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COMPOSITION OPERATORS ON $\mathcal{Q}_K$-TYPE SPACES AND A NEW COMPACTNESS CRITERION FOR COMPOSITION OPERATORS ON $\mathcal{Q}_s$ SPACES

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Abstract. For $-2 < \alpha < \infty$ and $0 < p < \infty$, the $\mathcal{Q}_K$-type space is the space of all analytic functions on the open unit disk $\mathbb{D}$ satisfying

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^\alpha K(g(z,a))dA(z) < \infty,$$

where $g(z,a) = \log\frac{1}{|\sigma_a(z)|}$ is the Green's function on $\mathbb{D}$ and $K : [0,\infty) \to [0,\infty)$, is a right-continuous and non-decreasing function. For $0 < s < \infty$, the space $\mathcal{Q}_s$ consists of all analytic functions on $\mathbb{D}$ for which

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2(g(z,a))^s dA(z) < \infty.$$

Boundedness and compactness of composition operators C_φ acting on $\mathcal{Q}_K$-type spaces and $\mathcal{Q}_s$ spaces is characterized in terms of the norms of φ^n. Thus the author announces a solution to the problem raised by Wulan, Zheng and Zhou.

1. Introduction

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For an analytic self-map φ of $\mathbb{D}$, the composition operator C_φ is defined by

$$C_\varphi f = f\circ\varphi, \quad f\in\mathcal{H}(\mathbb{D}).$$

Recently, there has been a lot of work on composition operators between Banach spaces of analytic functions, see for example [5, 7, 9, 13, 14, 15, 18]. One of the reasons is that it provides connections between operator theory and complex analysis and helps us to gain a deeper understanding of both areas.

Recall that the Green's function in $\mathbb{D}$ with singularity at $a\in\mathbb{D}$ is given by $g(z,a) = \log\frac{1}{|\sigma_a(z)|}$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius transformation of $\mathbb{D}$. Take a right-continuous and non-decreasing function $K : [0,\infty) \to [0,\infty)$. For

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$-2 < \alpha < \infty$, $0 < p < \infty$, the $\mathcal{Q}_K$-type space denoted by $\mathcal{Q}_K(p, \alpha)$ is the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{K,p,\alpha}^p := \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha K(g(z, a)) dA(z) < \infty,$$

where dA is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. For $1 \leqslant p < \infty$, $\mathcal{Q}_K(p, \alpha)$ equipped with the norm $\|f\|_{\mathcal{Q}_K(p,\alpha)} := |f(0)| + \|f\|_{K,p,\alpha}$, is a Banach space. When $\alpha + 2 = p$, $\mathcal{Q}_K(p, \alpha)$ is a Möbius invariant space, i.e., $\|f \circ \sigma_a\|_{K,p,\alpha} = \|f\|_{K,p,\alpha}$ for all $a \in \mathbb{D}$. The space $\mathcal{Q}_K(p, \alpha)$ is trivial, if it contains constant functions only. If

$$\int_0^1 (1 - r^2)^\alpha K(\log \tfrac{1}{r}) r dr, \tag{1.1}$$

is divergent, then the space $\mathcal{Q}_K(p, \alpha)$ is trivial [19]. The study of $\mathcal{Q}_K$-type spaces started in [19], by Wulan and Zhou. By [19, Theorem 2.1], if $K(1) > 0$, the kernel function K can be chosen as a bounded function, also $f \in \mathcal{Q}_K(p, \alpha)$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha K(1 - |\sigma_a(z)|^2) dA(z) < \infty.$$

From now on, we assume in this paper that $K(1) > 0$ and that the integral (1.1) is convergent.

