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Objective Bayesian multiple hypothesis testing for the shape parameter of generalized exponential distribution

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Abstract

This article deals with the problem of multiple hypothesis testing for the shape parameter in the generalized exponential distribution. We propose Bayesian hypothesis testing procedures for multiple hypotheses of the shape parameter with the noninformative prior. The Bayes factor with the noninformative prior is not well defined. The reason is that the most of the noninformative prior can be improper. Therefore we study the default Bayesian multiple hypothesis testing methods using the fractional and intrinsic Bayes factors with the reference priors. Simulation study is performed and an example is given.

Keywords: Fractional Bayes factor, generalized exponential distribution, intrinsic Bayes factor, reference prior, shape parameter.

1. Introduction

The generalized exponential distribution with two parameters is given by

$$f(x|\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}, x > 0, \tag{1.1}$$

where $\alpha > 0$ and $\lambda > 0$ are shape and scale parameters, respectively. We denote this distribution as $GE(\alpha, \lambda)$.

The generalized exponential distribution was proposed by Gupta and Kundu (1999). They showed that the generalized exponential distribution can be used in situations where a skewed distribution for a nonnegative random variable is needed and can be a good alternative for the use of the gamma or Weibull distributions for analysing lifetime data (see also Gupta and Kundu, 2001; Kundu and Gupta, 2007; Raqab, 2002; Raqab and Ahsanullah, 2001; Sarhan, 2007; Zheng, 2002).

The generalized exponential density is unimodal for $\alpha > 1$ and it is reverse J shaped for $\alpha < 1$ just like gamma density. When the shape parameter equals 1 it coincides with

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the exponential distribution. Also the generalized exponential distribution has an increasing hazard function if $\alpha > 1$ and it has a decreasing hazard function if $\alpha < 1$. Therefore the present paper considers the multiple hypothesis testing for the shape parameter in the generalized exponential distribution.

In objective Bayesian testing problem, the noninformative priors such as Jeffreys' prior and reference prior (Berger and Bernardo, 1989, 1992) can be use to compute the Bayes factors. However the noninformative priors are typically improper, and so Bayes factors with the noninformative priors can not defined. Therefore Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have been studied to overcome this problem. Spiegelhalter and Smith (1982) proposed the idea using imaginary training sample. Berger and Pericchi (1996) developed the intrinsic Bayes factor based on the method of a data-splitting idea. O'Hagan (1995) considered the fractional Bayes factor using a portion of the likelihood. These methods have been successful in many statistical applications (Kang *et al.*, 2013, 2014). An excellent work for the objective Bayesian testing methods is the study of Berger and Pericchi (2001).

In our works, we propose the objective Bayesian multiple hypothesis testing procedures for the shape parameter in the generalized exponential distribution based on the Bayes factors. The remaining sections is as follows. In Section 2, we develop the Bayesian multiple hypothesis testing methods using the fractional Bayes factor and the intrinsic Bayes factors with the reference priors. In Section 3, simulation study and an example are given.

2. Bayesian multiple hypothesis testing procedures

Let $X_i, i = 1, \cdots, n$ denote random samples from $GE(\alpha, \lambda)$ with the shape parameter α and the scale parameter λ Then likelihood function is given by

$$f(\mathbf{x}|\alpha,\lambda) = \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \prod_{i=1}^n (1 - e^{-\lambda x_i})^{\alpha - 1}, \tag{2.1}$$

where $\mathbf{x} = (x_1, \cdots, x_n)$. We are interested in testing multiple hypotheses $H_1 : \alpha < 1$ versus $H_2 : \alpha = 1$ versus $H_3 : \alpha > 1$ using the fractional Bayes factor and the intrinsic Bayes factors.

2.1. Bayesian multiple hypothesis testing using the fractional Bayes factor

From (2.1) the likelihood function under the hypothesis $H_1 : \alpha < 1$ is

$$L_1(\lambda|\mathbf{x}) \quad = \quad \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \prod_{i=1}^n (1 - e^{-\lambda x_i})^{\alpha - 1}, \tag{2.2}$$

where $\alpha < 1$. And the reference prior for (α, λ) under the hypothesis H_1 is

$$\pi_1^N(\alpha, \lambda) \propto \alpha^{-1} \lambda^{-1}, \tag{2.3}$$

where $\alpha < 1$. This reference prior developed by Moala *et al.* (2012), and also Kundo and Gupta (2008) used this prior when there is no prior information. Then from the likelihood (2.2) and the reference prior (2.3), the element $m_1^b(\mathbf{x})$ of the fractional Bayes facror (FBF) of O'Hagan (1995) under H_1 is given by

$$
\begin{aligned}
& m_1^b(\mathbf{x}) && (2.4)\\
= & \int_0^\infty \int_0^1 L_1^b(\alpha, \lambda|\mathbf{x}) \pi_1^N(\alpha, \lambda) d\alpha d\lambda \\
= & \int_0^\infty \lambda^{bn-1} \exp(-bn\bar{x}\lambda) \prod_{i=1}^n (1-\exp\{-\lambda x_i\})^{-b} \left[-b \sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right]^{-bn} \\
\times & \left(\Gamma[bn] - \Gamma[bn, -b\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\})] \right) d\lambda, && (2.5)
\end{aligned}
$$

where $\bar{x} = \sum_{i=1}^n x_i/n$ and $\Gamma[a,z] = \int_z^\infty t^{a-1}e^{-t}dt$. For the hypothesis $H_2 : \alpha = 1$, the reference prior for λ is

$$\pi_2^N(\lambda) \propto \lambda^{-1}. \qquad (2.6)$$

The likelihood function under the hypothesis H_2 from (2.1) is

$$L_2(\lambda|\mathbf{x}) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}. \qquad (2.7)$$

