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CONVERGENCE OF A NEW MULTISTEP ITERATION IN CONVEX CONE METRIC SPACES

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Abstract. In this paper, we propose a new multistep iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Then we show that our iteration converges to a common fixed point of this class of mappings under suitable conditions. Our result generalizes the corresponding result of Lee [5] from the closed convex subset of a convex cone metric space to whole space.

1. Introduction

Let X be a nonempty set and $T : X \to X$ be a mapping. A fixed point for T is a point $x \in X$ such that $Tx = x$. In the sequel, we will designate the set $\{1, 2, \ldots, r\}$ by I and the set of natural numbers by $\mathbb{N}$. We will also denote the set of fixed points of T by $F(T)$ and the set of common fixed points of a finite family of mappings $\{T_i : i \in I\}$ by $F := (\bigcap_{i=1}^{r} F(T_i))$. Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational inequalities, linear inequalities, optimization, and approximation theory. Takahashi [8] introduced the concept of convex metric space and the properties of the space. In 2007, Huang and Zhang [2] generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric spaces with the assumption of the normality of a cone. In [4], Lee introduced the concept of convex cone metric space and studied a convergence of two iterative sequences for uniformly quasi-lipschitzian mappings.

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In this paper, we will give convergence theorems of three new iteration scheme to fixed points of asymptotically quasi-nonexpansive mappings in convex cone metric spaces. Our convergence results generalize and refine many known results in the current literature [1, 3, 4, 5, 11].

2. Preliminaries

Before we start out discussing fixed point theorems in convex cone metric spaces, for the convenience of the reader we repeat some background and note some relevant material.

Let E be a normed vector space with a normal solid cone P, then we have the following definitions due to [2].

Definition 2.1 ([2])**.** *A* nonempty subset P of E is called a cone if P is closed, $P \neq \{\theta\}$ for $a, b \in \mathbb{R}^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap \{-P\} = \{\theta\}$. We define a partial ordering $\preceq$ in E as $x \preceq y$ if $y - x \in P$. $x \ll y$ indicates that $y - x \in intP$ and $x \prec y$ means that $x \preceq y$ but $x \neq y$. A cone P is said to be solid if its interior $intP$ is nonempty. A cone P is said to be normal if there exists a positive number k such that for $x, y \in P$, $0 \preceq x \preceq y$ implies $\|x\| \leq k \|y\|$. The least positive number k is called the normal constant of P.

Definition 2.2 ([2])**.** Let X be a nonempty set. A mapping $d : X \times X \to (E, P)$ is called a cone metric if (i) for $x, y \in X$, $0 \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$, (ii) for $x, y \in X$, $d(x, y) = d(y, x)$ and (iii) for $x, y, z \in X$, $d(x, y) \preceq d(x, z) + d(z, y)$. A nonempty set X with a cone metric $d : X \times X \to (E, P)$ is called a cone metric space denoted by (X, d), where P is a solid normal cone.

Definition 2.3. Let (X, d) be a cone metric space and $T : (X, d) \to (X, d)$ a mapping. The mapping T is said to be

(1) nonexpansive if
$$d(Tx, Ty) \preceq d(x, y) \text{ for all } x, y \in X.$$
(2) quasi-nonexpansive if $F(T) \neq \emptyset$ and
$$d(Tx, p) \preceq d(x, p) \text{ for all } x \in X \text{ and } p \in F(T).$$
(3) asymptotically nonexpansive if there exists $u_n \in [1, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that
$$d(T^n x, T^n y) \preceq u_n d(x, y) \text{ for all } x, y \in X \text{ and } n \in \mathbb{N}.$$
(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists $u_n \in [1, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 1$ such that
$$d(T^n x, p) \preceq u_n d(x, p) \text{ for all } x \in X,\ p \in F(T) \text{ and } n \in \mathbb{N}.$$

Remark 2.4. From the above definition, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. However, the inverse relation does not hold.

