find $N_2 \in \mathbb{N}$ such that

$$
\left(\frac{1+\lambda}{1-s\lambda^2}+s^{m-2}\alpha\lambda^{2m-2}\right) s\lambda^n d\left(Tx_0, Tx_1\right) \ll c
$$

for all $n > N_2$ and $m \geq 1$.

Thus, $d\left(Tx_n, Tx_{n+2m}\right) \preceq \left(\frac{1+\lambda}{1-s\lambda^2}+s^{m-2}\alpha\lambda^{2m-2}\right) s\lambda^n d\left(Tx_0, Tx_1\right) \ll c$ for all $n > N_2$ and $m \geq 1$. Let $N_0 = \max\{N_1, N_2\}$. Thus for each $c \in int(P)$, we have $d\left(Tx_n, Tx_{n+p}\right) \ll c$ for all $n > N_0$ and $p \geq 1$. Therefore, $\{Tx_n\}$ is a Cauchy sequence in X. Since, $T(X)$ is a complete subspace of X, then there exists a point z in $T(X)$ such that $\lim_{n\rightarrow\infty} Tx_{n+1} = \lim_{n\rightarrow\infty} Tfx_n = z$. Also, we can find $x \in X$ such that $z = Tx$. Let $\theta \ll c$ be given, we can choose natural numbers N_3 and N_4 such that $d\left(Tx_n, z\right) \ll \frac{c}{2s(1+\lambda)}$ for all $n > N_3$ and $d\left(Tx_n, Tx_{n+1}\right) \ll \frac{c}{2s}$ for all $n > N_4$. Let $N = \max\{N_3, N_4\}$.

Using b-rectangular inequality and (2.1) we get,

$$
d(z, Tfx) \preceq s\left[d\left(z, Tx_n\right)+d\left(Tx_n, Tfx_n\right)+d\left(Tfx_n, Tfx\right)\right]
$$

$$
\begin{aligned}
&\preceq sd\left(z,\, Tx_n\right) + sd\left(Tx_n,\, Tx_{n+1}\right) + s\lambda d\left(Tx_n, Tx\right)\\
&= sd\left(z,\, Tx_n\right) + sd\left(Tx_n,\, Tx_{n+1}\right) + s\lambda d\left(Tx_n, z\right)\\
&= s\left(1+\lambda\right) d\left(Tx_n, z\right) + sd\left(Tx_n,\, Tx_{n+1}\right)\\
&\ll \frac{c}{2} + \frac{c}{2} = c
\end{aligned}
$$

for all $n > N$.

Thus for each $c \in int(P)$, we have, $d\left(z,\, Tfx\right) \ll c$, since c is arbitrary we have $d\left(z,\, Tfx\right) = \theta$. Therefore, $Tfx = Tx = z$. Since T is one to one, we get that $x = fx$. Hence, x is a fixed point of f in X. Now, we prove the uniqueness of the fixed point of f. Let y be another fixed point of f, that is $y = fy$. Then,

$$d\left(Tx, Ty\right) = d\left(Tfx, Tfy\right) \preceq \lambda d\left(Tx, Ty\right) \prec \frac{1}{s} d\left(Tx, Ty\right),$$

which is a contradiction (since $s > 1$). Hence, $Tx = Ty$. Since T is one to one, we conclude that $x = y$. Since, f and T are commuting at the fixed point of f, $fTx = Tfx = Tx$. Therefore Tx is a fixed point of f. Since f has unique fixed point, $Tx = x$. Hence $Tx = fx = x$, that is x is the unique common fixed point of f and T in X. □

Taking $T = I$ (the identity mapping of X) in Theorem 2.1, we get an analogue of Banach contraction principle [6] in generalized cone b-metric space as follows.

Corollary 2.2. *Let (X, d) be a complete generalized cone b-metric space with $s > 1$ and $f : X \to X$ be a mapping satisfies the inequality:*

$$d\left(fx, fy\right) \preceq \lambda d\left(x, y\right) \tag{2.4}$$

for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{s}\right)$, then f has a unique fixed point in X.