Thus from the likelihood (2.7) and the reference prior (2.6), the element $m_2^b(\mathbf{x})$ of the FBF under H_2 is given as follows.

$$m_2^b(\mathbf{x}) = \int_0^\infty L_2^b(\lambda|\mathbf{x}) \pi_2^N(\lambda) d\lambda = \Gamma[bn][bn\bar{x}\lambda]^{-bn}. \qquad (2.8)$$

Under the hypothesis $H_3 : \alpha > 1$, the likelihood function is

$$L_3(\lambda|\mathbf{x}) = \alpha^n \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \prod_{i=1}^n (1-e^{-\lambda x_i})^{\alpha-1}, \qquad (2.9)$$

where $\alpha > 1$. The reference prior for (α, λ) is

$$\pi_3^N(\alpha, \lambda) \propto \alpha^{-1}\lambda^{-1}, \qquad (2.10)$$

where $\alpha > 1$. Then from the likelihood (2.9) and the reference prior (2.10), the element $m_3^b(\mathbf{x})$ of the FBF under H_3 is given by

$$
\begin{aligned}
& m_3^b(\mathbf{x}) && (2.11)\\
= & \int_0^\infty \int_1^\infty L_3^b(\alpha, \lambda|\mathbf{x}) \pi_3^N(\alpha, \lambda) d\alpha d\lambda \\
= & \int_0^\infty \lambda^{bn-1} \exp(-bn\bar{x}\lambda) \prod_{i=1}^n (1-\exp\{-\lambda x_i\})^{-b} \left[-b \sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right]^{-bn} \\
\times & \Gamma\left[bn, -b\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right] d\lambda. && (2.12)
\end{aligned}
$$

Therefore the element B_{21}^N of FBF is provided by

$$B_{21}^N = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}, \qquad (2.13)$$

where

$$\begin{aligned} S_1(\mathbf{x}) &= \int_0^\infty \lambda^{n-1} \exp(-n\bar{x}\lambda) \prod_{i=1}^n (1-\exp\{-\lambda x_i\})^{-1} \left[-\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right]^{-n} \\ &\times \left(\Gamma[n] - \Gamma[n, -\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\})] \right) d\lambda \end{aligned}$$

and

$$S_2(\mathbf{x}) = \Gamma[n](n\bar{x}\lambda)^{-n}.$$

The element B_{31}^N of FBF is given by

$$B_{31}^N = \frac{S_3(\mathbf{x})}{S_1(\mathbf{x})}, \tag{2.14}$$

where

$$\begin{aligned} S_3(\mathbf{x}) &= \int_0^\infty \lambda^{n-1} \exp(-n\bar{x}\lambda) \prod_{i=1}^n (1-\exp\{-\lambda x_i\})^{-1} \left[-\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right]^{-n} \\ &\times \Gamma\left[n, -\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right] d\lambda. \end{aligned}$$

Also the element B_{32}^N of FBF is given by

$$B_{32}^N = \frac{S_3(\mathbf{x})}{S_2(\mathbf{x})}. \tag{2.15}$$

And for the given fraction b, the ratio of marginal densities is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{S_1(\mathbf{x};b)}{S_2(\mathbf{x};b)}, \tag{2.16}$$

where

$$\begin{aligned} S_1(\mathbf{x};b) &= \int_0^\infty \lambda^{bn-1} \exp(-bn\bar{x}\lambda) \prod_{i=1}^n (1-\exp\{-\lambda x_i\})^{-b} \left[-b\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\}) \right]^{-n} \\ &\times \left(\Gamma[bn] - \Gamma[bn, -b\sum_{i=1}^n \log(1-\exp\{-\lambda x_i\})] \right) d\lambda \end{aligned}$$

and

$$S_2(\mathbf{x};b) = \Gamma[bn](bn\bar{x}\lambda)^{-n}.$$

The ratio of marginal densities with fraction b is given by

$$\frac{m_1^b(\mathbf{x})}{m_3^b(\mathbf{x};)} = \frac{S_1(\mathbf{x};b)}{S_3(\mathbf{x};b)}, \tag{2.17}$$

where

$$\begin{aligned} S_3(\mathbf{x};b) &= \int_0^\infty \lambda^{bn-1}\exp(-bn\bar{x}\lambda)\prod_{i=1}^n(1-\exp\{-\lambda x_i\})^{-b}\left[-\sum_{i=1}^n\log(1-\exp\{-\lambda x_i\})\right]^{-bn} \\ &\times\ \Gamma\left[bn,-b\sum_{i=1}^n\log(1-\exp\{-\lambda x_i\})\right]d\lambda. \end{aligned}$$

Also the ratio of marginal densities with fraction b given by

$$\frac{m_2^b(\mathbf{x})}{m_3^b(\mathbf{x})}=\frac{S_2(\mathbf{x};b)}{S_3(\mathbf{x};b)}. \tag{2.18}$$