The Banach space $\mathcal{Q}_K(p, \alpha)$ coincides with many well known analytic function spaces. For $0 < s < \infty$, if $K(t) = t^s$, then $\mathcal{Q}_K(p, \alpha) = F(p, \alpha, s)$, where a function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the general function space $F(p, \alpha, s)$ provided that

$$\|f\|_{F(p,\alpha,s)}^p = |f(0)|^p + \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha (1 - |\sigma_a(z)|^2)^s dA(z) < \infty.$$

The space $F(p, \alpha, s)$ was introduced by Zhao in [23]. By [19, Corollary 3.5], $F(p, \alpha, 0) \subset \mathcal{Q}_K(p, \alpha)$ and the equality holds if and only if $K(0) > 0$. If we take special parameters $p = 2, \alpha = 0$, we get the $\mathcal{Q}_K$ space. We refer to [3, 17] for a general theory of $\mathcal{Q}_K$ spaces. If we take $K(t) = t^s$, $0 < s < \infty$, $\mathcal{Q}_K$ space coincides with $\mathcal{Q}_s$ space. Thus $\mathcal{Q}_K$ spaces are generalizations of $\mathcal{Q}_s$ spaces which are themselves generalization of Bloch-type spaces or BMOA.

Recall that for $0 < \alpha < \infty$, a function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the α-Bloch space or Bloch type space $\mathcal{B}^\alpha$ provided that

$$b_\alpha(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The space $\mathcal{B}^\alpha$ becomes a Banach space under the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f).$$

If $\alpha = 1$, we obtain the well-known classical Bloch space $\mathcal{B}$. When $0 < \alpha < 1$, $\mathcal{B}^\alpha$ is the analytic Lipschitz space $\Lambda_{1-\alpha}$ where

$$\Lambda_{1-\alpha} = \{f \in \mathcal{H}(\mathbb{D}) : \sup\{\tfrac{|f(z)-f(w)|}{|z-w|^{1-\alpha}} : z, w \in \mathbb{D}, z \neq w\} < \infty\},$$

see [26]. If $\alpha > 1$, $\mathcal{B}^\alpha = H^\infty_{\alpha-1}$, the weighted Banach space of analytic functions that contains $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha-1}|f(z)| < \infty,$$

see [25]. By [19, Theorem 2.1], we know that for $0 < p < \infty$, and $-2 < \alpha < \infty$, $\mathcal{Q}_K(p,\alpha) \subset \mathcal{B}^{\frac{\alpha+2}{p}}$ and

$$b_{\frac{\alpha+2}{p}}(f) \leqslant C\|f\|_{K,p,\alpha}, \tag{1.2}$$

for some constant $C > 0$ independent to f. Also $\mathcal{Q}_K(p,\alpha) = \mathcal{B}^{\frac{\alpha+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2}K(\log\tfrac{1}{r})rdr < \infty.$$

The starting point of the present article is an article of Wulan, Zheng and Zhu published in 2009 (see [18]) in which compactness of composition operators on BMOA and the Bloch space is characterized in terms of the norms of φ^n (the iterates of φ) in the respective spaces as follows:

Theorem 1.1 ([18]). *Let X denote BMOA or the Bloch space on $\mathbb{D}$. Then a composition operator $C_\varphi : X \to X$ is compact if and only if $\|\varphi^n\|_X \to 0$ as $n \to \infty$.*

At the end of their paper, the authors ask the natural and not trivial problem of getting such a characterization for more general $\mathcal{Q}_s$ spaces.

We would like to point out that Montes-Rodrigues characterizing the compactness of C_φ on the Bloch and little Bloch space in terms of φ [10]. Moreover the essential norms of composition operators is even calculated in Montes-Rodriguez's paper.

Recall that the essential norm $\|T\|_e$ of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the space of compact operators from X to Y.

Recently, Zhao determined the essential norms of composition operators between Bloch type spaces by norms of φ^n as follows:

Theorem 1.2 ([24]). *Let $0 < \alpha, \beta < \infty$ and φ be an analytic self-map of $\mathbb{D}$. Then the essential norm of the composition operator $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is*

$$\|C_\varphi\|_e = (\frac{e}{2\alpha})^\alpha \limsup_{n\to\infty} n^{\alpha-1}\|\varphi^n\|_{\mathcal{B}^\beta}.$$

In particular $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if

$$\limsup_{n\to\infty} n^{\alpha-1}\|\varphi^n\|_{\mathcal{B}^\beta} = 0.$$

A natural question that arises from these results is whether we can get a compactness criterion for composition operators between $\mathcal{Q}_K$-type spaces in terms of the norms of φ^n. The author is interested in the study of composition operators acting on $\mathcal{Q}_K$-type spaces and announces a solution to the problem raised by Wulan, Zheng and Zhu about characterizing the compactness of composition operators on $\mathcal{Q}_s$ spaces.

