Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 29–38
https://doi.org/10.4134/CKMS.c150247
pISSN: 1225-1763 / eISSN: 2234-3024

CERTAIN INTEGRALS ASSOCIATED WITH GENERALIZED MITTAG-LEFFLER FUNCTION

Praveen Agarwal, Junesang Choi, Shilpi Jain,
and Mohammad Mehdi Rashidi

Abstract. The main objective of this paper is to establish certain unified integral formula involving the product of the generalized Mittag-Leffler type function $E^{(\gamma_j),(l_j)}_{(\rho_j),\lambda}[z_1,\ldots,z_r]$ and the Srivastava's polynomials $S^m_n[x]$. We also show how the main result here is general by demonstrating some interesting special cases.

1. Introduction and preliminaries

Throughout this paper let $\mathbb{C}$, $\mathbb{R}_0^+$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, non-negative real numbers, positive and non-positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Since, in 1903, the Swedish mathematician Gosta Mittag-Leffler [12] introduced the function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad \left(z \in \mathbb{C};\, \alpha \in \mathbb{R}_0^+\right), \tag{1.1}$$

$\Gamma(\cdot)$ being the familiar Gamma function (see, *e.g.*, [21, Section 1.1]), it has been actively investigated by many authors who have extended (or generalized) by adding parameters and variables to the previous extension and showed their importance by realizing a variety of applications in such research subjects as (for example) in physics, chemistry, biology, engineering and applied sciences (see, *e.g.*, [7, 10, 11, 17]). Mittag-Leffler function also occurs as the solution of fractional order differential equation or fractional order integral equations. It is easy to see that $E_0(z)$ in (1.1) reduces to a simple geometric series. For $0 < \alpha < 1$, the $E_\alpha(z)$ in (1.1) interpolates between the exponential function e^z and the geometric function $1/(1-z)$.

Here a brief historical account of the extensions of the Mittag-Leffler function $E_\alpha(z)$ in (1.1). In 1905, Wiman [25] introduced to investigate the following

Received December 26, 2015.
2010 *Mathematics Subject Classification.* Primary 65A05, 33C45; Secondary 44A15.
Key words and phrases. Gamma function, Srivastava polynomials, Hermite polynomials, Laguerre polynomials, generalized Mittag-Leffler type functions, Lauricella's multiple hypergeometric series, Psi (or Digamma) function $\psi(z)$.

function:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \quad (\min\{\Re(\alpha),\, \Re(\beta)\} > 0), \tag{1.2}$$

which is an obvious extension of the Mittag-Leffler function $E_\alpha(z)$ in (1.1) such as $E_{\alpha,1}(z) = E_\alpha(z)$. A lot of investigations on the Wiman's extension $E_{\alpha,\beta}(z)$ have been made subsequently by many authors including (for example) Mittag-Leffler [13], Agarwal [1], Humbert [5] and Humbert and Agarwal [6]. The main properties of these functions are given in the book by Erdélyi *et al.* [3, Section 18.1] and a more comprehensive and detailed account of the Mittag-Leffler functions is presented in Dzherbashyan [2, Chapter 2].

In 1971, Prabhakar [14] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \quad (\min\{\Re(\alpha),\, \Re(\beta),\, \Re(\gamma)\} > 0)\,, \tag{1.3}$$

which is an extension of the Wiman's function $E_{\alpha,\beta}(z)$ in (1.2) such as $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$. Here and in what follows $(\lambda)_n$ denotes the familiar Pochhammer symbol (see, *e.g.*, [21, p. 2 and p. 5]).

