Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 47–53
https://doi.org/10.4134/CKMS.c160017
pISSN: 1225-1763 / eISSN: 2234-3024

CERTAIN UNIFIED INTEGRAL FORMULAS INVOLVING THE GENERALIZED MODIFIED *k*-BESSEL FUNCTION OF FIRST KIND

Saiful Rahman Mondal and Kottakkaran Sooppy Nisar

Abstract. Generalized integral formulas involving the generalized modified k-Bessel function $J_{k,\nu}^{b,c,\gamma,\lambda}(z)$ of first kind are expressed in terms generalized Wright functions. Some interesting special cases of the main results are also discussed

1. Introduction

The integral formula involving various special functions have been studied by many researchers ([3, 7]). In 1888, Pincherle gave the integrals involving product of Gamma functions along vertical lines (see [16, 18, 19]). Barnes [2] and Cahen [4] extended some of these integrals in the study of Riemann zeta function and other Drichlet's series. The integral representation of Fox H-functions and hypergeometric ${}_2F_1$ functions studied by [11] and [1] respectively. Also, the integral representation of Bessel functions are given in many recent works (see [3, 5, 6, 22]).

In [21], the k-Bessel function of the first kind defined by the following series

$$J_{k,\nu}^{\gamma,\lambda}(z) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n + \upsilon + 1)} \frac{(-1)^n (z/2)^n}{(n!)^2}, \tag{1.1}$$

where $k \in \mathbb{R}$; $\alpha, \lambda, \gamma, \upsilon \in C$; $\Re(\lambda) > 0$ and $\Re(\upsilon) > 0$. Here $(\gamma)_{n,k}$ refer the well-known $k-$Pochhammer symbol defined as (see [8])

$$(\gamma)_{n,k} = \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C}\setminus\{0\}) \\ \gamma(\gamma+k)\cdots(\gamma+(n-1)k) & (n \in \mathbb{N}; \gamma \in \mathbb{C}) \end{cases} \tag{1.2}$$

Received January 25, 2016; Revised May 12, 2016.
2010 *Mathematics Subject Classification.* 33C20, 33B15, 33C05, 33C10.
Key words and phrases. k-Bessel function, Gamma function, hypergeometric function ${}_2F_1$, generalized hypergeometric function ${}_pF_q$, generalized (Wright) hypergeometric functions ${}_p\Psi_q$, Oberhettinger's integral formula.

while $\Gamma_k(z)$ denotes the k-gamma function defined by (see [8])

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \quad \Re(z) > 0, \quad k > 0. \tag{1.3}$$

Clearly, $\Gamma_k(z)$ reduces to the classical $\Gamma(z)$ function for $k = 1$.

In this paper, we introduce a generalization of the k-Bessel functions defined as:

Let $k \in \mathbb{R}$; $\alpha, \sigma, \gamma, \upsilon, c, b \in \mathbb{C}$; $\Re(\sigma) > 0$, $\Re(\upsilon) > 0$, the generalized k-Bessel function of the first kind given by the following series

$$J_{k,\nu}^{b,c,\gamma,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k\left(\sigma n + \upsilon + \frac{b+1}{2}\right)} \frac{(z/2)^{\upsilon+2n}}{(n!)^2}, \tag{1.4}$$

where to note that $J_{k,\nu}^{1,1,\gamma,\sigma}$ is the k-Bessel function and $J_{k,\nu}^{1,-1,\gamma,\sigma}$ denotes the modified k-Bessel function.

Here our aim is to establish two generalized integral formulas, which are expressed in terms of generalized k-Wright functions, by inserting newly generalized modified k-Bessel function.

The generalized Wright hypergeometric function ${}_p\Psi_q(z)$ is given by the series

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \middle| z\right] = \frac{\prod_{j=1}^q \Gamma(\beta_j)}{\prod_{i=1}^p \Gamma(\alpha_i)} \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}, \tag{1.5}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$. Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [10] and under the condition

$$\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1 \tag{1.6}$$

was found in the work of [23, 24]. Properties of this generalized Wright function were investigated in [13], (see also [14, 15]). In particular, it was proved [13] that ${}_p\Psi_q(z)$, $z \in \mathbb{C}$ is an entire function under the condition (1.6).

The generalized hypergeometric function represented as follows [20]:

$${}_pF_q\left[\begin{array}{c} (\alpha_p); \\ (\beta_q); \end{array} z\right] = \sum_{n=0}^\infty \frac{\Pi_{j=1}^p (\alpha_j)_n}{\Pi_{j=1}^q (\beta_j)_n} \frac{z^n}{n!}, \tag{1.7}$$

provided $p \le q; p = q + 1$ and $|z| < 1$, where $(\lambda)_n$ is well known Pochhammer symbol defined for $\lambda \in \mathbb{C}$ (see [20])

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \end{cases} \tag{1.8}$$

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \qquad \left(\lambda \in \mathbb{C} \backslash \mathbb{Z}_0^-\right), \tag{1.9}$$

where $\mathbb{Z}_0^-$ is the set of non positive integers.