Thus the FBFs of O'Hagan (1995) for H_2 versus H_1, H_3 versus H_1, and H_3 versus H_2 are given by

$$B_{21}^F=\frac{S_1(\mathbf{x};b)S_2(\mathbf{x})}{S_1(\mathbf{x})S_2(\mathbf{x};b)},\ B_{31}^F=\frac{S_1(\mathbf{x};b)S_3(\mathbf{x})}{S_1(\mathbf{x})S_3(\mathbf{x};b)}\ \text{and}\ B_{32}^F=\frac{S_2(\mathbf{x};b)S_3(\mathbf{x})}{S_2(\mathbf{x})S_3(\mathbf{x};b)}, \tag{2.19}$$

respectively. Note that the calculations of the FBFs require one dimensional integration.

2.2. Bayesian hypothesis testing procedure using the intrinsic Bayes factor

The elements B_{21}^N, B_{31}^N and B_{32}^N are needed for computation of the intrinsic Bayes factors of Berger and Pericchi (1996). These elements are already calculated in computation of FBF. Thus we only compute the marginal densities for the hypotheses H_1, H_2 and H_3 under the minimal training sample. Since the marginal density of (X_{j_1}, X_{j_2}) is proper for all $1\le j_1<j_2\le n$ under each hypothesis, the minimal training sample is any training sample of size 2.

The marginal densities $m_0^N(x_{j_1},x_{j_2})$ under hypothesis $H_0(=H_1\cup H_2\cup H_3):\alpha>0$ is computed by

$$\begin{aligned} m_0^N(x_{j_1},x_{j_2}) &= \int_0^\infty\int_0^\infty f(x_{j_1},x_{j_2}|\alpha,\lambda)\pi_0^N(\alpha,\lambda)d\alpha d\lambda \\ &= \int_0^\infty \lambda\exp\{-(x_{j_1}+x_{j_2})\lambda\}\prod_{k=1}^2(1-\exp\{-\lambda x_{j_k}\})^{-1} \\ &\times\ \left[-\sum_{k=1}^2\log(1-\exp\{-\lambda x_{j_2}\})\right]^{-2} d\lambda\equiv T_0(x_{j_1},x_{j_2}). \end{aligned}$$

For hypothesis H_1 the marginal density $m_1^N(x_{j_1},x_{j_2})$ is calculated by

$$\begin{aligned} &m_1^N(x_{j_1},x_{j_2}) \\ &= \int_0^\infty\int_0^1 f(x_{j_1},x_{j_2}|\alpha,\lambda)\pi_1^N(\alpha,\lambda)d\alpha d\lambda \\ &= \int_0^\infty \lambda\exp\{-(x_{j_1}+x_{j_2})\lambda\}\prod_{k=1}^2(1-\exp\{-\lambda x_{j_k}\})^{-1}\left[-\sum_{k=1}^2\log(1-\exp\{-\lambda x_{j_2}\})\right]^{-2} \\ &\times\ \left(1-\Gamma[2,-\sum_{k=1}^2\log(1-\exp\{-\lambda x_{j_k}\})]\right)d\lambda\equiv T_1(x_{j_1},x_{j_2}). \end{aligned}$$

Under H_2 the marginal density $m_2^N(x_{j_1},x_{j_2})$ is

$$m_2^N(x_{j_1}, x_{j_2}) = \int_0^\infty f(x_{j_1}, x_{j_2}|\lambda)\pi_2^N(\lambda)d\lambda = [x_{j_1} + x_{j_2}]^{-2} \equiv T_2(x_{j_1}, x_{j_2}).$$

And the marginal density $m_3^N(x_{j_1}, x_{j_2})$ under H_3 is given by

$$\begin{aligned} & m_3^N(x_{j_1}, x_{j_2}) \\ = & \int_0^\infty \int_1^\infty f(x_{j_1}, x_{j_2}|\alpha, \lambda)\pi_3^N(\alpha, \lambda)d\alpha d\lambda \\ = & \int_0^\infty \lambda \exp\{-(x_{j_1} + x_{j_2})\lambda\} \prod_{k=1}^{2}(1 - \exp\{-\lambda x_{j_k}\})^{-1} \\ \times & \left[-\sum_{k=1}^{2}\log(1 - \exp\{-\lambda x_{j_2}\})\right]^{-2} \Gamma\left[2, -\sum_{k=1}^{2}\log(1 - \exp\{-\lambda x_{j_k}\})\right] d\lambda \equiv T_3(x_{j_1}, x_{j_2}). \end{aligned}$$

Therefore the encompassing arithmetic intrinsic Bayes factor (EIBF) of Berger and Pericchi (1996) of hypotheses H_2 and H_1 is as below.

$$B_{21}^{EI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}\left[\frac{\sum_{j_1<j_2}^{n} T_1(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}{\sum_{j_1<j_2}^{n} T_2(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}\right]. \tag{2.20}$$

The EIBF of H_3 versus H_1 is as follows.

$$B_{31}^{EI} = \frac{S_3(\mathbf{x})}{S_1(\mathbf{x})}\left[\frac{\sum_{j_1<j_2}^{n} T_1(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}{\sum_{j_1<j_2}^{n} T_3(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}\right]. \tag{2.21}$$