Recently, in 2009, an extension of the Prabhakar's function $E^{\gamma}_{\alpha,\beta}(z)$ in (1.3) was investigated by Srivastava and Tomovski [24] as follows:

$$E^{\gamma,\mathbf{k}}_{\alpha,\beta}\,(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\mathbf{k}n}}{\Gamma\,(\alpha n+\beta)}\, \frac{z^n}{n!} \tag{1.4}$$

$$\big(\alpha,\, \beta,\, \gamma \in \mathbb{C};\; \Re\,(\alpha) > \max\{0, \Re\,(\mathbf{k}) - 1\}\,;\; \min\{\Re\,(\beta)\,, \Re\,(\mathbf{k})\} > 0\big),$$

whose special case with the following restrictions:

$$\mathbf{k} = \mathbf{q} \in (0,1) \cup \mathbb{N} \quad \text{and} \quad \min\{\Re(\alpha),\, \Re(\beta),\, \Re(\gamma)\} > 0 \tag{1.5}$$

was presented and studied, two years earlier in 2007, by Shukla and Prajapati [18].

A multivariable analogue of the Mittag-Leffler type function (1.3) has been, very recently, investigated by Goutam [4] and Saxena *et al.* [16, p. 536, Eq. 1.14] in the following form:

$$\begin{aligned} E^{(\gamma_j)}_{(\rho_j),\lambda}[z_1,\ldots,z_r] &= E^{(\gamma_1,\ldots,\gamma_r)}_{(\rho_1,\ldots,\rho_r),\lambda}[z_1,\ldots,z_r] \\ &= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1}\cdots(\gamma_r)_{k_r}}{\Gamma(\lambda+k_1\rho_1+\cdots+k_r\rho_r)}\, \frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \end{aligned} \tag{1.6}$$

$$\big(\lambda,\, \gamma_j,\, \rho_j \in \mathbb{C} \text{ with } \Re(\rho_j) > 0 \;(j \in \{1,\,\ldots,\,r\})\big).$$

The special case $\rho_1 = \rho_2 = \cdots = \rho_r = 1$ of (1.6) reduces to an important confluent form of Lauricella's multiple hypergeometric series as follows (see, *e.g.*, [22, p. 34, Eq. 1.4(8)]):

$$\begin{aligned} &\frac{1}{\Gamma(\lambda)}\Phi_2^{(r)}[\gamma_1,\ldots,\gamma_r;\lambda;z_1,\ldots,z_r] \\ &= \frac{1}{\Gamma(\lambda)}\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1}\cdots(\gamma_r)_{k_r}}{(\lambda)_{k_1+\cdots+k_r}}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \end{aligned} \tag{1.7}$$

$$\big(\gamma_j,\, z_j \in \mathbb{C}\ (j\in\{1,\ldots,r\});\ \max\{|z_1|,\ldots,|z_r|\}<1;\ \lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\big).$$

A further generalization of the multivariable analogue of Mittag-Leffler function (1.6) was given by Saxena *et al.* [16, p. 547, Eq. (7.1)]:

$$E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[z_1,\ldots,z_r] = \sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1 l_1}\cdots(\gamma_r)_{k_r l_r}}{\Gamma(\lambda+k_1\rho_1+\cdots+k_r\rho_r)}\frac{z_1^{k_1}\cdots z_r^{k_r}}{k_1!\cdots k_r!} \tag{1.8}$$

$$\big(\gamma_j,\, \rho_j\in\mathbb{C} \text{ with } \Re(\rho_j)>0 \text{ and } l_j\in\mathbb{N}\ (j\in\{1,\,\ldots,\,r\};\ \lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-\big).$$

Srivastava [20, p. 1, Eq. (1)] introduced the following very general polynomials $S_n^m[x]$ defined by

$$S_n^m[x] = \sum_{l=0}^{[n/m]}\frac{(-n)_{ml}}{l!}A_{n,l}\,x^l \quad (n\in\mathbb{N}_0;\ m\in\mathbb{N}), \tag{1.9}$$

where the coefficients $A_{n,l}$ $(n,\, l\in\mathbb{N}_0)$ are arbitrary constants, real or complex. The Srivastava polynomials $S_n^m[x]$ include, as special cases, a number of such known polynomials as (for example) Jacobi polynomials, Bessel polynomials, Laguerre polynomials and Brafman polynomials (for more several others, see [23]).