If we put $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ in (1.5), then (1.7) is a special case of the generalized Wright function:

(1.10)
$$ {}_p\Psi_q(z) = {}_p\Psi_q \left[\begin{array}{c} (\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1); \end{array} z \right] = \frac{\prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \, {}_pF_q \left[\begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{array} z \right]. $$

For the present investigation, we need the following result of Oberhettinger [17]

$$ \int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx = 2\lambda a^{-\lambda} \left(\frac{a}{2} \right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda - \mu)}{\Gamma(1 + \lambda + \mu)} \tag{1.11} $$

provided $0 < \Re(\mu) < \Re(\lambda)$.

Also, we need the following relation of Γ_k with the classical gamma Euler function (see [21]):

$$ \Gamma_k(z + k) = z\Gamma_k(z), \tag{1.12} $$

$$ \Gamma_k(z) = k^{\frac{z}{k} - 1} \Gamma\left(\frac{z}{k} \right), \tag{1.13} $$

$$ (\gamma)_{n,k} = k^n \left(\frac{\gamma}{k} \right)_n. \tag{1.14} $$

2. Main results

Two generalized integral formulas established here, which expressed in terms of generalized Wright functions (1.10) by inserting the generalized modified k-Bessel function of the first kind (1.4) with the suitable argument in the integrand of (1.11).

Theorem 1. *For $\lambda, \mu, \nu, c, \sigma \in \mathbb{C}$, $\Re(\lambda + \nu + 2) > \Re(\mu) > 0$ and $x > 0$. Then the following formula holds true*:

$$ \begin{aligned} &\int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{k,\upsilon}^{b,c,\gamma,\sigma} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \\ = \; & \frac{2^{1-\upsilon-\mu} y^{\upsilon} k^{1 - \frac{\upsilon}{k} - \frac{b+1}{2k}} a^{\mu - \lambda - \nu} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right)} \\ & \times {}_3\Psi_4 \left[\begin{array}{c} \left(\frac{\gamma}{k}, 1\right), (\lambda + \upsilon + k, 2), (\lambda + \nu - \mu, 2); \\ \left(\frac{\nu}{k} + \frac{b+1}{2k}, \frac{\sigma}{k}\right), (\lambda + \nu, 2), (1 + \lambda + \upsilon + \mu, 2), (1, 1); \end{array} \frac{c k^{1 - \frac{\sigma}{k}} y^2}{4a^2} \right]. \end{aligned} \tag{2.1} $$

Proof. By applying (1.4) to the LHS of (2.1) and interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions , we obtain

$$ \begin{aligned} I &= \int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{k,\upsilon}^{b,c,\gamma,\sigma} \left(\frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \\ &= \int_0^\infty x^{\mu-1} \left(x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \end{aligned} $$

$$\times \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k\left(\sigma n+\upsilon+\frac{b+1}{2}\right)} \frac{\left(\frac{y}{2}\right)^{\upsilon+2n}}{(n!)^2}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\upsilon+2n)} dx.$$

Now, using (1.13) and (1.14), we have

$$I = \sum_{n=0}^{\infty} \frac{(c)^n k^n \Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\gamma}{k}\right) k^{\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)} \frac{\left(\frac{y}{2}\right)^{\upsilon+2n}}{(n!)^2}$$
$$\times \int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\nu+2n)} dx.$$

In view of the conditions given in Theorem 1, since $\Re(\lambda+\nu) > \Re(\mu) > 0$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Applying (1.11) to the integrand of (2.1) and obtain the following expression:

$$I = \sum_{n=0}^{\infty} \frac{(c)^n k^n \Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\gamma}{k}\right) k^{\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)} \frac{\left(\frac{y}{2}\right)^{\upsilon+2n}}{(n!)^2}$$
$$\times 2(\lambda+\nu+2n)\, a^{-(\lambda+\nu+2n)}\left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\lambda+\nu+2n-\mu)}{\Gamma(1+\lambda+\nu+\mu+2n)}$$
$$= \frac{2^{1-\upsilon-\mu} y^{\upsilon} k^{1-\frac{\upsilon}{k}-\frac{b+1}{2k}} a^{\mu-\lambda-\nu}\Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(\lambda+\nu+1+2n)}{n!\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)\Gamma(\lambda+\nu+2n)}$$
$$\times \frac{\Gamma(\lambda+\nu-\mu+2n)}{\Gamma(1+\lambda+\nu+\mu+2n)\Gamma(n+1)}\left(\frac{ck^{1-\frac{\sigma}{k}}y^2}{4a^2}\right)^n.$$