And the EIBF of H_3 versus H_2 is given by

$$B_{32}^{EI} = \frac{S_3(\mathbf{x})}{S_2(\mathbf{x})}\left[\frac{\sum_{j_1<j_2}^{n} T_2(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}{\sum_{j_1<j_2}^{n} T_3(x_{j_1}, x_{j_2})/T_0(x_{j_1}, x_{j_2})}\right]. \tag{2.22}$$

Also the median intrinsic Bayes factor (MIBF) of Berger and Pericchi (1998) for H_2 versus H_1, the MIBF of H_3 versus H_1 and the MIBF of H_3 versus H_2 are given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}ME\left[\frac{T_1(x_{j_1}, x_{j_2})}{T_2(x_{j_1}, x_{j_2})}\right], B_{31}^{MI} = \frac{S_3(\mathbf{x})}{S_1(\mathbf{x})}ME\left[\frac{T_1(x_{j_1}, x_{j_2})}{T_3(x_{j_1}, x_{j_2})}\right] \tag{2.23}$$

and

$$B_{32}^{MI} = \frac{S_3(\mathbf{x})}{S_2(\mathbf{x})}ME\left[\frac{T_2(x_{j_1}, x_{j_2})}{T_3(x_{j_1}, x_{j_2})}\right], \tag{2.24}$$

respectively. Here ME represents the median in Bayes factors using all possible training sample. We know that the calculations of the EIBF and the MIBF require only one dimensional integration.

3. Numerical studies

In order to compare the Bayesian multiple hypothesis testing procedures, we compute the posterior probabilities for several values of parameters (α, λ) and the sample size n. For the given values of (α, λ) and n, we consider 1,000 independent random samples of $\mathbf{X}$ with sample size n from the model (1.1). We want to test the multiple hypotheses $H_1 : \alpha < 1$ versus $H_2 : \alpha = 1$ versus $H_3 : \alpha > 1$. When prior probabilities are equal, the posterior probabilities of $H_i, i = 1, 2, 3$, are computed.

The results of the averages posterior probabilities and the standard deviations in parentheses of posterior probabilities are given in Tables 3.1 and 3.2. From the results of Tables 3.1 and 3.2, $P^F(\cdot)$, $P^{EI}(\cdot)$ and $P^{MI}(\cdot)$ represent the posterior probabilities based on FBF, EIBF and MIBF, respectively, when each hypothesis $H_i, i = 1, 2, 3$, are true.

From the results of Tables 3.1 and 3.2, the EIBF and the MIBF provide the reasonable decisions for all parameters configurations. Also the EIBF and the MIBF have a similar behavior. But the FBF prefer to select the hypothesis H_2 or the hypothesis H_3 than the EIBF and the MIBF when the values of α are less than 1 and are greater than 1, respectively. That is, the FBF has the bias toward the hypothesis H_2 or H_3. This fact does not a surprising result. Berger and Mortera (1999) studied that the FBF gives the considerable bias in the direction of one of the hypotheses in non-symmetric situations. And so the FBF can not be used for the hypotheses testing in the case of non-symmetric circumstances.

Table 3.1 The averages and the standard deviations in parentheses of posterior probabilities