The following fairly well known integral formula is required (see, *e.g.*, [22, p. 275, Eq. (3)]):

$$\int\cdots\int u_1^{\alpha_1-1}\cdots u_n^{\alpha_n-1}(1-u_1-\cdots-u_n)^{\beta-1}du_1\cdots du_n = \frac{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)\Gamma(\beta)}{\Gamma(\alpha_1+\cdots+\alpha_n+\beta)} \tag{1.10}$$

$$\big(u_j\in\mathbb{R}_0^+ \text{ with } u_1+\cdots+u_n\le 1 \text{ and } \Re(\alpha_j)>0\ (j\in\{1,\,\ldots,\,n\});\ \Re(\beta)>0\big).$$

The Psi (or Digamma) function $\psi(z)$ is defined by (see, *e.g.*, [21, Section 1.3])

$$\psi(z) := \frac{d}{dz}\{\log\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,dt. \tag{1.11}$$

Integral formulas involving Mittag-Leffler type functions have been developed by many authors (see, *e.g.*, [8, 9, 19, 26]). In this sequel, here, we aim at establishing certain (presumably) new generalized integral formula involving the product of the generalized Mittag-Leffler type function $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[z_1,\ldots,z_r]$ and the Srivastava's polynomials $S_n^m[x]$. The main result presented here is general enough to be specialized to yield many interesting integral formulas involving the product of familiar polynomials and known hypergeometric series in one and several variables, some of which are demonstrated.

2. Main results

Here we begin by presenting a main integral formula stated in Theorem 1 below.

Theorem 1. *Let* $t,\, s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j \in \mathbb{C}$ *with* $\Re(t) > 0$ *and*

$$\min\{s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j\} > 0 \quad \text{and} \quad l_j \in \mathbb{N}\ (j \in \{1,\, \ldots,\, r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *with* $\Re(\lambda) > 0$. *Then the following integral formula holds true*:

$$\begin{aligned}
&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \cdots x_r^{s_r-1} (1-x_1-\cdots-x_r)^{t-1} \\
&\times S_n^m\left[x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1-x_1-\cdots-x_r)^{\lambda}\right] E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}\left[x_1^{\mu_1}(1-x_1-\cdots-x_r)^{\delta_1},\right. \\
&\quad \left. \ldots,\, x_r^{\mu_r}(1-x_1-\cdots-x_r)^{\delta_r}\right] dx_r \cdots dx_1 \\
&= \sum_{k=0}^{[n/m]} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + \cdots + k_r\rho_r)} \frac{1}{k_1! \cdots k_r!} \\
&\quad \times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1)\cdots\Gamma(s_r+\lambda_r k+\mu_r k_r)\Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r}(s_j+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)},
\end{aligned} \tag{2.1}$$

where $A_{n,k}$ *are arbitrary constants, real or complex, and* $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ *and* $S_n^m[\cdot]$ *are given as in* (1.6) *and* (1.9), *respectively.*

Proof. Using the defining series of $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ and $S_n^m[\cdot]$ in the integrand and then interchanging integral signs and summations, which may be verified under the given conditions, and, finally, evaluating the following integral:

$$\begin{aligned}
&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\ldots-u_{r-1}} x_1^{s_1+\lambda_1 k+\mu_1 k_1-1} \cdots x_r^{s_r+\lambda_r k+\mu_r k_r-1} \\
&\quad \times (1-x_1-\cdots-x_r)^{t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r-1}\, dx_r \cdots dx_1
\end{aligned}$$

with the aid of (1.10) proves the desired formula (2.1). □

By taking partial derivatives of both sides of (2.1) with respect to the variables which are suitably chosen and combined from the parameters $s_1,\, \ldots,\, s_r$, and t, among many integral formulas, we choose to present two formulas given in Corollaries 1 and 2.

