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Original Article

The extinction probability in systems randomly varying in time

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ABSTRACT

The extinction probability of a branching process (a neutron chain in a multiplying medium) is calculated for a system randomly varying in time. The evolution of the first two moments of such a process was calculated previously by the authors in a system randomly shifting between two states of different multiplication properties. The same model is used here for the investigation of the extinction probability. It is seen that the determination of the extinction probability is significantly more complicated than that of the moments, and it can only be achieved by pure numerical methods. The numerical results indicate that for systems fluctuating between two subcritical or two supercritical states, the extinction probability behaves as expected, but for systems fluctuating between a supercritical and a subcritical state, there is a crucial and unexpected deviation from the predicted behaviour. The results bear some significance not only for neutron chains in a multiplying medium, but also for the evolution of biological populations in a time-varying environment.

1. Introduction

One interesting characteristics of a branching process is the so-called extinction probability, i.e. the asymptotic probability that when time goes to infinity, the number of entities (particles) in the system is zero.

This paper discusses some aspects of the calculation of the extinction probability in settings other than the classical case of the extinction of family trees with constant reproduction probabilities, or neutron chains in a stationary multiplying medium. The setting discussed here is the extinction probability in systems randomly varying in time. Such systems were studied before [1–3], but only the temporal evolution of the first two moments was investigated. As will be seen here, the calculation of the extinction probability is a considerably more complicated task, which necessitates the use of numerical methods.

The dependence of the extinction probability on the multiplication properties of the system in the traditional case, i.e. in a system with constant parameters (constant multiplication properties) has long been well known. For subcritical and critical systems the extinction probability equals unity, whereas for supercritical systems it is less then unity. A similar behaviour was expected also for systems with multiplication properties varying in time, with the slight difference that the definition of criticality is different (more involved) for such systems. A system is defined critical in the mean if the expectation of the neutron number converges to a constant value as time goes to infinity [1], which requires that the time-averaged reactivity of the system be negative [4,3]. Defining the value of this time-averaged subcritical reactivity as the "critical reactivity", our expectation was that the extinction probability in time-varying systems would be unity for negative average reactivities up to the critical reactivity, and less than unity for time-averaged reactivities above this value. Much to our surprise, the calculations indicated that the extinction probability remains unity even if the average reactivity is zero, in which case the system is already supercritical in the mean (the expectation of the neutron number diverges asymptotically). This is a very unexpected new result, which constitutes a crucial difference in the properties of the extinction probability for constant and temporally fluctuating systems, respectively. This result has a significance also for branching processes other than neutron multiplication, such as the population dynamics of biological systems in a time-varying environment.

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2. Theory

Ever since the classic work of Galton and Watson on the extinction of family trees [5], the extinction probability of a branching process, started by one entity (individual/particle), has always been derived from a backward type master equation. One can write down a backward master equation for the generating function $g(z, t)$ of the probability distribution $p(n, t)$,

$$g(z,t) = \sum_{n=0}^{\infty} z^n p(n,t) \tag{1}$$

of having n particles in the system at time t, given that at $t = 0$ there was one neutron in the system as [3]

$$\frac{\partial g(z,t)}{\partial t} = Q\{q[g(z,t)] - g(z,t)\} \tag{2}$$

with the initial condition

$$g(z,0) = z. \tag{3}$$

Here, Q is the intensity of the reaction, and $q(z)$ is the generating function of the probability distribution $f(n)$ of having n particles from a reaction, i.e.

$$q(z) = \sum_{n=0}^{\infty} z^n f(n) \tag{4}$$

From this, it is immediately possible to obtain an equation for the probability $p(0, t) \equiv p_0(t)$ of extinction until time t, since $p_0(t) = g(0, t)$. The extinction probability

$$p_0 = \lim_{t\to\infty} p_0(t)$$

is obtained from (2) by assuming $dp_0(t)/dt = 0$ when $t\to\infty$, as the root of the equation

$$q(p_0) = p_0 \tag{5}$$

Actually, the above equation can be derived directly from a backward-type reasoning, considering the possible fate (=reaction) of the first individual (particle). This reasoning was given by the Dane Agner Krarup Erlang, a member of the famous Krarup family by his mother, which was about to become extinct. He published the formulation of the problem in the Danish journal Matematisk Tidsskrift in 1929 [6]. The reasoning goes as follows. The extinction probability p_0 is equal to the sum of the probabilities of the mutually exclusive events that the first particle either will not have any secondaries, with probability f(0), or will have one descendant, with probability $f(1)$, which will have to die out (with probability p_0), or will have two descendants (probability$f(2)$) which both will have to die out (probability p_0^2) etc. That is,

$$p_0 = f(0) + f(1)p_0 + f(2)p_0^2 + \ldots = q(p_0) \tag{6}$$

More generally, one can also derive a similar backward type equation for the number distribution of the total number of neutrons $p(n)$ generated in the system, due to one starting neutron as

$$p(n) = \sum_{k=0}^{\infty} f(k) \prod_{n_1+n_2+\ldots n_k=n} p(n_1)p(n_2)\ldots p(n_k) \tag{7}$$

This yields for the generating function the equation

$$g(z) = q[g(z)] \tag{8}$$

from which Eq. (6) is immediately recovered by substituting $z = 0$.

Although in the above derivation time does not appear explicitly, it is clear that the equation is of a backward type. This is because the construction of the equation is based on the summing up of the probabilities of the mutually exclusive events that can happen with the starting particle on its first collision (the multiplication of the first entity/individual in the family chain). A forward equation would correspond to the summing up of the probabilities of the events of the particle(s) on their last collisions which, given the fact that all particle numbers in the system are possible, could only be given as an infinite system of coupled equations.