We remark that if $C_\varphi f \in \mathcal{Q}_{K_2}(q,\beta)$ for every f in $\mathcal{Q}_{K_1}(p,\alpha)$, then by the closed graph theorem, the composition operator $C_\varphi : \mathcal{Q}_{K_1}(p,\alpha) \to \mathcal{Q}_{K_2}(q,\beta)$ is bounded. Throughout this paper, constants are denoted by C, they are positive and not necessarily the same in all occurrences.

2. Bounded composition operators on $\mathcal{Q}_K$-type spaces

Let $\alpha, p > 0$, $n \in \mathbb{N}$, $0 \leqslant x \leqslant 1$ and $f_{n,\alpha,p}(x) = n^\alpha x^{(n-1)p}(1-x^2)^\alpha$. Then by simple calculation, $\max_{0\leqslant x\leqslant 1} f_{1,\alpha,p}(x) = f_{1,\alpha,p}(0) = 1$ and for $n \geqslant 2$,

$$\max_{0\leqslant x\leqslant 1} f_{n,\alpha,p}(x) = f_{n,\alpha,p}(t_n) = n^\alpha\left(\tfrac{2\alpha}{(n-1)p+2\alpha}\right)^\alpha\left(\tfrac{(n-1)p}{(n-1)p+2\alpha}\right)^{\frac{(n-1)p}{2}}, \tag{2.1}$$

where $t_n = \left(\frac{(n-1)p}{(n-1)p+2\alpha}\right)^{\frac{1}{2}}$. For $n \geqslant 1$, $f_{n,\alpha,p}(x)$ is decreasing on $[t_n, t_{n+1}]$, and so

$$\lim_{n\to\infty} \min_{x\in[t_n,t_{n+1}]} f_{n,\alpha,p}(x) = \lim_{n\to\infty} f_{n,\alpha,p}(t_{n+1}) = \left(\tfrac{2\alpha}{ep}\right)^\alpha. \tag{2.2}$$

Using the above notation we now establish the following result.

Theorem 2.1. *Let $0 < p, q < \infty$, $-2 < \alpha, \beta < \infty$ and φ be an analytic self-map of $\mathbb{D}$.*

(i) *For $\alpha > 0$, if $C_\varphi(\mathcal{Q}_{K_1}(p,\alpha)) \subset \mathcal{Q}_{K_2}(q,\beta)$, then*

$$\sup_{n\in\mathbb{N}} n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)} < \infty.$$

(ii) *For $\frac{1}{q} + \frac{\alpha+2}{p} < 1$, if $\sup_{n\in\mathbb{N}} \|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)} < \infty$, then*

$$C_\varphi(\mathcal{Q}_{K_1}(p,\alpha)) \subset \mathcal{Q}_{K_2}(q,\beta).$$

Proof. (i) For $n \geqslant 2$, consider the functions $f_n(z) = \frac{z^n}{n^{1-\frac{\alpha}{p}}}$, $z \in \mathbb{D}$. By (2.1), we have

$$\begin{aligned}\left\|\frac{z^n}{n^{1-\frac{\alpha}{p}}}\right\|^p_{\mathcal{Q}_{K_1}(p,\alpha)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} n^\alpha |z|^{(n-1)p}(1-|z|^2)^\alpha K_1(g(z,a))\,dA(z)\\ &\leqslant n^\alpha\left(\tfrac{2\alpha}{(n-1)p+2\alpha}\right)^\alpha\left(\tfrac{(n-1)p}{(n-1)p+2\alpha}\right)^{\frac{(n-1)p}{2}} K_1(1).\end{aligned}$$