Corollary 1. *Let* $t,\, s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j \in \mathbb{C}$ *with* $\Re(t) > 0$ *and*

$$\min\{s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j\} > 0 \quad \text{and} \quad l_j \in \mathbb{N}\ (j \in \{1,\, \ldots,\, r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *with* $\Re(\lambda) > 0$. *Then the following integral formula holds true*:

$$\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} \cdot \log(x_1) \cdot x_2^{s_2-1} \cdots x_r^{s_r-1} (1-x_1-\cdots-x_r)^{t-1}$$
$$\times S_n^m \left[x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1-x_1-\cdots-x_r)^{\lambda}\right] E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)} \left[x_1^{\mu_1} (1-x_1-\cdots-x_r)^{\delta_1},\right.$$
$$\left.\ldots, x_r^{\mu_r} (1-x_1-\cdots-x_r)^{\delta_r}\right] dx_r \cdots dx_1$$

$$\begin{aligned} &= \sum_{k=0}^{[n/m]} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + \cdots + k_r\rho_r)} \frac{1}{k_1! \cdots k_r!} \\ &\quad \times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1) \cdots \Gamma(s_r+\lambda_r k+\mu_r k_r) \Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r} (s_j+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)} \\ &\quad \times \left[\psi(s_1+\lambda_1 k+\mu_1 k_1) - \psi\left(\sum_{j=1}^{r} (s_j+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)\right], \end{aligned} \tag{2.2}$$

where $A_{n,k}$ *are arbitrary constants, real or complex, and* $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ *and* $S_n^m[\cdot]$ *are given as in* (1.6) *and* (1.9), *respectively, and* $\psi(\cdot)$ *is the Psi function in* (1.11).

Corollary 2. *Let* $t,\ s_j,\ \lambda_j,\ \gamma_j,\ \rho_j,\ \delta_j,\ \mu_j \in \mathbb{C}$ *with* $\Re(t) > 0$ *and*

$$\min\{s_j,\ \lambda_j,\ \gamma_j,\ \rho_j,\ \delta_j,\ \mu_j\} > 0 \quad \text{and} \quad l_j \in \mathbb{N}\ (j \in \{1,\ldots,r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *with* $\Re(\lambda) > 0$. *Then the following integral formula holds true*:

$$\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \cdots x_r^{s_r-1} (1-x_1-\cdots-x_r)^{t-1}$$
$$\times \log(1-x_1-\cdots-x_r) \cdot S_n^m \left[x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1-x_1-\cdots-x_r)^{\lambda}\right]$$
$$\times E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)} \left[x_1^{\mu_1} (1-x_1-\cdots-x_r)^{\delta_1}, \ldots, x_r^{\mu_r} (1-x_1-\cdots-x_r)^{\delta_r}\right] dx_r \cdots dx_1$$

$$\begin{aligned} &= \sum_{k=0}^{[n/m]} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + \cdots + k_r\rho_r)} \frac{1}{k_1! \cdots k_r!} \\ &\quad \times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1) \cdots \Gamma(s_r+\lambda_r k+\mu_r k_r) \Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r} (s_j+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)} \\ &\quad \times \left[\psi(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r) - \psi\left(\sum_{j=1}^{r} (s_i+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)\right], \end{aligned} \tag{2.3}$$

where $A_{n,k}$ *are arbitrary constants, real or complex, and* $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ *and* $S_n^m[\cdot]$ *are given as in* (1.6) *and* (1.9)*, respectively, and* $\psi(\cdot)$ *is the Psi function in* (1.11).

Proof. Differentiating both sides of (2.1) with respect to the parameters s_1 and t, respectively, gives the formulas (2.2) and (2.3). □

3. Further special cases and remarks

Since the polynomials $A_{n,k}$ in (1.9) and the generalized Mittag-Leffler type function $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ in (1.6) are very general, the main result (2.1) can be specialized to yield a large number of integral formulas involving familiar polynomials and special functions, three of which are demonstrated as in the following examples.