λ	α	n	$P_1^F(H_1\|\mathbf{x})$	$P_2^F(H_2\|\mathbf{x})$	$P_3^F(H_3\|\mathbf{x})$	$P_1^{EI}(H_1\|\mathbf{x})$	$P_2^{EI}(H_2\|\mathbf{x})$	$P_3^{EI}(H_3\|\mathbf{x})$	$P_1^{MI}(H_1\|\mathbf{x})$	$P_2^{MI}(H_2\|\mathbf{x})$	$P_3^{MI}(H_3\|\mathbf{x})$
0.5	0.4	5	0.448 (0.267)	0.327 (0.147)	0.226 (0.145)	0.590 (0.316)	0.209 (0.173)	0.201 (0.223)	0.539 (0.303)	0.199 (0.150)	0.250 (0.230)
		10	0.673 (0.303)	0.232 (0.207)	0.094 (0.109)	0.791 (0.264)	0.142 (0.175)	0.068 (0.119)	0.766 (0.274)	0.136 (0.162)	0.096 (0.139)
		15	0.778 (0.274)	0.172 (0.205)	0.051 (0.073)	0.864 (0.220)	0.104 (0.162)	0.032 (0.069)	0.854 (0.225)	0.102 (0.155)	0.043 (0.080)
		20	0.879 (0.216)	0.097 (0.169)	0.023 (0.053)	0.929 (0.163)	0.057 (0.127)	0.014 (0.045)	0.926 (0.166)	0.056 (0.122)	0.018 (0.051)
	0.6	5	0.298 (0.185)	0.392 (0.098)	0.310 (0.139)	0.414 (0.258)	0.332 (0.167)	0.254 (0.209)	0.399 (0.242)	0.310 (0.145)	0.282 (0.205)
		10	0.360 (0.251)	0.429 (0.153)	0.212 (0.126)	0.496 (0.287)	0.358 (0.196)	0.146 (0.141)	0.487 (0.278)	0.334 (0.178)	0.176 (0.145)
		15	0.434 (0.293)	0.411 (0.200)	0.155 (0.123)	0.564 (0.301)	0.336 (0.220)	0.100 (0.117)	0.566 (0.295)	0.317 (0.206)	0.116 (0.120)
		20	0.502 (0.307)	0.384 (0.225)	0.113 (0.101)	0.625 (0.298)	0.309 (0.237)	0.066 (0.087)	0.630 (0.292)	0.292 (0.223)	0.078 (0.091)
	0.8	5	0.228 (0.124)	0.405 (0.079)	0.367 (0.142)	0.312 (0.202)	0.390 (0.146)	0.298 (0.208)	0.315 (0.191)	0.363 (0.125)	0.315 (0.201)
		10	0.227 (0.173)	0.471 (0.109)	0.302 (0.155)	0.328 (0.231)	0.463 (0.160)	0.209 (0.169)	0.338 (0.225)	0.431 (0.146)	0.230 (0.165)
		15	0.238 (0.204)	0.502 (0.136)	0.260 (0.160)	0.337 (0.248)	0.492 (0.179)	0.172 (0.159)	0.353 (0.246)	0.461 (0.169)	0.186 (0.157)
		20	0.238 (0.205)	0.539 (0.139)	0.223 (0.137)	0.334 (0.243)	0.526 (0.182)	0.139 (0.128)	0.353 (0.242)	0.495 (0.174)	0.152 (0.128)
	1.0	5	0.189 (0.090)	0.396 (0.085)	0.415 (0.149)	0.251 (0.156)	0.410 (0.137)	0.338 (0.211)	0.263 (0.151)	0.386 (0.116)	0.346 (0.204)
		10	0.148 (0.100)	0.466 (0.110)	0.386 (0.168)	0.214 (0.149)	0.510 (0.132)	0.276 (0.188)	0.229 (0.151)	0.478 (0.119)	0.292 (0.181)
		15	0.134 (0.104)	0.508 (0.125)	0.358 (0.179)	0.196 (0.149)	0.561 (0.142)	0.242 (0.187)	0.216 (0.153)	0.530 (0.131)	0.253 (0.181)
		20	0.132 (0.109)	0.551 (0.118)	0.317 (0.164)	0.194 (0.150)	0.602 (0.132)	0.204 (0.159)	0.214 (0.156)	0.570 (0.126)	0.215 (0.157)
	1.2	5	0.164 (0.068)	0.385 (0.089)	0.452 (0.145)	0.209 (0.114)	0.422 (0.134)	0.369 (0.205)	0.226 (0.119)	0.400 (0.116)	0.368 (0.200)
		10	0.114 (0.075)	0.427 (0.133)	0.459 (0.189)	0.163 (0.115)	0.504 (0.151)	0.333 (0.216)	0.182 (0.121)	0.477 (0.135)	0.341 (0.208)
		15	0.093 (0.064)	0.464 (0.145)	0.442 (0.192)	0.136 (0.096)	0.555 (0.150)	0.308 (0.206)	0.156 (0.106)	0.531 (0.137)	0.313 (0.199)
		20	0.083 (0.061)	0.485 (0.164)	0.432 (0.209)	0.123 (0.091)	0.579 (0.165)	0.298 (0.218)	0.141 (0.100)	0.555 (0.151)	0.303 (0.210)
	1.5	5	0.143 (0.059)	0.359 (0.104)	0.498 (0.158)	0.181 (0.094)	0.417 (0.141)	0.402 (0.213)	0.200 (0.100)	0.402 (0.125)	0.393 (0.209)
		10	0.085 (0.049)	0.382 (0.142)	0.533 (0.185)	0.120 (0.070)	0.487 (0.163)	0.393 (0.218)	0.140 (0.082)	0.471 (0.146)	0.389 (0.210)
		15	0.063 (0.045)	0.377 (0.172)	0.560 (0.212)	0.092 (0.065)	0.495 (0.192)	0.413 (0.242)	0.109 (0.074)	0.482 (0.175)	0.409 (0.233)
		20	0.047 (0.036)	0.360 (0.187)	0.593 (0.220)	0.070 (0.052)	0.484 (0.213)	0.446 (0.255)	0.083 (0.061)	0.473 (0.199)	0.443 (0.249)
	2.0	5	0.124 (0.051)	0.329 (0.109)	0.547 (0.158)	0.157 (0.073)	0.403 (0.143)	0.440 (0.207)	0.177 (0.085)	0.397 (0.130)	0.421 (0.206)
		10	0.060 (0.038)	0.298 (0.153)	0.642 (0.190)	0.086 (0.053)	0.412 (0.189)	0.502 (0.237)	0.102 (0.062)	0.410 (0.176)	0.487 (0.232)
		15	0.039 (0.029)	0.272 (0.169)	0.690 (0.198)	0.059 (0.041)	0.394 (0.214)	0.547 (0.252)	0.071 (0.049)	0.394 (0.202)	0.535 (0.248)