The above derivation is completely analogous with that of the first of the so-called Böhnel equations of nuclear safeguards [7], which specify the probability distribution of the number of neutrons leaving a multiplying sample due to one starting neutron. These are analogous to the above equations in that they do not contain time; but also in that it is not possible to derive a forward equation for any of the number distributions, for the reasons stated above. As an illustration, we quote the Böhnel equation for the probability distribution due to one starting particle, and its generating function, respectively, as [8].

$$p(n) = (1 - \mathrm{p})\delta_{n,1} + \mathrm{p}\sum_{k=0}^{\infty} f(k) \prod_{n_1+n_2+\ldots n_k=n} p(n_1)p(n_2)\ldots p(n_k) \tag{9}$$

and

$$h(z) = (1 - \mathrm{p})z + \mathrm{p}\, q[h(z)] \tag{10}$$

Here, p is the probability that the initial neutron will have a first collision before leaving the sample, and $h(z)$ is the generating function of $p(n)$, where the usual notation was chosen for the generating function, for easier distinction from the usual extinction problem.

This latter equation is useful to illustrate the suitability of the backward equation for the calculation of the whole probability distribution in a simple recursive manner. First, the "extinction probability" $p(0)$ of no neutrons leaving the sample is obtained in a form very similar to the traditional extinction equation as

$$p(0) = \mathrm{p}\, q[p(0)] \tag{11}$$

This is still the same transcendental equation as for the traditional extinction probability. However, as it was shown in [8], the higher order probabilities $p(n)$, $n\geq 1$ can be obtained by solving linear algebraic equations, in which polynomial combinations of the (already known) lower order moments appear. Hence, in principle, the terms of the probability distribution $p(n)$ can be determined analytically to any arbitrary order of n.

It is thus seen that with the backward formalism, one can derive an equation directly for the extinction probability (or, for that matter, for the asymptotic number distribution of the neutrons in the system or those leaving the system), without the need of first deriving an equation for $g(z,t)$ and then substitute $z = 0$ and take the limit $t\to\infty$.

However, for systems varying randomly in time, the backward equation is not applicable. The main reason, as discussed in [1–3], is that the factorisation ansatz of the backward equation cannot be applied, because the evolution of the chains started by neutrons born simultaneously will not be independent (will be influenced simultaneously by the changing properties of the material). Hence

the only possibility for the calculation of the extinction probability in a temporally randomly varying medium is to use the forward equation.

Application of the forward equation for the determination of the extinction probability is though much more cumbersome than that of the backward equation, and is not to be found in the literature. The difficulties will be illustrated with the case of the traditional branching process in a stationary medium. The forward equation for this case reads as

$$\frac{\partial g(z,t)}{\partial t} = Q[q(z) - z]\frac{\partial g(z,t)}{\partial z}. \tag{12}$$

Substituting $z = 0$ in Eq. (12) shows that there is a closure problem; the resulting equation contains both $p_0(t)$ and $p_1(t)$; differentiation with respect to z and substituting $z = 0$ will lead to and equation containing $p_1(t)$ and $p_2(t)$, and so on. Besides, being a forward equation, operating on the final co-ordinates, one cannot take the limit $t \to \infty$ in the equation itself, only in the solution.

In contrast, no closure problem exists when calculating the moments of the distribution, as it was seen in [1–3]. This is because, although the r.h.s. of Eq. (12) contains a first derivative with respect to z, differentiating both sides of the equation will lead to equal derivatives on both sides, due to the factor $q(z) - z$ on the right hand side, which vanishes for $z = 1$.

It is also obvious that the heuristic reasoning of the type of Eq. (6) is not applicable here; the asymptotic value of p_0 requires the knowledge of the asymptotic value of p_1, which requires the knowledge of p_2 and so on, illustrating again the problem of closure and the need for the knowledge of the full solution of $p(n,t)$.

The above shows that a suitable starting point is to first investigate the possibilities of determining the extinction probability from a forward master equation for the classic case of the static medium, after which the solution method may be attempted to be generalised to the case of the randomly varying system. Two basic possibilities appeared to be worth trying. The first is to Laplace transform in time the forward equation, and then seek the asymptotic value of the extinction probability with the help of the Tauberian theorem. In other words, taking the Laplace transform

$$\overline{g}(z,s) = \int_0^\infty e^{-st} g(z,t)dt \tag{13}$$

will convert Eq. (12) to

$$s\,\overline{g}(z,s) - z = Q[q(z) - z]\frac{\partial \overline{g}(z,s)}{\partial z}, \tag{14}$$

This differential equation may be solved for $g(z,s)$. Since the extinction probability is defined as

$$p_0 = \lim_{t\to\infty} g(z=0,t), \tag{15}$$

from the solution for $g(z,s)$, this can be recovered as

$$p_0 = \lim_{s\to 0} s\,\overline{g}(z=0,s) \tag{16}$$

If such a solution can be obtained, one might try to generalise it to the case of a medium randomly varying in time.

If this method should not work, then one can restrict the branching process to a quadratic one, such that the total number of new-born particles is either zero, one, or at most two, i.e.

$$f(n) = f_0\delta_{n,0} + f_1\delta_{n,1} + f_2\delta_{n,2} \tag{17}$$

and hence

$$q(z) = f_0 + f_1 z + f_2 z^2 \tag{18}$$

It was shown in [3] that in a static system, for such a case a complete time-dependent solution can be obtained for the full generating function, and hence for the extinction probability. Hence this method appeared to have larger potential to be applicable for the randomly varying system.

In the case of a system randomly varying between two states, such as in those treated in [3], the notations of which will be used here, one seeks the generating function $g_{j,i}(z,t)$, $\{i,j = 1,2\}$ of the probability that at time t there are n particles in the system, and the system is in state j, on the condition that at time $t = 0$, the system was in state i and there was one particle in the multiplying medium. Since we are only interested in the asymptotic behaviour of the neutron population irrespective of the state of the system, we seek the extinction probability

$$p_{0,i} \equiv p_i = \lim_{t\to\infty}\left[g_{1,i}(0,t) + g_{2,i}(0,t)\right] \tag{19}$$

It is this quantity whose calculation is attempted in this paper.

As will be seen below, none of the above expectations worked, and only a pure numerical scheme made it possible to calculate the extinction probability in systems randomly varying in time.