Since

$$\lim_{n\to\infty} n^\alpha\left(\tfrac{2\alpha}{(n-1)p+2\alpha}\right)^\alpha\left(\tfrac{(n-1)p}{(n-1)p+2\alpha}\right)^{\frac{(n-1)p}{2}} = \left(\tfrac{2\alpha}{ep}\right)^\alpha,$$

hence, there is a constant $C > 0$, independent of n, such that $\|\frac{z^n}{n^{1-\frac{\alpha}{p}}}\|_{\mathcal{Q}_{K_1}(p,\alpha)} \leqslant C$. Thus $(\frac{z^n}{n^{1-\frac{\alpha}{p}}})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Q}_{K_1}(p,\alpha)$. By assumption

$$C > \|C_\varphi\|\|f_n\|_{\mathcal{Q}_{K_1}(p,\alpha)} \geqslant \|C_\varphi f_n\|_{\mathcal{Q}_{K_2}(q,\beta)} = n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}.$$

(ii) Let $L := \sup_{n\in\mathbb{N}} \|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)} < \infty$. Then obviously $\|\varphi\|_{\mathcal{Q}_{K_2}(q,\beta)} < \infty$. If $\sup_{z\in\mathbb{D}} |\varphi(z)| < 1$, then there is a number r, with $0 < r < 1$, such that $\sup_{z\in\mathbb{D}} |\varphi(z)| < r$. In this case for $f \in \mathcal{Q}_{K_1}(p,\alpha)$, we have

$$\begin{aligned}\|C_\varphi f\|^q_{K_2,q,\beta} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^q |\varphi'(z)|^q (1-|z|^2)^\beta K_2(g(z,a))dA(z)\\ &\leqslant b^q_{\frac{\alpha+2}{p}}(f) \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^q(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\frac{(\alpha+2)q}{p}}} K_2(g(z,a))dA(z)\\ &\leqslant C\frac{1}{(1-r^2)^{\frac{(\alpha+2)q}{p}}} \|f\|^q_{K_1,p,\alpha} \|\varphi\|^q_{K_2,q,\beta},\end{aligned}$$

which implies that $C_\varphi f \in \mathcal{Q}_{K_2}(q,\beta)$, thus $C_\varphi(\mathcal{Q}_{K_1}(p,\alpha)) \subset \mathcal{Q}_{K_2}(q,\beta)$.

In the following we assume that $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$. For any integer $n \geqslant 1$, let

$$D_n = \{z \in \mathbb{D} : (\tfrac{(n-1)p}{(n-1)p+2\alpha})^{\frac{1}{2}} \leqslant |\varphi(z)| \leqslant (\tfrac{np}{np+2\alpha})^{\frac{1}{2}}\}.$$

Let m be the smallest positive integer such that $D_m \neq \phi$. Since $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$, D_n is not empty for every integer $n \geqslant m$, and $\mathbb{D} = \cup_{n=m}^\infty D_m$. By (2.2), for every $n \geqslant m$,

$$\lim_{n\to\infty} \min_{z\in D_n} n^{\alpha+2} |\varphi(z)|^{(n-1)p} (1-|\varphi(z)|^2)^{\alpha+2} = (\tfrac{2\alpha+4}{ep})^{\alpha+2}.$$

Therefore, there exists a constant $K > 0$ such that, for any $n \geqslant m$,

$$\min_{z\in D_n} n^{\alpha+2} |\varphi(z)|^{(n-1)p} (1-|\varphi(z)|^2)^{\alpha+2} \geqslant K.$$