Definition 2.5 ([4]). Let (X,d) be a cone metric space. A mapping $W: X^2 \times I \to X$ is called a convex structure on X if $d(W(x,y,\lambda),u) \preceq \lambda d(x,u) + (1-\lambda) d(y,u)$ for all $x,y,z,u \in X$ and λ in $I=[0,1]$. A cone metric space (X,d) with a convex structure W is called a convex cone metric space and denoted as (X,d,W). A nonempty subset C of a convex cone metric space (X,d,W) is said to be convex if $W(x,y,\lambda) \in C$ for all $x,y \in C$ and λ in $\lambda \in I$.

Definition 2.6. A sequence $\{x_n\}$ in a cone metric space (X,d) is said to converge to $x \in X$ and denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$) if for any $c \in intP$, there exists a natural number N such that for all $n > N$, $c - d(x_n, x) \in intP$. A sequence $\{x_n\}$ in (X,d) is called a Cauchy sequence if for any $c \in intP$, there exists a natural number N such that for all $n,m > N$, $c - d(x_n, x_m) \in intP$. A cone metric space (X,d) is said to be complete if every Cauchy sequence converges.

In other words, $\{x_n\}$ is said to converge to x, if there exists a natural number N such that $d(x_n,x) \ll c$ for all $n > N$ and for any $c \in E$ with $0 \ll c$. $\{x_n\}$ is called a Cauchy sequence in X, if there exists a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m > N$ and for any $c \in E$ with $0 \ll c$.

Proposition 2.7 ([2]). *Let $\{x_n\}$ be a sequence in a cone metric space (X,d) and P be a normal cone. Then*

(1) *$\{x_n\}$ converges to x in X if and only if $d(x_n,x) \to \theta$ (as $n \to \infty$) in E.*
(2) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_m) \to \theta$ (as $n,m \to \infty$) in E.*

Let (X,d,W) be a convex cone metric space and $T_i : X \to X$ $(i \in I)$ be a finite family of asymptotically quasi-nonexpansive mappings. Let $\{\alpha_{ni}\}$ and $\{\beta_{ni}\}$ be real sequences in $[0,1]$ such that $\alpha_{ni} + \beta_{ni} < 1$ for all $i \in I$. For any given $x_1 \in X$ and $n \geq 1$, a sequence $\{x_n\}$ defined as

$$\begin{aligned}
y_n &= W\left(T_r^n x_n, x_n, \alpha_{nr}\right),\\
y_{n+1} &= W\left(T_{r-1}^n x_n, W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), \alpha_{n(r-1)}\right),\\
y_{n+2} &= W\left(T_{r-2}^n y_n, W\left(T_{r-2}^n y_{n+1}, x_n, \frac{\beta_{n(r-2)}}{1-\alpha_{n(r-2)}}\right), \alpha_{n(r-2)}\right),\\
&\vdots\\
y_{n+r-2} &= W\left(T_2^n y_{n+r-4}, W\left(T_2^n y_{n+r-3}, x_n, \frac{\beta_{n2}}{1-\alpha_{n2}}\right), \alpha_{n2}\right),\\
x_{n+1} &= W\left(T_1^n y_{n+r-3}, W\left(T_1^n y_{n+r-2}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right),
\end{aligned} \tag{1}$$

is called the modified multistep iteration in convex cone metric spaces. Restricting a normed linear space (E,P) to a real number system $(\mathbb{R},[0,\infty))$ and using "$W(x,y,0)=y$ for any $x,y\in X$ ([9], Proposition 1.2(a))", the modified multistep iteration (1) reduces in a convex cone metric space to Khan and Ahmed [3] iteration (2) with $\alpha_{n1}=\alpha_{n2}=\cdots=\alpha_{n(r-1)}=0$. Their sequence is defined as follows:

$$
\begin{aligned}
x_{n+1} &= W\left(T_r^n y_{(r-1)n}, x_n, \beta_{rn}\right), \\
y_{(r-1)n} &= W\left(T_{r-1}^n y_{(r-2)n}, x_n, \beta_{(r-1)n}\right), \\
y_{(r-2)n} &= W\left(T_{r-2}^n y_{(r-3)n}, x_n, \beta_{(r-2)n}\right), \\
&\vdots \\
y_{2n} &= W\left(T_2^n y_{1n}, x_n, \beta_{2n}\right), \\
y_{1n} &= W\left(T_1^n y_{0n}, x_n, \beta_{1n}\right),
\end{aligned}
\tag{2}
$$