3. Solutions

3.1. General solution in a static medium

We start with the forward equation

$$\frac{\partial g(z,t)}{\partial t} = Q[q(z) - z]\frac{\partial g(z,t)}{\partial z}, \tag{20}$$

which can be re-written by introducing $\tau = Qt$ as

$$\frac{\partial g(z,\tau)}{\partial \tau} = [q(z) - z]\frac{\partial g(z,\tau)}{\partial z} \tag{21}$$

Taking a Laplace transform in time we have

$$\overline{g}(z,s) = \int_0^\infty d\tau e^{-s\tau} g(z,\tau) \tag{22}$$

from which one can deduce the following condition which may be of use later, viz:

$$s\overline{g}(z,s)_{\lim s\to 0} = g(z,\infty) \tag{23}$$

Then Eq. (21) becomes

$$(q(z) - z)\frac{d\overline{g}}{dz} = s\overline{g} - z \tag{24}$$

Rearranging this we have

$$\frac{d\overline{g}}{dz} - \frac{s}{q(z) - z}\overline{g} = -\frac{z}{q(z) - z} \tag{25}$$

Introducing the integrating factor leads to

$$\frac{d}{dz}\left(\overline{g}e^{-sz\int\frac{dz'}{q(z')-z'}}\right)=-\frac{z}{q(z)-z}e^{-sz\int\frac{dz'}{q(z')-z'}} \tag{26}$$

Integrating from z to unity, we find

$$\overline{g}(1,s)e^{-s^1\int\frac{dz'}{q(z')-z'}}-\overline{g}(z,s)e^{-sz\int\frac{dz'}{q(z')-z'}}=-\int_z^1\frac{z'dz'}{q(z')-z'}e^{-sz'\int\frac{dz''}{q(z''')-z''}} \tag{27}$$

After some re-arrangement this becomes

$$\overline{g}(z,s)=\frac{1}{s}e^{-s\int_z^1\frac{dz'}{q(z')-z'}}+\int_z^1\frac{z'dz'}{q(z')-z'}e^{-s\int_z^{z'}\frac{dz''}{q(z'')-z''}} \tag{28}$$

where we have set $\overline{g}(1,s)=1/s$. Again we may write

$$s\overline{g}(z,s)=e^{-s\int_z^1\frac{dz'}{q(z')-z'}}+s\int_z^1\frac{z'dz'}{q(z')-z'}e^{-s\int_z^{z'}\frac{dz''}{q(z'')-z''}} \tag{29}$$

We may now be tempted to use the relation Eq. (23). However this leads on first sight to $g(z,\infty)=1$, for all z, which is not helpful. The fact that this result occurs means that we have not dealt with singularities appearing in the integrands of Eq. (9). These occur at the zeros of

$$q(z)-z=0 \tag{30}$$

It would appear that before taking the limit in Eq. (29) we should use the properties of $q(z)$. Indeed, if one specifies the process as quadratic, i.e. only absorption, scattering and binary fission can take place, the problem can be solved, as it was shown in [3]. From now on we restrict ourselves to such processes, and turn to the binary random medium.

3.2. Quadratic process in a time-varying medium

We will now employ the quadratic process, characterised with the number distribution (17) and its quadratic generating function (18). Note that since

$$f_0+f_1+f_2=1, \tag{31}$$

one of the f_i can be expressed with the other two, which simplifies the notations. As is known, in the classical case, the extinction probability is unity for subcritical and critical systems, i.e. when

$$\mathbf{E}\{n\}=\left[\frac{dq(z)}{dz}\right]_{z=1}=q'(1)\equiv\nu\le 1, \tag{32}$$

whereas it becomes less than unity for $\nu>1$. From (17) or (18) one obtains

$$\nu=f_1+2f_2, \tag{33}$$

hence using (31) leads to the condition of criticality as

$$f_0=f_2. \tag{34}$$

The distribution can be defined by two parameters, e.g. by f_0 and $f_2, f_0+f_2\le 1$, or by the mean ν and either the variance or the second factorial moment $q_2\equiv q''(1)$ of the number of secondary particles per reaction.

The temporally randomly varying multiplying system will be the same as that used in our previous work [1–3]. It is assumed that the system has two states, with reaction intensities Q_i and the generating functions of the distribution neutrons per reaction, $q_i(z)$, $i=1,2$. The generating functions will be defined in terms of the parameters ν_i and $q_{2,i}$, $i=1,2$, which are defined as the first and second factorial moments of the corresponding number distributions:

$$\nu_i=q_i'(0) \tag{35}$$

and

$$q_{2,i}=q_i''(0) \tag{36}$$

We will seek the generating functions $g_{j,i}(z,t)$, $j,i=1,2$ of the probability that at time t, the system is in state j and the number of neutrons in the system equals n, given that at time $t=0$ the system was in state i and there was one neutron in the system. It is also assumed that the probability that during time Δt the system changes from state 1 to state 2, or vice versa, is equal to $\lambda\Delta t+o(\Delta t)$.

As it is shown e.g. in [3], the generating functions $g_{j,i}(z,t)$ obey the following coupled differential equation system:

$$\frac{\partial g_{1,i}(z,t)}{\partial t}=Q_1[q_1(z)-z]\frac{\partial g_{1,i}(z,t)}{\partial z}+\lambda\Big[g_{2,i}(z,t)-g_{1,i}(z,t)\Big], \tag{37}$$

and

$$\frac{\partial g_{2,i}(z,t)}{\partial t}=Q_2[q_2(z)-z]\frac{\partial g_{2,i}(z,t)}{\partial z}+\lambda\Big[g_{1,i}(z,t)-g_{2,i}(z,t)\Big], \tag{38}$$

with the initial conditions

$$g_{j,i}(z,0)=z\,\delta_{ij},\quad i,j=1,2.$$

The expectation is that substituting quadratic forms for the $q_i(z)$, taking temporal Laplace transforms and eliminating, say, $g_{2,i}(z,s)$, a differential equation in z can be derived for $g_{1,i}(z,s)$. Having solved this differential equation, one can obtain the corresponding extinction probability by the Tauberian theorem as

$$\lim_{t\to\infty}g_{1,i}(0,t)=\lim_{s\to 0}s\,g_{1,i}(0,s) \tag{39}$$

Unfortunately, this strategy does not work because the arising differential equation for $g_{1,i}(z,s)$ is not amenable for an analytic solution. A Laplace transform in time yields

$$Q_1[q_1(z)-z]\frac{\partial\overline{g}_{1,i}(z,s)}{\partial z}+\lambda\,\overline{g}_{2,i}(z,s)-[\lambda+s]\overline{g}_{1,i}(z,s)=z\,\delta_{1,i} \tag{40}$$

and

$$Q_2[q_2(z)-z]\frac{\partial\overline{g}_{2,i}(z,s)}{\partial z}+\lambda\overline{g}_{1,i}(z,s)-[\lambda+s]\overline{g}_{2,i}(z,s)=z\,\delta_{2,i} \tag{41}$$