Let $f \in \mathcal{Q}_{K_1}(p,\alpha)$. Then

$$\begin{aligned}&\|C_\varphi f\|^q_{\mathcal{Q}_{K_2}(q,\beta)}\\ =\ & \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(\varphi(z))|^q |\varphi'(z)|^q (1-|z|^2)^\beta K_2(g(z,a))dA(z)\\ =\ & \sup_{a\in\mathbb{D}} \sum_{n=m}^\infty \int_{D_n} [\frac{|f'(\varphi(z))|^q (1-|\varphi(z)|^2)^{\frac{(\alpha+2)q}{p}} n^{\frac{(\alpha+2)q}{p}} |\varphi(z)|^{q(n-1)} |\varphi'(z)|^q}{n^{\frac{(\alpha+2)q}{p}} |\varphi(z)|^{q(n-1)} (1-|\varphi(z)|^2)^{\frac{(\alpha+2)q}{p}}}\\ & \times (1-|z|^2)^\beta K_2(g(z,a))]dA(z)\\ \leqslant\ & b^q_{\frac{\alpha+2}{p}}(f) (\frac{1}{K})^{\frac{q}{p}} \sup_{a\in\mathbb{D}} \sum_{n=m}^\infty n^{\frac{(\alpha+2)q}{p}-q} \int_{D_n} [n^q |\varphi(z)|^{q(n-1)} |\varphi'(z)|^q\\ & \times (1-|z|^2)^\beta K_2(g(z,a))]dA(z)\\ \leqslant\ & C(\tfrac{1}{K})^{\frac{q}{p}} \|f\|^q_{K_1,p,\alpha} \sum_{n=m}^\infty \frac{\|\varphi^n\|^q_{\mathcal{Q}_{K_2}(q,\beta)}}{n^{q-\frac{(\alpha+2)q}{p}}}\\ \leqslant\ & C(\tfrac{L}{K^{\frac{1}{p}}})^q \|f\|^q_{K_1,p,\alpha} \sum_{n=m}^\infty \frac{1}{n^{q-\frac{(\alpha+2)q}{p}}}.\end{aligned}$$

Since $\frac{1}{q}+\frac{\alpha+2}{p}<1$, thus $\sum_{n=m}^{\infty}\frac{1}{n^{q-\frac{(\alpha+2)q}{p}}}<\infty$. Hence, $C_\varphi f\in\mathcal{Q}_{K_2}(q,\beta)$. Therefore C_φ is bounded from $\mathcal{Q}_{K_1}(p,\alpha)$ to $\mathcal{Q}_{K_2}(q,\beta)$. The proof is complete. □

3. Compact composition operators on $\mathcal{Q}_K$-type spaces

In this section, we first give some auxiliary results. We recall the following estimate which will use in the proof of Lemma 3.1.

For functions $f\in\mathcal{B}_\alpha$ and $z\in\mathbb{D}$, we have

$$|f(z)|\leqslant\begin{cases} C\|f\|_{\mathcal{B}_\alpha} & 0<\alpha<1\\ |f(0)|+\frac{1}{2}b(f)\log\frac{1+|z|}{1-|z|} & \alpha=1\\ C\frac{\|f\|_{\mathcal{B}_\alpha}}{(1-|z|^2)^{\alpha-1}} & \alpha>1\end{cases}\tag{3.1}$$

for some constant C, independent of f. Using (1.2) and (3.1), the proof of following lemma is similar to [9, Lemma 2.1], thus we omit the proof here.

Lemma 3.1. *Let $0<p,q<\infty$, $-2<\alpha,\beta<\infty$. The operator $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact if and only if for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}_{K_1}(p,\alpha)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|C_\varphi f_n\|_{\mathcal{Q}_{K_2}(q,\beta)}\to 0$ as $n\to\infty$.*

Theorem 3.2. *Let $0<p,q<\infty$, $-2<\alpha,\beta<\infty$ and φ be an analytic self-map of $\mathbb{D}$.*

(i) *For $\alpha>0$, if $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact, then*

$$\lim_{n\to\infty}n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}=0.$$

(ii) *For $\frac{1}{q}+\frac{\alpha+2}{p}<1$, if $\lim_{n\to\infty}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}=0$, then $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact.*