where $y_{0n}=x_n$ and $\beta_{in}\in[0,1]$ for all $n\in\mathbb{N}$, $i\in I$. For $r=3$, (1) reduces to the modified three-step iteration:

$$
\begin{aligned}
y_n &= W\left(T_3^n x_n, x_n, \alpha_{n3}\right) \\
y_{n+1} &= W\left(T_2^n x_n, W\left(T_2^n y_n, x_n, \frac{\beta_{n2}}{1-\alpha_{n2}}\right), \alpha_{n2}\right) \\
x_{n+1} &= W\left(T_1^n y_n, W\left(T_1^n y_{n+1}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right),
\end{aligned}
\tag{3}
$$

where $\{\alpha_{ni}\},\{\beta_{ni}\}$ are sequences in $[0,1]$ such that $\alpha_{ni}+\beta_{ni}<1$ for each $i\in\{1,2,3\}$ and $n\geq 1$. Restricting a normed linear space (E,P) to a real number system $(\mathbb{R},[0,\infty))$ and the metric space (X,d) to a Banach space with $W(x,y,\alpha)=\alpha x+(1-\alpha)y$, we show that the modified multistep iteration (1) coincides with the iterative scheme of Yang [11] in Banach space setting. The iteration scheme (3) coincides with the modified Noor iterative scheme defined by Suantai [7] in Banach spaces under same restriction, when $T_i=T$ for all $i\in\{1,2,3\}$. By the similar way iteration (1) and (3) reduce in a Banach space to Xu-Noor algorithm [10], Ishikawa algorithm, Mann algorithm and their convex metric spaces version.

Next lemma plays a key role in the proof of our main results.

Lemma 2.8 ([6]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n<\infty,\ \ a_{n+1}=(1+b_n)\,a_n,\ \ n\geq 0.$$

Then

i) $\lim_{n\to\infty} a_n$ *exists,*

ii) *if either* $\liminf_{n\to\infty} a_n=0$ *or* $\limsup_{n\to\infty} a_n=0$, *then* $\lim_{n\to\infty} a_n=0$.

3. Main results

Let's start with a proposition. It can be proved easily by following the steps in the proof of [3, Proposition 2.1].

Proposition 3.1. *Let X be a cone metric space and $T_i : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F := (\bigcap_{i=1}^{r} F(T_i)) \neq \emptyset$. Then, there exist a point $p \in F$ and a sequence $\{u_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} u_n = 1$ such that*

$$d\left(T_i^n x, p\right) \preceq u_n d\left(x, p\right)$$

for all $x \in X$, for each $i \in I$.

We now prove convergence theorem of the iterative scheme (1) in convex cone metric spaces.

Theorem 3.2. *Let (X, d, W) be a convex cone metric space with a cone metric $d : X \times X \to (E, P)$, where P is a solid normal cone with the normal constant k. Let $T_i : X \to X$ $(i \in I)$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} (u_n - 1) < \infty$ and $\{x_n\}$ is the modified multistep iteration in* (1). (i) *If $\{x_n\}$ converges to a unique point in F, then $\liminf_{n\to\infty} d(x_n, F) = \theta$.* (ii) *$\{x_n\}$ converges to a unique point in F, if X is complete and $\liminf_{n\to\infty} d(x_n, F) = \theta$.*

Proof. The statement (i) is obvious. We consider only the statement (ii). Let $p \in F$. Then, from Proposition 3.1 and (1), we have

$$\begin{aligned} d\left(y_n, p\right) &= d\left(W\left(T_r^n x_n, x_n, \alpha_{nr}\right), p\right) \\ &\preceq \alpha_{nr} d\left(T_r^n x_n, p\right) + \left(1 - \alpha_{nr}\right) d\left(x_n, p\right) \\ &\preceq \alpha_{nr} u_n d\left(x_n, p\right) + \left(1 - \alpha_{nr}\right) d\left(x_n, p\right) \\ &\preceq u_n d\left(x_n, p\right). \end{aligned} \tag{4}$$