		20	0.024 (0.022)	0.222 (0.170)	0.753 (0.192)	0.039 (0.032)	0.341 (0.224)	0.620 (0.255)	0.048 (0.039)	0.343 (0.215)	0.609 (0.252)
1.0	0.4	5	0.452 (0.267)	0.325 (0.147)	0.224 (0.144)	0.597 (0.310)	0.207 (0.173)	0.196 (0.218)	0.542 (0.300)	0.196 (0.150)	0.250 (0.227)
		10	0.658 (0.302)	0.243 (0.205)	0.099 (0.106)	0.783 (0.264)	0.149 (0.174)	0.068 (0.119)	0.757 (0.272)	0.144 (0.161)	0.096 (0.135)
		15	0.805 (0.260)	0.150 (0.193)	0.045 (0.077)	0.885 (0.204)	0.087 (0.148)	0.028 (0.074)	0.876 (0.209)	0.086 (0.141)	0.038 (0.083)
		20	0.872 (0.221)	0.105 (0.176)	0.024 (0.046)	0.925 (0.164)	0.063 (0.135)	0.012 (0.032)	0.922 (0.165)	0.061 (0.128)	0.017 (0.040)
	0.6	5	0.286 (0.173)	0.393 (0.094)	0.321 (0.145)	0.400 (0.257)	0.332 (0.166)	0.269 (0.219)	0.384 (0.238)	0.309 (0.139)	0.299 (0.216)
		10	0.376 (0.260)	0.418 (0.161)	0.206 (0.131)	0.514 (0.289)	0.344 (0.199)	0.142 (0.149)	0.504 (0.280)	0.322 (0.180)	0.172 (0.152)
		15	0.442 (0.297)	0.404 (0.202)	0.154 (0.125)	0.571 (0.304)	0.328 (0.222)	0.100 (0.123)	0.573 (0.298)	0.310 (0.208)	0.116 (0.126)
		20	0.495 (0.311)	0.388 (0.227)	0.117 (0.104)	0.616 (0.303)	0.316 (0.239)	0.069 (0.087)	0.622 (0.297)	0.298 (0.226)	0.079 (0.092)
	0.8	5	0.223 (0.126)	0.399 (0.085)	0.379 (0.152)	0.306 (0.206)	0.386 (0.152)	0.308 (0.220)	0.309 (0.191)	0.361 (0.130)	0.324 (0.213)
		10	0.217 (0.158)	0.481 (0.104)	0.303 (0.148)	0.316 (0.214)	0.476 (0.151)	0.208 (0.163)	0.325 (0.208)	0.442 (0.136)	0.232 (0.157)
		15	0.222 (0.183)	0.512 (0.123)	0.266 (0.154)	0.320 (0.232)	0.508 (0.168)	0.172 (0.153)	0.336 (0.230)	0.476 (0.157)	0.188 (0.152)
		20	0.245 (0.210)	0.533 (0.140)	0.223 (0.139)	0.343 (0.249)	0.520 (0.185)	0.137 (0.127)	0.362 (0.248)	0.489 (0.178)	0.149 (0.126)
	1.0	5	0.186 (0.088)	0.395 (0.086)	0.419 (0.150)	0.247 (0.151)	0.417 (0.140)	0.337 (0.213)	0.263 (0.153)	0.387 (0.118)	0.345 (0.207)
		10	0.149 (0.094)	0.472 (0.105)	0.379 (0.164)	0.218 (0.147)	0.516 (0.131)	0.266 (0.186)	0.236 (0.150)	0.483 (0.119)	0.281 (0.180)
		15	0.138 (0.107)	0.508 (0.119)	0.354 (0.175)	0.203 (0.154)	0.562 (0.135)	0.236 (0.178)	0.223 (0.159)	0.530 (0.125)	0.247 (0.173)
		20	0.133 (0.122)	0.539 (0.130)	0.328 (0.177)	0.193 (0.160)	0.596 (0.144)	0.211 (0.172)	0.214 (0.165)	0.565 (0.137)	0.221 (0.168)
	1.2	5	0.169 (0.070)	0.389 (0.090)	0.443 (0.149)	0.222 (0.121)	0.432 (0.131)	0.347 (0.203)	0.239 (0.123)	0.407 (0.113)	0.350 (0.198)
		10	0.118 (0.080)	0.435 (0.128)	0.447 (0.183)	0.169 (0.119)	0.513 (0.144)	0.318 (0.205)	0.189 (0.125)	0.486 (0.127)	0.324 (0.196)
		15	0.096 (0.065)	0.470 (0.144)	0.434 (0.192)	0.140 (0.098)	0.563 (0.150)	0.297 (0.204)	0.160 (0.106)	0.536 (0.135)	0.304 (0.196)
		20	0.083 (0.065)	0.484 (0.158)	0.433 (0.204)	0.123 (0.094)	0.586 (0.157)	0.291 (0.208)	0.143 (0.104)	0.562 (0.145)	0.295 (0.202)
	1.5	5	0.146 (0.061)	0.364 (0.102)	0.490 (0.157)	0.187 (0.096)	0.426 (0.136)	0.387 (0.207)	0.206 (0.103)	0.407 (0.120)	0.382 (0.201)
		10	0.087 (0.052)	0.380 (0.147)	0.534 (0.194)	0.123 (0.077)	0.484 (0.168)	0.393 (0.226)	0.142 (0.087)	0.467 (0.150)	0.391 (0.217)
		15	0.063 (0.044)	0.377 (0.170)	0.560 (0.210)	0.092 (0.064)	0.496 (0.189)	0.412 (0.240)	0.108 (0.074)	0.481 (0.174)	0.411 (0.232)
		20	0.047 (0.036)	0.361 (0.188)	0.593 (0.221)	0.070 (0.051)	0.486 (0.213)	0.444 (0.255)	0.084 (0.061)	0.476 (0.197)	0.440 (0.247)
	2.0	5	0.122 (0.050)	0.327 (0.111)	0.551 (0.160)	0.155 (0.071)	0.402 (0.147)	0.444 (0.211)	0.176 (0.082)	0.399 (0.136)	0.420 (0.213)
		10	0.060 (0.038)	0.300 (0.155)	0.640 (0.192)	0.086 (0.052)	0.413 (0.192)	0.500 (0.240)	0.102 (0.061)	0.410 (0.177)	0.488 (0.234)
		15	0.037 (0.030)	0.259 (0.171)	0.705 (0.200)	0.056 (0.042)	0.379 (0.217)	0.564 (0.256)	0.068 (0.050)	0.381 (0.205)	0.551 (0.252)
		20	0.023 (0.021)	0.216 (0.164)	0.760 (0.185)	0.038 (0.031)	0.336 (0.220)	0.626 (0.249)	0.047 (0.037)	0.340 (0.212)	0.613 (0.247)