Substitution of a quadratic form for the $q_i(z)$ yields for the factors multiplying the derivatives in (40) and (41)

$$q_i(z) - z = (1-z)\left(1 - \nu_i + \frac{1}{2}q_{2,i}(1-z)\right) \tag{42}$$

Putting these into the equations, differentiating (40) w.r.t. z and eliminating $g_{2,i}(z,s)$ leads to

$$\frac{d^2\overline{g}_1(z,s)}{dz^2} + \left[\frac{\nu_1 - 1}{q_1(z)-z} - \frac{\lambda+s}{q_1(z)-z} - \frac{\lambda+s}{q_2(z)-z}\right]\frac{d\overline{g}_1(z,s)}{dz} + \frac{s(s+2\lambda)}{(q_1(z)-z)(q_2(z)-z)}\overline{g}_1(z,s) + \frac{q_2(z) - z(2\lambda+1+s)}{(q_1(z)-z)(q_2(z)-z)} = 0 \tag{43}$$

This equation shows the basic difference between the determination of the moments and that of the extinction probability. When calculating the moments, the substitution $z = 1$ can be made already in the defining equations. Hence after a temporal Laplace transform, there remain only algebraic equations to be solved with constant coefficients for the $g_{j,i}^{(n)}(1,s)$. These can be readily handled, even with a general (not quadratic) $q_i(z)$, since only the moments of this distribution occur. For the extinction probability, the substitution $z = 0$ would lead to a closure problem, hence the differential equations first need to be solved for the $g_{j,i}(z,s)$, and the substitution $z = 0$ can only be made afterwards. The differential equation is not of constant coefficients, rather the coefficients are highly nonlinear functions of z. This shows that the derivation of the extinction probability in a randomly varying medium is substantially more complicated than calculating the first two moments of the neutron distribution.

3.3. Numerical solution

Since there is very little hope that Eq. (42) can be solved analytically, we chose a numerical solution. Since a numerical solution is not suitable for the application of the Tauberian theorem, instead of solving the Laplace-transformed equations (40)–(41) or (43), the original equations (37)–(38) were solved. Two different solution methods were used: one based on the Chebyshev-Gauss-Lobatto collocation algorithm, which will be referred to as "the numerical scheme", and one based on the numerical partial differential equation system solver NDSolve of the symbolic manipulation code Mathematica [9].

Since $p_0 = 1$ is always a root of the extinction equation (5), which is the non-physical root for the case of subcritical systems where two roots exist in $z \in [0,1]$, the question may arise whether the numerical schemes yield the physical root. This question was investigated in Ref. [10], where it was shown that unless starting with the pathological initial conditions of no particles in the system, a numerical solution will find the physical root.

In the numerical scheme, it is convenient to transform the variable $z(0,1)$ to $x(-1,1)$ via the substitution $x = 2z-1$, so that it conforms to the space of the Chebyshev polynomials. Thus, Eqs. (37) and (38) then become

$$\frac{\partial g_1(x,t)}{\partial t} = Q_1(1-x)\left(1 - \nu_1 + \frac{1}{4}q_{2,1}(1-x)\right)\frac{\partial g_1(x,t)}{\partial x} + \lambda[g_2(x,t) - g_1(x,t)] \tag{44}$$

and

$$\frac{\partial g_2(x,t)}{\partial t} = Q_2(1-x)\left(1 - \nu_2 + \frac{1}{4}q_{2,2}(1-x)\right)\frac{\partial g_2(x,t)}{\partial x} + \lambda[g_1(x,t) - g_2(x,t)] \tag{45}$$

and the initial conditions become

$$g_{1,i}(x,0) = \frac{1}{2}(1+x)\delta_{1i} \text{ and } g_{2,i}(x,0) = \frac{1}{2}(1+x)\delta_{2i}. \tag{46}$$

We solve equations (44) and (45) by replacing the derivative $\partial/\partial x$ by a Chebyshev-Gauss-Lobatto collocation in the form

$$\frac{\partial y(x)}{\partial x}|_{x_k} \approx \sum_{j=0}^{N} D_{k,j}y(x_j) \tag{47}$$

where $x_j = \cos(\pi j/N)$ and the $D_{k,j}$ are defined in [11] (but see the original paper by Don and Solomonoff [12] for correct form). Then one has

$$\frac{dg_{1k,i}(t)}{dt} = Q_1(1-x_k)\left(1 - \nu_1 + \frac{1}{4}q_{2,1}(1-x_k)\right) \times \left[\sum_{j=1}^{N} D_{k,j}g_{1,j,i}(t) + D_{k,0}g_{1,0,i}(t)\right] + \lambda_1\left(g_{2,k,i}(t) - g_{1,k,i}(t)\right) \tag{48}$$

and

$$\frac{dg_{2k,i}(t)}{dt} = Q_2(1-x_k)\left(1 - \nu_2 + \frac{1}{4}q_{2,2}(1-x_k)\right) \times \left[\sum_{j=1}^{N} D_{k,j}g_{2,j,i}(t) + D_{k,0}g_{2,0,i}(t)\right] + \lambda_1\left(g_{1,k,i}(t) - g_{2,k,i}(t)\right) \tag{49}$$

Eqs (48) and (49), together with the initial and boundary conditions, allow a numerical solution, from which the extinction probability is extracted at $z = 0$ or $x = -1$. Since we are interested in the extinction probability irrespective of the final system state, we use Eq. (19). In terms of the collocation points this is expressed as

$$p_i = g_i(0,t) = g_{1,N,i}(t) + g_{2,N,i}(t) \tag{50}$$

This way the whole time dependence of the probability $g_i(0,t)$ of no particles being in the system is recovered, and the extinction probability is equal to its value for a t for which the asymptotic state is reached. The same procedure is used when finding the solution with the application of the Mathematica routine NDSolver, namely the extinction probability is extracted from the asymptotic value of the solution for large times at $z = 0$. This is a rather challenging task for both methods due to the large time scales involved in the calculations.