By a similar way, using (4), we get

$$\begin{aligned} d\left(y_{n+1}, p\right) &= d\left(W\left(T_{r-1}^n x_n, W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), \alpha_{n(r-1)}\right), p\right) \\ &\preceq \alpha_{n(r-1)} d\left(T_{r-1}^n x_n, p\right) + \left(1 - \alpha_{n(r-1)}\right) \\ &\quad d\left(W\left(T_{r-1}^n y_n, x_n, \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right), p\right) \\ &\preceq \alpha_{n(r-1)} u_n d\left(x_n, p\right) + \left(1 - \alpha_{n(r-1)}\right) \\ &\quad \left(\frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}} d(T_{r-1}^n y_n, p) + \left(1 - \frac{\beta_{n(r-1)}}{1-\alpha_{n(r-1)}}\right) d\left(x_n, p\right)\right) \\ &\preceq \alpha_{n(r-1)} u_n d\left(x_n, p\right) + \beta_{n(r-1)} u_n d\left(y_n, p\right) \\ &\quad + (1 - \alpha_{n(r-1)} - \beta_{n(r-1)}) d\left(x_n, p\right) \\ &\preceq \alpha_{n(r-1)} u_n d\left(x_n, p\right) + \beta_{n(r-1)} u_n^2 d\left(x_n, p\right) \end{aligned}$$

$$+ (1-\alpha_{n(r-1)}-\beta_{n(r-1)})d(x_n,p)$$
$$\preceq u_n^2 d(x_n,p).$$

By induction, considering (1) we have

$$d(y_{n+j},p) \preceq u_n^{j+1} d(x_n,p) \tag{5}$$

for some $j = 0, 1, \ldots, r-2$. Hence, it derives from (1) and (5) that

$$\begin{aligned}
& d(x_{n+1},p) \\
&= d\left(W\left(T_1^n y_{n+r-3}, W\left(T_1^n y_{n+r-2}, x_n, \frac{\beta_{n1}}{1-\alpha_{n1}}\right), \alpha_{n1}\right), p\right) \\
&\preceq \alpha_{n1} d(T_1^n y_{n+r-3}, p) \\
&\quad + (1-\alpha_{n1})\left(\frac{\beta_{n1}}{1-\alpha_{n1}} d(T_1^n y_{n+r-2}, p) + \left(1-\frac{\beta_{n1}}{1-\alpha_{n1}}\right) d(x_n,p)\right) \\
&\preceq \alpha_{n1} u_n d(y_{n+r-3},p) + \beta_{n1} u_n d(y_{n+r-2},p) + (1-\alpha_{n1}-\beta_{n1}) d(x_n,p) \\
&\preceq u_n^r d(x_n,p) \\
&= (1+(u_n^r-1))\, d(x_n,p).
\end{aligned}$$

Thus, by the normality of P, for the normal constant $k > 0$

$$\|d(x_{n+1},p)\| \leq k(1+(u_n^r-1))\|d(x_n,p)\|. \tag{6}$$

Since p is an arbitrary point in F, we have

$$\|d(x_{n+1},F)\| \leq k(1+(u_n^r-1))\|d(x_n,F)\|.$$

Since $0 \leq t^r - 1 \leq rt^{r-1}(t-1)$ for all $t \geq 1$, the assumption $\sum_{n=1}^{\infty}(u_n - 1) < \infty$ implies that $\{u_n\}$ is bounded, then $u_n \in [1, M]$, $\forall n \geq 1$ and for some M. Hence, $u_n^r - 1 \leq rM^{r-1}(u_n - 1)$ holds for all $n \geq 1$. Therefore, $\sum_{n=1}^{\infty}(u_n^r - 1) < \infty$. By Lemma 2.8, $\lim_{n\to\infty}\|d(x_{n+1},F)\|$ exists. Now $\liminf_{n\to\infty}\|d(x_n,F)\| = 0$ implies $\lim_{n\to\infty}\|d(x_n,F)\| = 0$. Since $\lim_{n\to\infty}\|d(x_n,F)\| = 0$, for any positive real number ε, there exists a natural number $N_0 \in N$ such that $\|d(x_n,F)\| \leq \frac{\varepsilon}{(1+KM)}$ for $n \geq N_0$. In particular, there exists a point $p_0 \in F$ such that $\|d(x_n,p_0)\| \leq \frac{\varepsilon}{(1+KM)}$ for $n \geq N_0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X. For all $x \geq 0$, we know that $1 + x \leq e^x$. Considering it for the inequality (6), for each corresponding normal constant k_i $(i = 1, 2, \ldots, m)$, we get