Proof. (i) From the proof of Theorem 2.1, $(\frac{z^n}{n^{1-\frac{\alpha}{p}}})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Q}_{K_1}(p,\alpha)$. It is clear that for $\frac{\alpha}{p}\leqslant 1$, $(\frac{z^n}{n^{1-\frac{\alpha}{p}}})_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Also it is an easy exercise in calculus that for $|a|<1$ and $t>0$, $\lim_{n\to\infty}n^ta^n=0$. It implies that for $\frac{\alpha}{p}>1$, $(\frac{z^n}{n^{1-\frac{\alpha}{p}}})_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Consequently for $\alpha,p>0$, $(\frac{z^n}{n^{1-\frac{\alpha}{p}}})_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Q}_{K_1}(p,\alpha)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence, by Lemma 3.1, the compactness of C_φ implies that

$$\lim_{n\to\infty}\|C_\varphi(\frac{z^n}{n^{1-\frac{\alpha}{p}}})\|_{\mathcal{Q}_{K_2}(q,\beta)}=\lim_{n\to\infty}n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}=0.$$

(ii) Suppose $\lim_{n\to\infty}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}=0$, then by Theorem 2.1, C_φ is a bounded operator from $\mathcal{Q}_{K_1}(p,\alpha)$ to $\mathcal{Q}_{K_2}(q,\beta)$. Define D_n and m as in the proof of Theorem 2.1. Given $\varepsilon>0$, there exists a positive integer $N>m$, such that for $k\geqslant N$,

$$\|\varphi^k\|_{\mathcal{Q}_{K_2}(q,\beta)}<\varepsilon.\tag{3.2}$$

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{Q}_{K_1}(p,\alpha)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Using analogue of the proof of Theorem 2.1, we have

$$\begin{aligned}
&\|C_\varphi f_n\|^q_{\mathcal{Q}_{K_2}(q,\beta)}\\
&= \sup_{a\in\mathbb{D}}\Big(\sum_{k=m}^{N}+\sum_{k>N}\Big)\int_{D_k}|f_n'(\varphi(z))|^q|\varphi'(z)|^q(1-|z|^2)^\beta K_2(g(z,a))dA(z)\\
&\leqslant \sup_{a\in\mathbb{D}}\sum_{k=m}^{N}\int_{D_k}|f_n'(\varphi(z))|^q|\varphi'(z)|^q(1-|z|^2)^\beta K_2(g(z,a))dA(z)\\
&\quad + C\|f_n\|^q_{\mathcal{Q}_{K_1}(p,\alpha)}\sum_{k>N}\Big(\frac{\|\varphi^k\|_{\mathcal{Q}_{K_2}(q,\beta)}}{k^{1-\frac{\alpha+2}{p}}}\Big)^q\\
&:= I_1+I_2.
\end{aligned}$$

As a simple consequence of convergence of the sequence $(f_n)_{n\in\mathbb{N}}$ to zero on compact subsets of $\mathbb{D}$, there exist integers n_k, $m\leqslant k\leqslant N$, such that for $n>n_k$, $\sup_{z\in D_k}|f_n'(\varphi(z))|^q<\varepsilon$. For $n>N_1=\max\{n_m,\dots,n_N\}$, we have

$$\begin{aligned}
I_1 &\leqslant \sup_{a\in\mathbb{D}}\sum_{k=m}^{N}\sup_{z\in D_k}|f_n'(\varphi(z))|^q\int_{D_k}|\varphi'(z)|^q(1-|z|^2)^\beta K_2(g(z,a))dA(z)\\
&\leqslant \varepsilon\|\varphi\|^q_{\mathcal{Q}_{K_2}(q,\beta)}.
\end{aligned}$$

Obviously $\|\varphi\|_{\mathcal{Q}_{K_2}(q,\beta)}<\infty$, therefore $I_1\leqslant C\varepsilon$.