Table 3.2 The averages and the standard deviations in parentheses of posterior probabilities

λ	α	n	$P_1^F(H_1\|\mathbf{x})$	$P_2^F(H_2\|\mathbf{x})$	$P_3^F(H_3\|\mathbf{x})$	$P_1^{EI}(H_1\|\mathbf{x})$	$P_2^{EI}(H_2\|\mathbf{x})$	$P_3^{EI}(H_3\|\mathbf{x})$	$P_1^{MI}(H_1\|\mathbf{x})$	$P_2^{MI}(H_2\|\mathbf{x})$	$P_3^{MI}(H_3\|\mathbf{x})$
5.0	0.4	5	0.438 (0.264)	0.331 (0.145)	0.230 (0.143)	0.584 (0.308)	0.227 (0.178)	0.190 (0.208)	0.538 (0.293)	0.217 (0.156)	0.234 (0.211)
		10	0.661 (0.309)	0.240 (0.208)	0.100 (0.109)	0.777 (0.274)	0.158 (0.188)	0.065 (0.111)	0.759 (0.279)	0.150 (0.173)	0.089 (0.125)
		15	0.788 (0.266)	0.165 (0.201)	0.047 (0.069)	0.873 (0.205)	0.099 (0.156)	0.027 (0.060)	0.866 (0.210)	0.096 (0.148)	0.038 (0.073)
		20	0.870 (0.223)	0.106 (0.177)	0.024 (0.048)	0.923 (0.167)	0.065 (0.139)	0.012 (0.032)	0.921 (0.167)	0.063 (0.131)	0.016 (0.039)
	0.6	5	0.297 (0.179)	0.391 (0.094)	0.312 (0.142)	0.416 (0.258)	0.342 (0.168)	0.242 (0.203)	0.408 (0.244)	0.315 (0.141)	0.269 (0.199)
		10	0.368 (0.257)	0.419 (0.156)	0.213 (0.139)	0.503 (0.291)	0.363 (0.204)	0.134 (0.140)	0.503 (0.281)	0.336 (0.185)	0.159 (0.141)
		15	0.428 (0.291)	0.417 (0.199)	0.155 (0.117)	0.556 (0.298)	0.357 (0.230)	0.088 (0.101)	0.564 (0.291)	0.332 (0.214)	0.103 (0.105)
		20	0.498 (0.312)	0.385 (0.226)	0.117 (0.105)	0.615 (0.304)	0.322 (0.244)	0.064 (0.084)	0.625 (0.296)	0.301 (0.229)	0.074 (0.086)
	0.8	5	0.219 (0.119)	0.405 (0.081)	0.377 (0.143)	0.298 (0.190)	0.412 (0.143)	0.290 (0.204)	0.308 (0.184)	0.378 (0.122)	0.308 (0.197)
		10	0.219 (0.165)	0.473 (0.104)	0.308 (0.154)	0.317 (0.225)	0.478 (0.156)	0.205 (0.166)	0.331 (0.220)	0.442 (0.143)	0.226 (0.163)
		15	0.233 (0.193)	0.515 (0.123)	0.252 (0.137)	0.331 (0.238)	0.521 (0.175)	0.148 (0.124)	0.351 (0.234)	0.485 (0.165)	0.163 (0.125)
		20	0.243 (0.212)	0.530 (0.146)	0.227 (0.148)	0.337 (0.249)	0.531 (0.191)	0.132 (0.131)	0.360 (0.248)	0.497 (0.183)	0.143 (0.131)
	1.0	5	0.188 (0.085)	0.399 (0.085)	0.414 (0.147)	0.252 (0.148)	0.430 (0.137)	0.318 (0.206)	0.267 (0.146)	0.400 (0.116)	0.328 (0.200)
		10	0.149 (0.100)	0.469 (0.111)	0.382 (0.169)	0.216 (0.145)	0.535 (0.134)	0.249 (0.183)	0.239 (0.149)	0.497 (0.121)	0.264 (0.177)
		15	0.135 (0.100)	0.514 (0.115)	0.351 (0.169)	0.198 (0.144)	0.584 (0.129)	0.218 (0.167)	0.222 (0.149)	0.548 (0.120)	0.229 (0.162)
		20	0.128 (0.111)	0.544 (0.127)	0.328 (0.174)	0.186 (0.147)	0.617 (0.139)	0.196 (0.163)	0.211 (0.153)	0.584 (0.130)	0.204 (0.158)
	1.2	5	0.162 (0.069)	0.384 (0.093)	0.455 (0.149)	0.207 (0.112)	0.424 (0.134)	0.368 (0.204)	0.224 (0.118)	0.403 (0.117)	0.368 (0.199)
		10	0.123 (0.088)	0.436 (0.131)	0.441 (0.188)	0.176 (0.129)	0.508 (0.148)	0.316 (0.210)	0.196 (0.134)	0.482 (0.134)	0.322 (0.201)
		15	0.096 (0.067)	0.467 (0.147)	0.436 (0.197)	0.142 (0.101)	0.553 (0.156)	0.305 (0.213)	0.161 (0.110)	0.527 (0.142)	0.312 (0.207)
		20	0.083 (0.074)	0.478 (0.161)	0.439 (0.206)	0.122 (0.102)	0.573 (0.164)	0.305 (0.214)	0.140 (0.110)	0.550 (0.152)	0.310 (0.209)
	1.5	5	0.142 (0.060)	0.359 (0.102)	0.499 (0.156)	0.181 (0.095)	0.417 (0.139)	0.403 (0.208)	0.201 (0.103)	0.403 (0.125)	0.392 (0.205)
		10	0.085 (0.049)	0.380 (0.149)	0.534 (0.194)	0.121 (0.072)	0.480 (0.174)	0.400 (0.232)	0.139 (0.084)	0.460 (0.157)	0.401 (0.225)
		15	0.065 (0.045)	0.389 (0.167)	0.546 (0.207)	0.095 (0.063)	0.507 (0.185)	0.397 (0.235)	0.113 (0.075)	0.493 (0.169)	0.395 (0.227)
		20	0.049 (0.037)	0.373 (0.186)	0.578 (0.219)	0.074 (0.052)	0.497 (0.207)	0.429 (0.250)	0.087 (0.061)	0.485 (0.193)	0.428 (0.244)
	2.0	5	0.124 (0.051)	0.330 (0.109)	0.546 (0.158)	0.156 (0.073)	0.403 (0.144)	0.441 (0.209)	0.177 (0.083)	0.397 (0.132)	0.422 (0.209)
		10	0.057 (0.036)	0.287 (0.152)	0.656 (0.187)	0.082 (0.050)	0.401 (0.193)	0.517 (0.240)	0.098 (0.060)	0.402 (0.180)	0.500 (0.237)
		15	0.037 (0.029)	0.263 (0.164)	0.700 (0.191)	0.057 (0.040)	0.385 (0.208)	0.558 (0.246)	0.069 (0.048)	0.387 (0.198)	0.544 (0.242)
		20	0.025 (0.023)	0.225 (0.171)	0.750 (0.193)	0.040 (0.033)	0.345 (0.225)	0.615 (0.256)	0.049 (0.040)	0.349 (0.216)	0.602 (0.254)