$$\begin{aligned}
\|d(x_{n+m},p)\| &\leq k_1\left(1+\left(u_{n+m-1}^r-1\right)\right)\|d(x_{n+m-1},p)\| \\
&\leq k_1 e^{u_{n+m-1}^r-1}\|d(x_{n+m-1},p)\| \\
&\leq k_1 e^{u_{n+m-1}^r-1}\left[k_2\left(1+\left(u_{n+m-2}^r-1\right)\right)\|d(x_{n+m-2},p)\|\right] \\
&\leq k_1 k_2 e^{u_{n+m-1}^r-1+u_{n+m-2}^r-1} d(x_{n+m-2},p) \\
&\quad \vdots
\end{aligned}$$

$$\leq \prod_{j=1}^{m} k_j e^{\sum_{j=n}^{n+m-1}\left(u_j^r-1\right)} \left\|d\left(x_n,p\right)\right\|$$
$$\leq KM \left\|d\left(x_n,p\right)\right\|,$$

where $K=\prod_{j=1}^{m} k_j$ and $M=e^{\sum_{j=n}^{n+m-1}\left(u_j^r-1\right)}<\infty$. Thus, we have

$$\left\|d\left(x_{n+m},p\right)\right\| \leq KM \left\|d\left(x_n,p\right)\right\|$$

for all $m,n\in\mathbb{N}$ and $p\in F$.

On the other hand, we have for $m+n,n>N_1$,

$$\begin{aligned}\left\|d\left(x_{n+m},x_n\right)\right\| &\leq \left\|d\left(x_{n+m},p_0\right)\right\| + \left\|d\left(x_n,p_0\right)\right\| \\ &\leq KM\left\|d\left(x_n,p_0\right)\right\| + \left\|d\left(x_n,p_0\right)\right\| \\ &= (1+KM)\left\|d\left(x_n,p_0\right)\right\| \\ &< (1+KM)\frac{\varepsilon}{(1+KM)} = \varepsilon.\end{aligned}$$

Now, for $c\in E$ with $0\ll c$, there exists a positive number δ such that for $d\in E$ with $\|d\|<\delta$, $c-d\in intP$. From the fact that $d\left(x_n,x_m\right)\to\theta$ (as $n,m\to\infty$) in E, for such δ, there exists a natural number N such that $\left\|d\left(x_n,x_m\right)\right\|$ for all $n,m\geq N$. Thus $c-d\left(x_n,x_m\right)\in intP$, which means that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some point q in X.

Finally, we show that $q\in F$. Let $\{q_n\}$ be a sequence in F such that $q_n\to q$. Since

$$\begin{aligned}d(q,T_iq) &\preceq d(q,q_n)+d(q_n,T_iq) \\ &= d(q,q_n)+d(T_iq_n,T_iq) \\ &\preceq d(q,q_n)+(1+u_i)\,d(q_n,q),\end{aligned}$$

letting also $n\to\infty$ in above inequality, we have $q\in F$ for all $i\in I$. Thus $q\in F$, which means that F is closed. Since $d(p_0,F)=d(\lim_{n\to\infty}x_n,F)=\lim_{n\to\infty}d(x_n,F)=\theta$ by Proposition 2.7(i), we have $p_0\in F$. In other words, $\{x_n\}$ converges to a common fixed point in F. □

Remark 3.3. Our result generalizes the corresponding result of Lee [5] from the closed convex subset of a convex cone metric space to whole space.

Remark 3.4. We can obtain the above result for iterations defined as (2) and (3) with a finite family of asymptotically nonexpansive mappings in convex cone metric spaces.

Remark 3.5. In view of Remark 2.4, our results are valid not only asymptotically nonexpansive mappings but also nonexpansive (and also quasi-nonexpansive) mappings for iterations (1), (2) and (3).

Remark 3.6. Our result generalizes and extends the corresponding result of Gunduz [1] and many others (see, references in [1]).

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