The following examples support Theorem 2.1.

Example 2.3. Let $X = A \cup B$, where $A = \{0\} \cup \left\{\frac{1}{n} : n \in \{2, 3, 4, 5, 6\}\right\}$ and $B = [1, 2]$. Let $E = C_{\mathbb{R}}\left(X\right)$ be the set of all continuous functions defined on X to $\mathbb{R}$ and $P = \{\phi \in E : \phi\left(t\right) \geq 0,\ t \in X\} \subset E$. It is known that P is a solid cone in E. Define $d : X \times X \to E$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$
\begin{cases}
d\left(0, \frac{1}{2}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0.6e^t;\\
d\left(0, \frac{1}{3}\right) = d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = 0.4e^t;\\
d\left(0, \frac{1}{4}\right) = d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0.1e^t;\\
d\left(0, \frac{1}{5}\right) = d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = 0.5e^t;\\
d\left(0, \frac{1}{6}\right) = d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = 0.3e^t;\\
d\left(x, y\right) = |x - y|^2 e^t, \text{ otherwise,}
\end{cases}
$$

where $e^t \in E$. Then (X, d) is not a cone metric space with respect to P as $d(0, 1/2) = 0.6e^t \succ d(0, 1/4) + d(1/4, 1/2) = 0.1e^t + 0.3e^t = 0.4e^t$ and (X, d) is not a cone rectangular metric space with respect to P as $d(0, 1/2) = 0.6e^t \succ d(0, 1/4) + d(1/4, 1/6) + d(1/6, 1/2) = 0.1e^t + 0.1e^t + 0.1e^t = 0.3e^t$. However, it is easy to see that (X,d) is a complete generalized cone b-metric space with coefficient $s=2>1$.

Further, let f and $T:X\to X$ be the mappings defined by:

$$f(x)=\begin{cases}\frac{1}{4} & \text{if } x\in A\\ \frac{1}{2} & \text{if } x\in B,\end{cases}$$

$$\text{and } T(x)=\begin{cases}\frac{1}{2}-x & \text{if } x\in\{0\}\cup\left\{\frac{1}{n}:n\in\{2,3,4\}\right\}\\ \frac{1}{3} & \text{if } x=\frac{1}{5}\\ \frac{1}{5} & \text{if } x=\frac{1}{6}\\ x & \text{if } x\in B.\end{cases}$$

It is clear that, T is one to one, f satisfies T-contraction (2.1) with $\lambda=\frac{2}{5}$. In fact, if $x\in A$ and $y\in B$, then $d(Tfx,Tfy)=d(T(1/4),T(1/2))=d(1/4,0)=0.1e^t$.

Case (i). If $x\in\{0\}\cup\left\{\frac{1}{n}:n\in\{2,3,4\}\right\}$ and $y\in B$, then $d(Tx,Ty)=d\left(\frac{1}{2}-x,y\right)=\left|\frac{1}{2}-(x+y)\right|^2e^t$. Then clearly we can find $\lambda=\frac{2}{5}\in\left(0,\frac{1}{2}\right)$ satisfying T-contraction (2.1).

Case (ii). If $x\in\{0\}\cup\left\{\frac{1}{n}:n\in\{2,3,4\}\right\}$ and $y\in\left\{\frac{1}{5},\frac{1}{6}\right\}$, then we can find $\lambda=\frac{2}{5}\in\left(0,\frac{1}{2}\right)$ satisfying T-contraction (2.1).

Case (iii). If $x\in B$ and $y\in\left\{\frac{1}{5},\frac{1}{6}\right\}$, then $d(Tx,Ty)=d\left(Tx,T(\frac{1}{5})\right)=d\left(x,\frac{1}{3}\right)=\left|x-\frac{1}{3}\right|^2e^t$ and $d(Tx,Ty)=d\left(Tx,T(\frac{1}{6})\right)=d\left(x,\frac{1}{5}\right)=\left|x-\frac{1}{5}\right|^2e^t$. Then we can find $\lambda=\frac{2}{5}\in\left(0,\frac{1}{2}\right)$ satisfying T-contraction (2.1).