Example 3.1. This example taken from Kundu and Gupta (2008), and the data presented here arose in tests on endurance of deep groove ball bearings (Lawless, 1982). The data presented are the number of million revolution before failure for each of the 23 ball bearings in the life test. The data set are 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40. For this data set, the maximum likelihood estimates of α and λ can be obtained as 5.2836 and 0.0323, respectively.

We consider to test the multiple hypotheses $H_1 : \alpha < 1$ versus $H_2 : \alpha = 1$ versus $H_3 : \alpha > 1$. The results of the Bayes factors and the posterior probabilities of H_1 are presented in Table 3.3. From the results of Table 3.3, the posterior probabilities using all Bayes factors have the same conclusion, and so the FBF, the EIBF and the MIBF select the hypothesis H_3.

Table 3.3 Bayes factors and posterior probabilities of H_1, H_2 and H_3

	B_{12}	B_{13}	B_{23}	$P(H_1\|\mathbf{x})$	$P(H_2\|\mathbf{x})$	$P(H_3\|\mathbf{x})$
FBF	0.07182	0.00009	0.00129	0.00009	0.00129	0.99861
EIBF	0.07015	0.00021	0.00295	0.00021	0.00294	0.99685
MIBF	0.07846	0.00031	0.00396	0.00031	0.00394	0.99575

4. Concluding remarks

In this paper, we consider the objective Bayesian multiple hypotheses testing procedures for the shape parameter in the generalized exponential distribution. We proposed the testing methods using the fractional Bayes factor and the intrinsic Bayes factors under the reference prior. From our numerical results, the developed testing methods give fairly reasonable answers for all configurations of parameters. However the FBF clearly chooses the hypothesis H_2 or H_3 rather than the EIBF and the MIBF, and so the FBF has considerable biases to the hypothesis H_2 or H_3. Thus in the case of clearly non-symmetric situations, the FBF can not be applied. From our results of simulation and example, we recommend the methods of the EIBF and the MIBF rather than the FBF in practical applications.

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