4. Quantitative results and discussion

Quantitative results for some characteristic cases will be shown below. The selection of the cases is based on the knowledge gained through the study of the behaviour of the moments of the neutron population in systems randomly varying in time [1–3]. For easy reference, some notations and basic properties of the time-varying systems will be re-capitulated. First of all, in the continuation, the reaction intensities will be assumed equal, i.e. we will assume that $Q_1 = Q_2 = Q$, so that some properties of the systems randomly fluctuating in time, reported in the previous publications where a similar assumption was used, become applicable. For the definition of the reactivity, instead of the first moment quantities ν_i, it is more convenient to introduce the parameters

$$\alpha_i = Q_i(\nu_i - 1), \tag{51}$$

In such a simple multiplying system without delayed neutrons, the parameters ν_i can be identified with the multiplication factor, and the parameters α_i correspond to the reactivities of the corresponding static systems. In the case of a constant system, $\alpha < 0$ represents a subcritical and $\alpha > 0$ a supercritical system.

However, for the case of temporally varying systems, the situation is somewhat more involved. An illustration of a system randomly jumping between two states, as described by the parameters α_1 and α_2 is shown in Fig. 1. In the Figure, $\alpha_1 > 0$ and $\alpha_2 < 0$, i.e. the system is fluctuating between a supercritical and a subcritical state. As it was discussed in the above publications, a system, both of whose states are subcritical or supercritical, are called strongly subcritical and strongly supercritical, respectively. Those systems which fluctuate between a subcritical and a supercritical state fall into three categories. Defining the parameter

$$\lambda_{cr} = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}, \tag{52}$$

it was found that systems that fulfil the condition

$$\lambda = \lambda_{cr} \tag{53}$$

such that $\alpha_1\ \alpha_2 < 0$, are critical in the mean, in the sense that the expectation of the population is constant. Systems where $\lambda < \lambda_{cr}$ are supercritical in the mean, that is the expectation of the population diverges in time, whereas if $\lambda > \lambda_{cr}$, the system is subcritical in the mean, that is the expectation dies out asymptotically.

A particular feature, found already in 1971 [4] is that if the system fluctuates randomly between a supercritical and a subcritical state such that the average reactivity is zero, i.e.

$$\alpha_1 + \alpha_2 = 0 \quad \text{or} \quad \frac{\nu_1 + \nu_2}{2} = 1, \tag{54}$$

such systems are supercritical in the mean (except for the pathological case when λ diverges). In order that the system be critical in the mean, the α_1 and α_2 values must fulfil (52) and (53), for which one finds that

$$\overline{\alpha} = \frac{\alpha_1 + \alpha_2}{2} = \alpha_{crit} < 0 \tag{55}$$

Such a negative α_{crit} is denoted by the broken green line in Fig. 1, i.e. the figure illustrates the case of a system critical in the mean.

Based on these preliminaries, and on the known dependence of the traditional extinction probability on the criticality of the system, one would expect that in a time-varying system, the extinction probability is unity for $\overline{\alpha} \le \alpha_{crit}$, whereas it is less than unity for $\overline{\alpha} > \alpha_{crit}$. That is, the point when the extinction probability starts deviating from unity is when the system becomes supercritical in the mean. However, the numerical results indicate otherwise, as will be seen in the quantitative results shown below.

Fig. 2 shows the case of a strongly subcritical system, for which both α_1 and α_2 are negative. The upper part of the figure illustrates the random variation of α. This subfigure is only schematic both here and in the subsequent figures, i.e. the α values do not correspond quantitatively to those used in the calculations of the extinction probability in the subfigures below it. The blue line

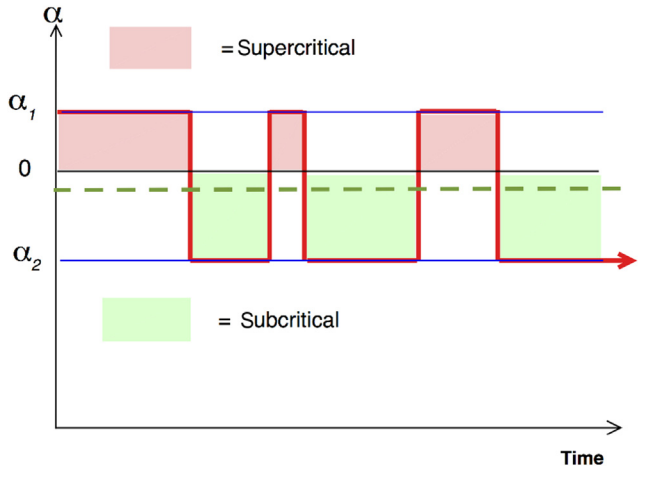

Fig. 1. Illustration of a multiplying system randomly varying between two states.