Using $\frac{1}{q}+\frac{\alpha+2}{p}<1$ and (3.2), the boundedness of $(f_n)_{n\in\mathbb{N}}$ implies that

$$\begin{aligned}
I_2 &= C\|f_n\|^q_{\mathcal{Q}_{K_1}(p,\alpha)}\sum_{k>N}\Big(\frac{\|\varphi^k\|_{\mathcal{Q}_{K_2}(q,\beta)}}{k^{1-\frac{\alpha+2}{p}}}\Big)^q\\
&\leqslant C\varepsilon^q\sum_{k>N}\Big(\frac{1}{k^{1-\frac{\alpha+2}{p}}}\Big)^q\\
&\leqslant C\varepsilon^q.
\end{aligned}$$

Since ε is arbitrary, we conclude that $\|C_\varphi f_n\|_{\mathcal{Q}_{K_2}(q,\beta)}\to 0$ as $n\to\infty$. By Lemma 3.1, $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact. □

Corollary 3.3. *Let $0<\alpha,p,q<\infty$, $-2<\beta<\infty$ and φ be an analytic self map of $\mathbb{D}$. If $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact, then $C_\varphi:\mathcal{B}^{\frac{\alpha}{p}}\to\mathcal{B}^{\frac{\beta+2}{q}}$ is compact.*

Proof. Since $C_\varphi:\mathcal{Q}_{K_1}(p,\alpha)\to\mathcal{Q}_{K_2}(q,\beta)$ is compact, by Theorem 3.2,

$$\lim_{n\to\infty}n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{Q}_{K_2}(q,\beta)}=0.$$

Therefore $\limsup_{n\to\infty}n^{\frac{\alpha}{p}-1}\|\varphi^n\|_{\mathcal{B}^{\frac{\beta+2}{q}}}=0$. Hence by Theorem 1.2, $C_\varphi:\mathcal{B}^{\frac{\alpha}{p}}\to\mathcal{B}^{\frac{\beta+2}{q}}$ is compact. □

4. Composition operators on $\mathcal{Q}_s$ spaces

For $0 < s < \infty$, the space $\mathcal{Q}_s$ consists of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (g(z,a))^s dA(z) < \infty.$$

We know that $\mathcal{Q}_0$ is the Dirichlet space $\mathcal{D}$ and $\mathcal{Q}_1$=BMOA, the space of all analytic functions with bounded mean oscillation on $\mathbb{D}$ [2]. Furthermore the spaces $\mathcal{Q}_s$ are the same for all $1 < s < \infty$, and each equals the classical Bloch space $\mathcal{B}$. For a summary of recent research for $\mathcal{Q}_s$ spaces we refer to [11, 20, 21].

Bounded and compact composition operators on $\mathcal{Q}_s$ spaces were characterized by Lou in [8], Wirths and Xiao in [16], and Li in [6]. In this section we give a new and simple boundedness and compactness criterion for composition operators between $\mathcal{Q}_s$ spaces. Thus we announces a solution to the question posed by Wulan, Zheng and Zhu in [18].

In the sequel the symbol $u \simeq v$ will mean that there are two constants C_1 and C_2, independent of said or implied variables or functions such that $C_1 v \leqslant u \leqslant C_2 v$, and $u \leqslant C_2 v$ will be simply written as $u \lesssim v$.

By [22], we have

$$\|f\|_{\mathcal{Q}_s} \lesssim |f(0)| + [\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\frac{f(z)-f(a)}{1-|z|^2}|^2 (1-|\sigma_a(z)|^2)^s dA(z)]^{\frac{1}{2}}.$$

It is well-known that for $a \in \mathbb{D}$,

$$\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \simeq \frac{1}{(1-|z|^2)^2}, \quad z \in \mathbb{D}. \tag{4.1}$$

Also as a simple case of [26, Theorem 1.12], for $c > 0$, $t > -1$ and $a \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-\bar{a}z|^{t+2+c}} dA(z) \simeq \frac{1}{(1-|a|^2)^c}. \tag{4.2}$$