In view of definition (1.5), we obtain the desired result. $\square$

Corollary 2.1 ([6]). *Let the conditions of theorem 1 be satisfied and let $k = \gamma = \sigma = 1$ and $c = -c$ in (2.1). Then the following integral formula holds:*

$$\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{k,\upsilon}^{b,c,\gamma,\sigma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= 2^{1-\nu-\mu} a^{\mu-\lambda-\upsilon} y^{\upsilon}\Gamma(2\mu)$$

$$\times {}_2\Psi_3\left[\begin{array}{c}(1+\lambda+\upsilon,2),(\nu+\lambda-\mu,2);\\ \left(\nu+\frac{b+1}{2},1\right),(1+\lambda+\upsilon+\mu,2),(\lambda+\nu,2);\end{array} \frac{-cy^2}{4a^2}\right]. \tag{2.2}$$

Theorem 2. *For $\sigma, \mu, \nu, c, b \in \mathbb{C}$, $0 < \Re(\mu+\nu+2) < \Re(\lambda+\nu+2)$ and $x > 0$, then the following integral formula holds true:*

$$\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{k,\upsilon}^{b,c,\gamma,\sigma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \frac{2^{1-2\nu-\mu} y^{\nu} a^{\mu-\lambda}\Gamma(\lambda-\mu)}{k^{\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\gamma}{k}\right)}$$

$$\times {}_3\Psi_4\left[\begin{array}{c}\left(\frac{\gamma}{k},1\right),(\lambda+\upsilon+1,2),(2\mu+2\nu,4);\\ \left(\frac{\upsilon}{k}+\frac{b+1}{2k},\frac{\sigma}{k}\right),(\nu+\lambda,2),(\lambda+\mu+2\nu+1,4),(1,1);\end{array} \frac{ck^{1-\frac{\sigma}{k}}y^2}{4}\right]. \tag{2.3}$$

Proof. By applying (1.4) to the LHS of (2.3) and interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we obtain

$$\begin{aligned} I_1 &= \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{k,\upsilon}^{b,c,\gamma,\sigma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx \\ &= \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \\ &\quad \times \sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\sigma n+\upsilon+\frac{b+1}{2}\right)} \frac{(2)^{-\upsilon-2n}(xy)^{\upsilon+2n}}{(n!)^2}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\upsilon+2n)}dx. \end{aligned}$$

Using the (1.13) and (1.14), we have

$$\begin{aligned} I_1 &= \sum_{n=0}^\infty \frac{(c)^n k^n \Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)} \frac{\left(\frac{y}{2}\right)^{\upsilon+2n}}{(n!)^2} \\ &\quad \times \int_0^\infty x^{(\mu+\upsilon+2n)-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+\upsilon+2n)}dx. \end{aligned}$$

Applying (1.11) to the integrand of (2.3), we obtain the following expression:

$$\begin{aligned} I_1 &= \sum_{n=0}^\infty \frac{(c)^n k^n \Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)} \frac{\left(\frac{y}{2}\right)^{\upsilon+2n}}{(n!)^2} \\ &\quad \times 2(\lambda+\upsilon+2n)a^{-(\lambda+\upsilon+2n)}\left(\frac{a}{2}\right)^{\mu+\upsilon+2n}\frac{\Gamma(2\mu+2\nu+4n)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+2\nu+\mu+4n)}. \end{aligned}$$

By making the use of (1.12) it follows that

$$\begin{aligned} I_1 &= \frac{2^{1-2\nu-\mu}y^\nu a^{\mu-\lambda}\Gamma(\lambda-\mu)}{k^{\frac{\upsilon}{k}+\frac{b+1}{2k}-1}\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^\infty \frac{\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(\lambda+\upsilon+1+2n)}{n!\Gamma\left(\frac{\sigma}{k}n+\frac{\upsilon}{k}+\frac{b+1}{2k}\right)\Gamma(\lambda+\upsilon+2n)} \\ &\quad \times \frac{\Gamma(2\mu+2\nu+4n)}{\Gamma(\lambda+2\nu+\mu+1+4n)\Gamma(n+1)}\left(\frac{ck^{1-\frac{\sigma}{k}}y^2}{4}\right)^n, \end{aligned}$$

and the desired result follows from the definition (1.5). □

Corollary 2.2 ([6])**.** *Let the conditions given in theorem* 2 *satisfied and set* $k=\sigma=\gamma=1$ *and* $c=-c$*, Theorem* 2 *reduces to the following form*

$$\begin{aligned} &\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} J_{1,\upsilon}^{b,c,1,1}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx \\ &= 2^{1-2\nu-\mu}y^\nu a^{\mu-\lambda}\Gamma(\lambda-\mu) \\ &\quad \times {}_2\Psi_3\left[\begin{array}{c} (\lambda+\upsilon+1,2),(2\mu+2\nu,4); \\ \left(\upsilon+\frac{b+1}{2},1\right),(\nu+\lambda,2),(\lambda+\mu+2\nu+1,4); \end{array} -\frac{cy^2}{4}\right]. \end{aligned}$$

Remark 2.1. Setting $b=c=1$ in (2.1) and (2.3) with some appropriate parameter replacements, we respectively get two integral formula for the Bessel function $J_\upsilon(z)$ given by Choi and Agarwal [5].