Choosing $m = 2$ and $A_{n,k} = (-1)^k$ in (1.9), the polynomials $S_n^2(x)$ become the Hermite polynomials $H_n(x)$ (see [23]; see also [15, p. 187]):

$$S_n^2(x) = x^{n/2}\, H_n\left(\frac{1}{2\sqrt{x}}\right). \tag{3.1}$$

Example 1. Let $t,\, s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and

$$\min\{s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j\} > 0 \quad \text{and} \quad l_j \in \mathbb{N}\ (j \in \{1,\, \ldots,\, r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned}
&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \cdots x_r^{s_r-1} (1 - x_1 - \cdots - x_r)^{t-1} \\
&\times \left[x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1 - x_1 - \cdots - x_r)^{\lambda}\right] H_n\left(\frac{1}{x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1 - x_1 - \cdots - x_r)^{\lambda}}\right) \\
&\times E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}\left[x_1^{\mu_1} (1 - x_1 - \cdots - x_r)^{\delta_1}, \ldots, x_r^{\mu_r} (1 - x_1 - \cdots - x_r)^{\delta_r}\right] dx_r \cdots dx_1
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{[n/2]} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^k\, (-n)_{2k}}{k!} \, \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + \cdots + k_r\rho_r)} \, \frac{1}{k_1! \cdots k_r!} \\
&\quad \times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1)\cdots\Gamma(s_r+\lambda_r k+\mu_r k_r)\Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r}(s_j + \lambda_j k + \mu_j k_j + \delta_j k_j) + t + \lambda k\right)},
\end{aligned} \tag{3.2}$$

where $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ are given as in (1.6) and $H_n(x)$ are Hermite polynomials.

Setting $m = 1$ and $A_{n,k} = \binom{n+\alpha}{n} \frac{1}{(\alpha+1)_k}$ in (1.9), we have

$$S_n^1(x) = L_n^{(\alpha)}(x)\,, \tag{3.3}$$

where $L_n^{(\alpha)}(x)$ are Laguerre polynomials (see [23]; see also [15, p. 200]). Applying (3.3) to the main result (2.1) yields an integral formula asserted by the following example.

Example 2. Let $t,\, s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and

$$\min\{s_j,\, \lambda_j,\, \gamma_j,\, \rho_j,\, \delta_j,\, \mu_j\} > 0 \quad \text{and} \quad l_j \in \mathbb{N}\ (j \in \{1,\, \ldots,\, r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned}
&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \cdots x_r^{s_r-1} (1 - x_1 - \cdots - x_r)^{t-1} \\
&\times L_n^{(\alpha)}\left(x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1 - x_1 - \cdots - x_r)^{\lambda} \right) \\
&\times E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)} \left[x_1^{\mu_1} (1 - x_1 - \cdots - x_r)^{\delta_1}, \ldots,\, x_r^{\mu_r} (1 - x_1 - \cdots - x_r)^{\delta_r} \right] dx_r \cdots dx_1 \\
&= \sum_{k=0}^{n} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-n)_k}{k!\,(\alpha+1)_k} \binom{n+\alpha}{n} \frac{(\gamma_1)_{k_1 l_1} \cdots (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + \cdots + k_r\rho_r)} \frac{1}{k_1! \cdots k_r!} \\
&\quad \times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1)\cdots\Gamma(s_r+\lambda_r k+\mu_r k_r)\Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r} (s_j + \lambda_j k + \mu_j k_j + \delta_j k_j) + t + \lambda k \right)},
\end{aligned} \tag{3.4}$$

where $E_{(\rho_j),\lambda}^{(\gamma_j),(l_j)}[\cdot]$ are given as in (1.6) and $L_n^{(\alpha)}(x)$ are Laguerre polynomials.

The special case $\rho_1 = \cdots = \rho_r = 1$ and $l_1 = \cdots = l_r = 1$ of the generalized Mittag-Leffler function (1.8) yields the confluent hypergeometric series (1.7) (see [22, p. 34, Eq. 1.4(8)]). Thus the same special case of the main result (2.1) yields an integral formula as given in Example 3.