Similarly, If $x\in A$ and $y\in A$, then $d(Tfx,Tfy)=d(T(1/4),T(1/4))=\theta$ and if $x\in B$ and $y\in B$, then $d(Tfx,Tfy)=d(T(1/2),T(1/2))=\theta$. Hence f satisfies T-contraction (2.1). However f does not satisfy Banach contraction (2.4) at $x=\frac{1}{5}$ and $y=1$, as $d(fx,fy)=d\left(\frac{1}{4},\frac{1}{2}\right)=0.3e^t\succ\frac{2}{5}d(x,y)=\frac{2}{5}d\left(\frac{1}{5},1\right)=\frac{2}{5}\left|\frac{1}{5}-1\right|^2e^t=0.256e^t$. Thus f satisfies all the conditions of Theorem 2.1 and f has unique fixed point $\frac{1}{4}$. Moreover, f and T are commuting at $\frac{1}{4}$ and hence $x=\frac{1}{4}$ is the unique common fixed point of the mappings f and T.

Example 2.4. Let $X=\{a,b,c,d\}$, where a,b,c,d are distinct natural numbers. Let $E=\mathcal{M}_{n\text{x}n}(\mathbb{R})$ be the space of real matrices of order $n\geq 1$ and $P=\left\{M=(a_{ij})_{1\leq i,j\leq n}:a_{ij}\geq 0,\forall i,j\right\}$ is a solid cone in E. Define $d:X\times X\to E$ such that $d(x,y)=d(y,x)$ for all $x,y\in X$ and

$$\begin{cases}d(x,x)=\mathrm{O}_{n\times n} \quad \text{for all } x\in X;\\ d(a,b)=0.1I_n;\\ d(a,c)=d(b,c)=0.01I_n;\\ d(a,d)=d(b,d)=d(c,d)=0.03I_n,\end{cases}$$

where I_n is the identity matrix. In this case, (X,d) is not a cone metric space with respect to P since, $d(a,b)=0.1I_n\succ d(a,c)+d(c,b)=0.01I_n+0.01I_n=0.02I_n$ and (X,d) is not a cone rectangular metric space with respect to P since, $d(a,b)=0.1I_n\succ d(a,c)+d(c,d)+d(d,b)=0.01I_n+0.03I_n+0.03I_n=0.07I_n$.

However, it is easy to see that (X, d) is a complete generalized cone b-metric space with coefficient $s = 1.5 > 1$. Further, let f and $T : X \to X$ be the mappings defined by:

$$f(x) = \begin{cases} c & \text{if } x \neq d \\ a & \text{if } x = d, \end{cases}$$

$$\text{and } T(x) = \begin{cases} b & \text{if } x = a \\ a & \text{if } x = b \\ x & \text{if } x \in \{c, d\}. \end{cases}$$

Since, $Tf(d) = T(f(d)) = T(a) = b$ and $fT(d) = f(T(d)) = f(d) = a$. Therefore $Tf(d) \neq fT(d)$. That is, f and T are non-commuting self maps. It is clear that, T is one to one, f satisfies T-contraction (2.1) on generalized cone b-metric space (X, d) with $\lambda = \frac{1}{3} < \frac{1}{s} = \frac{2}{3}$. Thus f satisfies all the conditions of Theorem 2.1 and f has unique fixed point c. Moreover, f and T are commuting at c and hence $x = c$ is the unique common fixed point of the mappings f and T.

Conclusion.

In this article we have proved that the existence and uniqueness of common fixed point theorem for T-contraction in generalized cone b-metric spaces. We note that the results of this paper generalize the results of R. George, et al. [6] on generalized cone b-metric spaces.

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