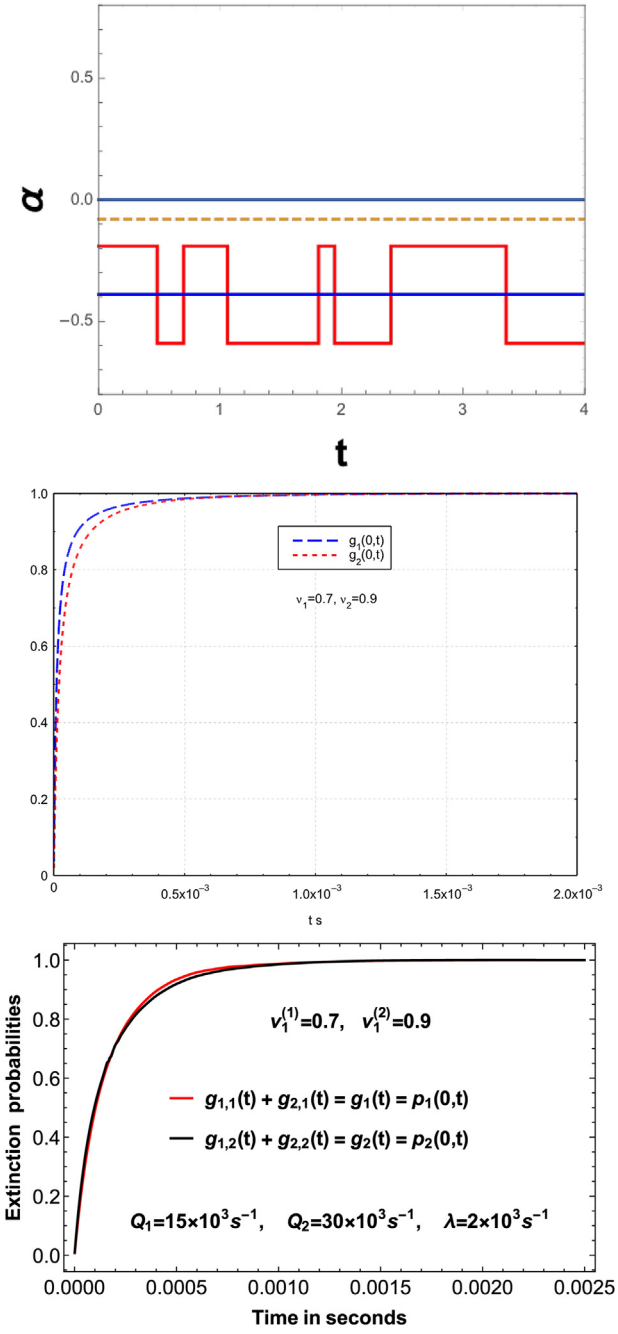

Fig. 2. Time dependence of the extinction probability in a strongly subcritical randomly varying system. Middle subfigure: solution with a numerical scheme; lowermost subfigure: solution obtained by the NDSolver routine of Mathematica 11.

represents the mean reactivity $\overline{\alpha}$ of the system, whereas the broken line stands for the critical value $\alpha_{crit} < 0$. The system is critical in the mean when $\overline{\alpha} = \alpha_{crit}$, i.e. when the broken line and the solid blue line coincide. The middle and the lower subfigures show the quantities $p_i = g_i(0,t)$ for $i = 1,2$ as obtained from the numerical scheme and from the Mathematica solver, respectively. The subscript i defines the initial conditions, i.e with which α value the system started. The two curves in the figures correspond to these two different initial conditions. It is seen that, as expected, the extinction probabilities converge to unity for both solution methods.

As it was noted already in the earlier work concerning the expectations, the initial conditions have a strong influence on the system behaviour, which in most cases prevails even in the asymptotic state. Such a difference is seen here too, but it is restricted to the speed of attaining the asymptotic values. However, asymptotically, both extinction probabilities (starting from the deeper or shallower subcritical state) tend to unity, as expected. It is seen that $g_1(0,t)$, which corresponds to the case when the system started from the deeper subcritical case, reaches the asymptotic value faster. This is in accordance with the findings on the expectations in [3], where also a dependence of the asymptotic values on the initial conditions was found, and which is clear intuitively.

In Fig. 3 the time dependence of the system parameters, as well as that of the extinction probabilities are shown for the case of two strongly supercritical systems. The results from both the numerical scheme and the Mathematica solver are fully consistent with each other and with the expectations. They both yield extinction probabilities less than unity. Moreover, the differences in the time

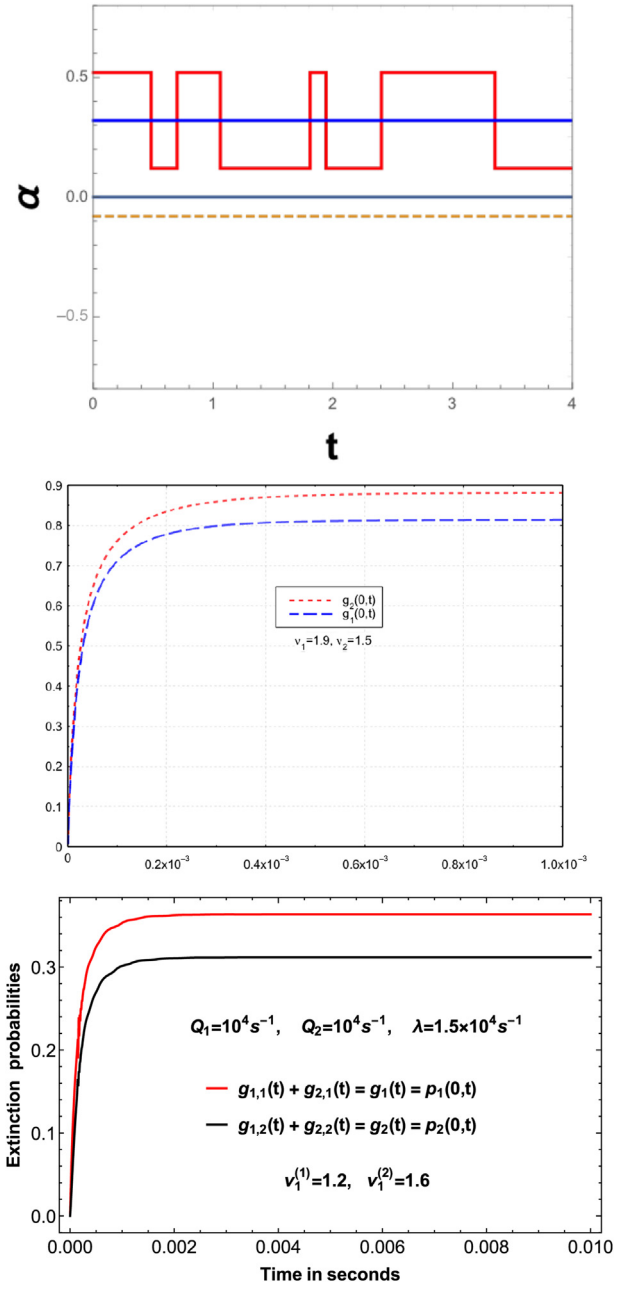

Fig. 3. Time dependence of the extinction probability in a strongly supercritical randomly varying system.