Example 3. Let $t,\, s_j,\, \lambda_j,\, \gamma_j,\, \delta_j,\, \mu_j \in \mathbb{C}$ with $\Re(t) > 0$ and

$$\min\{s_j,\, \lambda_j,\, \gamma_j,\, \delta_j,\, \mu_j\} > 0\ (j \in \{1,\, \ldots,\, r\}).$$

Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned}
&\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{r-1}} x_1^{s_1-1} x_2^{s_2-1} \cdots x_r^{s_r-1} (1 - x_1 - \cdots - x_r)^{t-1} \\
&\times S_n^m \left[x_1^{\lambda_1} \cdots x_r^{\lambda_r} (1 - x_1 - \cdots - x_r)^{\lambda} \right] \Phi_2^{(r)} \left[\gamma_1, \ldots, \gamma_r; \lambda; \right. \\
&\quad \left. x_1^{\mu_1} (1 - x_1 - \cdots - x_r)^{\delta_1}, \ldots,\, x_r^{\mu_r} (1 - x_1 - \cdots - x_r)^{\delta_r} \right] dx_r \cdots dx_1
\end{aligned}$$

$$\begin{aligned}(3.5)\qquad &= \sum_{k=0}^{[n/m]} \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\gamma_1)_{k_1} \cdots (\gamma_r)_{k_r}}{(\lambda)_{k_1+\cdots+k_r}} \frac{1}{k_1! \cdots k_r!} \\ &\times \frac{\Gamma(s_1+\lambda_1 k+\mu_1 k_1)\cdots\Gamma(s_r+\lambda_r k+\mu_r k_r)\Gamma(t+\lambda k+\delta_1 k_1+\cdots+\delta_r k_r)}{\Gamma\left(\sum_{j=1}^{r}(s_j+\lambda_j k+\mu_j k_j+\delta_j k_j)+t+\lambda k\right)},\end{aligned}$$

where $A_{n,k}$ are arbitrary constants, real or complex, and $\Phi_2^{(r)}[\cdot]$ and $S_n^m[\cdot]$ are given as in (1.7) and (1.9), respectively.

Setting $r = 1$ in the main result (2.1), after a little simplification, gives a simper integral formula as in Example 4.

Example 4. Let t, s, α, γ, ρ, δ, $\mu \in \mathbb{C}$ with

$$\min\{\Re(t), \Re(s), \Re(\alpha), \Re(\gamma), \Re(\rho), \Re(\delta), \Re(\mu)\} > 0$$

and $\ell \in \mathbb{N}$. Also let $\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-$ with $\Re(\lambda) > 0$. Then the following integral formula holds true:

$$\begin{aligned}(3.6)\qquad &\int_0^1 x^{s-1}(1-x)^{t-1} S_n^m\left[x^\alpha(1-x)^\lambda\right] E_{(\rho),\lambda}^{(\gamma),(\ell)}[x^\mu(1-x)^\delta]\,dx \\ &= \sum_{k=0}^{[n/m]} \sum_{j=0}^{\infty} \frac{(-n)_{mk}}{k!} A_{n,k} \frac{(\gamma)_{j\ell}}{\Gamma(\lambda+\rho j)} \frac{1}{j!}\, B(s+\alpha k+\mu j, t+\lambda k+\delta j),\end{aligned}$$

where $A_{n,k}$ are arbitrary constants, real or complex, and $S_n^m[\cdot]$ are given as in (1.9), and $B(\cdot,\cdot)$ is the Beta function (see, e.g., [21, p. 7]).

The main result (2.1) is general enough to be further specialized to give many integral formulas involving the product of diverse known polynomials and various known hypergeometric series in one and several variables. Detailed demonstrations of those specialized integral formulas may be left to the interested reader.

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Praveen Agarwal
Department of Mathematics
Anand International College of Engineering
Jaipur-303012, India
E-mail address: praveen.agarwal@anandice.ac.in; goyal.praveen2011@gmail.com

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 780-714, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Shilpi Jain
Department of Mathematics
Poornima College of Engineering
Sita Pura, Jaipur 302022, India
E-mail address: shilpijain1310@gmail.com

Mohammad Mehdi Rashidi
Department of Mechanical Engineering
Bu-Ali Sina University
P.O. Box 65175-4161 Hamedan, Iran
E-mail address: mm_rashidi@sjtu.edu.cn