3. Conclusion

The integral formulas for generalized modified k-Bessel function of first kind is derived and the results expressed in term of Fox-Wright function. Some of interesting special cases also derived from the main results. Using some suitable parametric replacement, theorem 1 and theorem 2 gives the unified integral representation of Bessel function when $c = -1 = -b$ and integral representation of modified Bessel function by putting $b = c = 1$.

References

[1] S. Ali, *On some new unified integrals*, Advances in Computational Mathematics and its Applications **1** (2012), no. 3, 151–153 .
[2] E. W. Barnes, *A new development of the theory of the hypergeometric functions*, Proc. Lond. Math. Soc. **6** (1908), 141–177.
[3] Y. A. Brychkov, *Handbook of Special Functions*, CRC, Boca Raton, FL, 2008.
[4] E. Cahen, *Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues*, Ann. Sci. École Norm. Sup. (3) **11** (1894), 75–164.
[5] J. Choi and P. Agarwal, *Certain unified integrals associated with Bessel functions*, Bound. Value Probl. **2013** (2013), 95, 9 pp.
[6] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc. **51** (2014), no. 4, 995–1003.
[7] J. Choi, A. Hasanov, H. M. Srivastava, and M. Turaev, *Integral representations for Srivastava's triple hypergeometric functions*, Taiwanese J. Math. **15** (2011), no. 6, 2751–2762.
[8] R. Diaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.
[9] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vols. I, II*, McGraw-Hill Book Company, Inc., New York, 1953. NewYork, Toronto, London, 1953.
[10] C. Fox, *The asymptotic expansion of generalized hypergeometric functions*, Proc. Lond. Math. Soc. **27** (1928), no. 4, 389–400.
[11] M. Garg and S. Mittal, *On a new unified integral*, Proc. Indian Acad. Sci. Math. Sci. **114** (2004), no. 2, 99–101.
[12] A. A. Kilbas and M. Saigo, *H-transforms*, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[13] A. A. Kilbas, M. Saigo, and J. J. Trujillo, *On the generalized Wright function*, Fract. Calc. Appl. Anal. **5** (2002), no. 4, 437–460.
[14] A. A. Kilbas and N. Sebastian, *Generalized fractional integration of Bessel function of the first kind*, Integral Transforms Spec. Funct. **19** (2008), no. 11-12, 869–883.
[15] ———, *Fractional integration of the product of Bessel function of the first kind*, Fract. Calc. Appl. Anal. **13** (2010), no. 2, 159–175.
[16] F. Mainardi, G. Pagnini, and S. Pincherle, *The pioneer of the Mellin-Barnes integrals*, J. Comput. Appl. Math. **153** (2003), no. 1-2, 331–342.
[17] F. Oberhettinger, *Tables of Mellin Transforms*, Springer, New York, 1974.
[18] S. Pincherle, *Sulle funzioni ipergeometriche generalizzate Nota I*, Atti della Reale Accademia dei Lincei. Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali, Serie 4, Vol. 4 (1888), 694–700.
[19] ———, *Sulle funzioni ipergeometriche generalizzate. Nota I-II*, In: Opere Scelte-I, Unione Matematica Italiana, Ed. Cremonese, Rome (1954), 223–239.
[20] E. D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[21] L. G. Romero, G. A. Dorrego, and R. A. Cerutti, *The k-Bessel function of first kind*, International Mathematical Forum **7** (2012), no. 38, 1859–1864.
[22] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library Edition. Camdridge University Press, Camdridge, 1995.
[23] E. M. Wright, *The asymptotic expansion of integral functions defined by Taylor series*, Philos. Trans. Roy. Soc. London, Ser. A. **238** (1940), 423–451.
[24] ———, *The asymptotic expansion of the generalized hypergeometric function*, Proc. Lond. Math. Soc. (2) **46** (1940), 389–408.

Saiful Rahman Mondal
Department of Mathematics and Statistics
College of Science
King Faisal University
Al-Ahsa, Pin. 31982, Saudi Arabia
E-mail address: saiful786@gmail.com, smondal@kfu.edu.sa

Kottakkaran Sooppy Nisar
Department of Mathematics
College of Arts and Science
Prince Sattam bin Abdulaziz University
Wadi Aldawaser, Riyadh region 11991, Saudi Arabia
E-mail address: ksnisar1@gmail.com, n.sooppy@psau.edu.sa