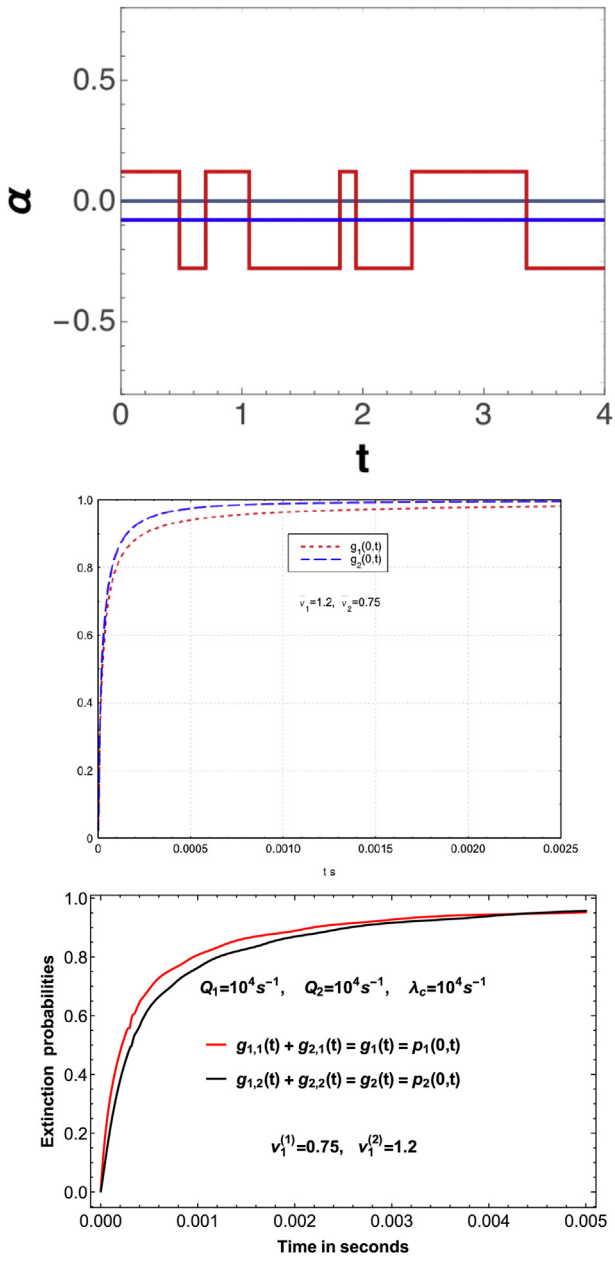

Fig. 4. Time dependence of the extinction probability in a randomly varying system which is critical in the mean.

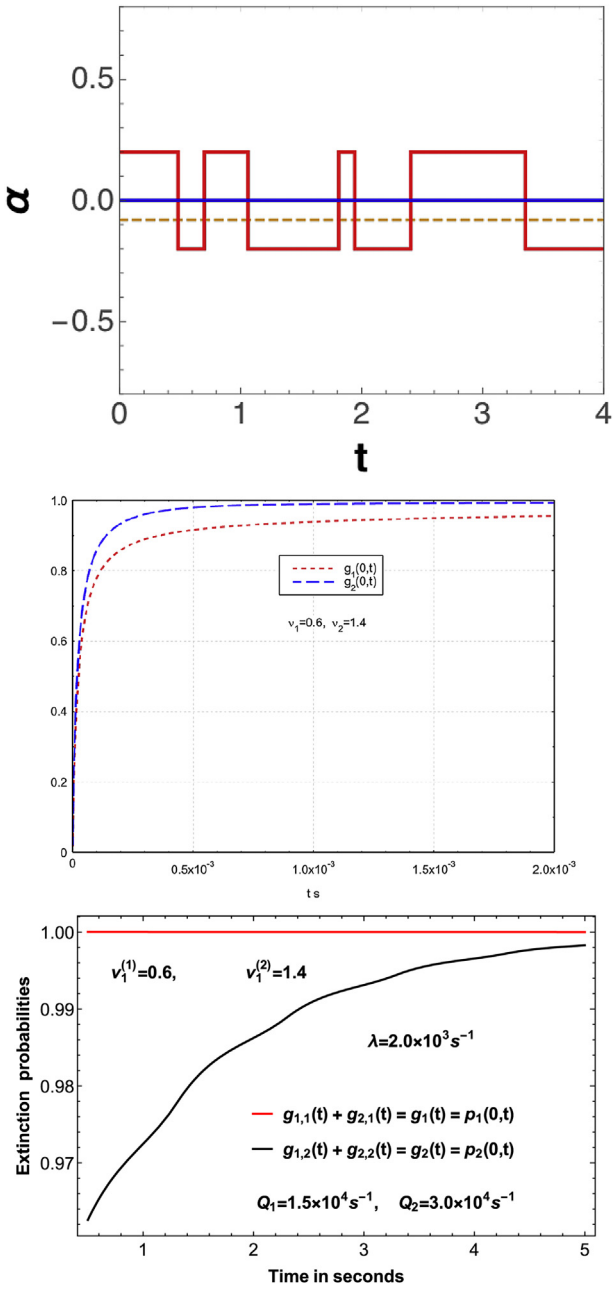

Fig. 5. Time dependence of the extinction probability in a randomly varying system with a zero time-averaged reactivity, which is thus supercritical in the mean.

evolution of the extinction probabilities, due to the different initial conditions, remain non-zero even asymptotically. As expected, the extinction probability is larger for the cases which started from the less supercritical state.

The results so far correspond to what one would expect from the similar results for traditional (constant) systems, and from physical intuition. Since for the constant systems, the extinction probability is unity even for critical systems, it is natural to expect that this will also be the case for time-varying systems which are critical in the mean. For such systems the values α_1 and α_2 must be chosen such that $\overline{\alpha} = \alpha_{crit} < 0$.