Using (4.1) and (4.2), we have

$$\begin{aligned}
\|z^n\|_{\mathcal{Q}_s} &\lesssim [\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\frac{z^n-a^n}{1-|z|^2}|^2 (1-|\sigma_a(z)|^2)^s dA(z)]^{\frac{1}{2}} \\
&\lesssim [\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(|z|^n+|a|^n)^2}{(1-|z|^2)^2} (1-|\sigma_a(z)|^2)^s dA(z)]^{\frac{1}{2}} \\
&\lesssim 2[\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^2} \frac{(1-|a|^2)^s(1-|z|^2)^s}{|1-\bar{a}z|^{2s}} dA(z)]^{\frac{1}{2}} \\
&\lesssim [\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \frac{(1-|a|^2)^s(1-|z|^2)^s}{|1-\bar{a}z|^{2s}} dA(z)]^{\frac{1}{2}} \\
&\lesssim [\sup_{a\in\mathbb{D}} (1-|a|^2)^{s+2} \int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{a}z|^{2s+4}} dA(z)]^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim [\sup_{a\in\mathbb{D}}(1-|a|^2)^{s+2}\frac{1}{(1-|a|^2)^{s+2}}]^{\frac{1}{2}}$$
$$\lesssim 1.$$

Therefore for $s > 0$, $(z^n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{Q}_s$ space which converges to zero uniformly on compact subsets of $\mathbb{D}$. Employing the sequence $(z^n)_{n\in\mathbb{N}}$ and using the same argument as in the proof of Theorem 3.2, one can prove that compactness of $C_\varphi : \mathcal{Q}_s \to \mathcal{Q}_r$ implies $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{Q}_r} = 0$.

It should be pointed out that for $s, r > 0$, if we take special spaces $\mathcal{Q}_{K_1}(p,\alpha) = \mathcal{Q}_s$, and $\mathcal{Q}_{K_2}(q,\beta) = \mathcal{Q}_r$, since $p = q = 2$, $\alpha = \beta = 0$, we get $\frac{1}{q} + \frac{\alpha+2}{p} > 1$. In this case, by simple calculation for each $n \geqslant 2$ the function

$$f_n(x) = nx^{n-1}(1-x), \quad \tfrac{n-2}{n-1} \leqslant x \leqslant \tfrac{n-1}{n},$$

is increasing, so for any x between $\frac{n-2}{n-1}$ and $\frac{n-1}{n}$ we have

$$\tfrac{1}{e} \leqslant f(\tfrac{n-2}{n-1}) \leqslant f_n(x) \leqslant f(\tfrac{n-1}{n}) \leqslant \tfrac{1}{2}. \tag{4.3}$$

Consider the set

$$D_n = \{z \in \mathbb{D} : \tfrac{n-2}{n-1} \leqslant |\varphi(z)| \leqslant \tfrac{n-1}{n}\}.$$

From (4.3) we deduce that for each $n \geqslant 2$,

$$\inf_{z\in D_n} n|\varphi(z)|^{n-1}(1-|\varphi(z)|) \geqslant \tfrac{1}{e}. \tag{4.4}$$

Imposing a stronger condition, a slightly modified argument of the proofs of Theorems 2.1 and 3.2, and equation (4.4) implies that:

Theorem 4.1. *Let $s, r > 0$ and φ be an analytic self-map of $\mathbb{D}$. Then*

(i) *If $\sup_{n\in\mathbb{N}} n\|\varphi^n\|_{\mathcal{Q}_r} < \infty$, then $C_\varphi : \mathcal{Q}_s \to \mathcal{Q}_r$ is bounded.*
(ii) *If $C_\varphi : \mathcal{Q}_s \to \mathcal{Q}_r$ be bounded, then $\sup_{n\in\mathbb{N}} \|\varphi^n\|_{\mathcal{Q}_r} < \infty$.*

Theorem 4.2. *Let $s, r > 0$ and φ be an analytic self-map of $\mathbb{D}$. Then*

(i) *If $\lim_{n\to\infty} n\|\varphi^n\|_{\mathcal{Q}_r} = 0$, then $C_\varphi : \mathcal{Q}_s \to \mathcal{Q}_r$ is compact.*
(ii) *If $C_\varphi : \mathcal{Q}_s \to \mathcal{Q}_r$ be compact, then $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{Q}_r} = 0$.*

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