Such a case is shown in Fig. 4. The uppermost subfigure illustrates that this is the case when $\overline{\alpha} = \alpha_{crit} < 0$, i.e. when the solid and the broken lines, representing these two values in the uppermost subfigure, coincide. With $Q_1 = Q_2 = Q = 10^4\ s^{-1}$ and $\lambda = 10^4\ s^{-1}$, the values $\nu_1 = 1.2$ and $\nu_2 = 0.75$ fulfil (52) and (53). Indeed, as the Figure shows, and also the numerical values confirm to four significant digits, the extinction probabilities tend to unity even in this case. However, the convergence is much slower than in the strongly subcritical system.

It is now also interesting to investigate the case when the system parameters vary in such a manner that $\alpha_1 + \alpha_2 = 0$, i.e. the time-averaged reactivity of the system is zero. From earlier works [4,1] we know that in such systems the expectation of the population diverges, i.e. the system is supercritical in the mean. One would therefore expect that the extinction probability is less then unity.

Very surprisingly, the numerical results indicate it otherwise, as shown in Fig. 5. The figure shows the case when the system is fluctuating between a subcritical and a supercritical state with $\nu_1 = 0.6$ and $\nu_2 = 1.4$, with the values $\lambda = 10^3 s^{-1}$ and $Q_1 = Q_2 = 10^4 s^{-1}$. This system fulfils the condition expressed in (54), i.e. its average criticality is zero in the traditional sense.

It is seen that the extinction probability, as calculated by the two different methods, converges to unity. However, these calculations are rather complicated because the numerical schemes become unstable for long times. As the lowest subfigure shows, the convergence is only seen for rather long times. It is also seen that the extinction probability for the case when the system started from the subcritical state has already converged when the

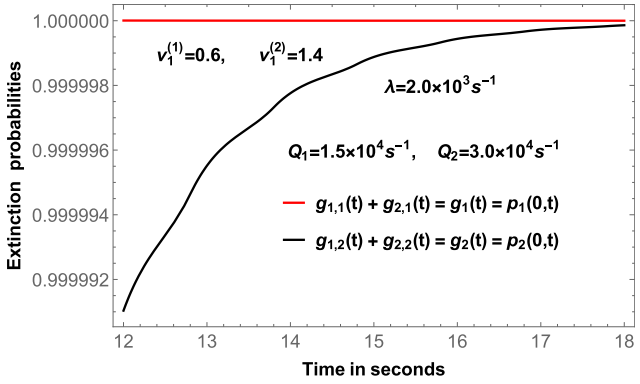

Fig. 6. Time dependence of the extinction probability in a randomly varying system with a zero time-averaged reactivity, which is supercritical in the mean, for long times.

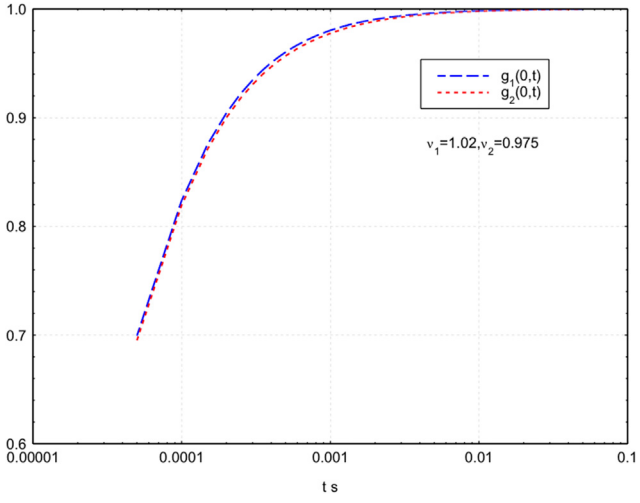

Fig. 7. Asymptotic time dependence of the extinction probability in a randomly varying system with a zero time-averaged reactivity, i.e. which is supercritical in the mean.

corresponding probability for the case when the process started from the supercritical state, has still not converged. The proof of the convergence of this latter requires calculations for very long times. This is illustrated in Fig. 6. It is seen that for $t = 18$ sec the extinction probability deviates from unity only in the seventh digit even for the case when the system started from the strongly supercritical state. Another case of a system with $\overline{\alpha} = \alpha_{crit}$, i.e. which is supercritical in the mean, but with different values of α_1, α_2 and α_{crit} is shown in Fig. 7. In this case the two values of α are much closer to each other, and the positive α is much smaller than in the previous case, hence the asymptotic regime is reached much faster (but note the logarithmic scale on the x-axis).

These results mean that in the domain $\alpha_{crit} \leq \overline{\alpha} \leq 0$, the extinction probability is still unity, despite the fact that the temporally varying system is supercritical in the mean. This is an unexpected result, revealing interesting new insight into the physics of branching processes in temporally randomly varying media. It has though to be noted that this result was obtained by numerical methods, and has to be substantiated by further numerical and analytical investigations.

5. Conclusions

It was shown that the extinction probability in systems randomly varying in time can only be calculated with purely numerical methods, even if the generation function of the reproduction multiplicity is quadratic, i.e. at most two new particles/entities can be produced per reaction. Two methods were used in this paper to calculate the extinction probability, a scheme based on the Chebyshev-Gauss-Lobatto collocation algorithm, and one based on the numerical partial differential equation system solver NDSolve of the symbolic manipulation code Mathematica [9].

The results show that the properties of the extinction probability in systems fluctuating in time are in agreement with those of the lower order moments, but appear to show some novel features as well. In particular, it was found that for systems with a time-averaged reactivity equal to zero, the extinction probability is still equal to unity. Such a result may have some relevance with the evolution of biological processes, such as the population dynamics of bacteria colonies.

It has also to be added that the results presented in the paper are based on numerical calculations, which has to be kept in mind when interpreting them. Whether the extinction probability is exactly unity or only extremely close, is not possible to show by pure numerical methods. However, the observation of the possibility of an extinction probability being unity in supercritical systems is a rather interesting one, and further work will be performed to confirm the present results, as well as to get a deeper insight into the characteristics of the extinction probability in systems randomly varying in time.

Conflicts of interest

There is no conflict of interest.

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