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GEOMETRIC CLASSIFICATION OF ISOMETRIES ACTING ON HYPERBOLIC 4-SPACE

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Abstract. An isometry of hyperbolic space can be written as a composition of the reflection in the isometric sphere and two Euclidean isometries on the boundary at infinity. The isometric sphere is also used to construct the Ford fundamental domains for the action of discrete groups of isometries. In this paper, we study the isometric spheres of isometries acting on hyperbolic 4-space. This is a new phenomenon which occurs in hyperbolic 4-space that the two isometric spheres of a parabolic isometry can intersect transversally. We provide one geometric way to classify isometries of hyperbolic 4-space using the isometric spheres.

1. Introduction

Hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ is the unique complete simply connected $(n+1)$-dimensional Riemannian manifold with constant sectional curvature -1. The upper half space model of $\mathbb{H}^{n+1}$ is $\{(x_1,\ldots,x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$ with the metric $ds = \frac{dx}{x_{n+1}}$. Hyperbolic $(n+1)$-space $\mathbb{H}^{n+1}$ has the boundary at infinity $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$. Any isometry of $\mathbb{H}^{n+1}$ extends continuously to a Möbius transformation of $\hat{\mathbb{R}}^n$ and vice-versa. The converse is so-called Poincaré extension [5, 11]. Thus we identify an isometry of $\mathbb{H}^{n+1}$ with a Möbius transformation of $\hat{\mathbb{R}}^n$. In what follows, we assume for simplicity that all maps are orientation-preserving.

An isometry of hyperbolic space can be written as a composition of the reflection in the isometric sphere and two Euclidean isometries on the boundary at infinity. In this paper, we study the isometric spheres of isometries acting on hyperbolic 4-space. We observe a new phenomenon which occurs in hyperbolic 4-space that the two isometric spheres of a parabolic isometry can intersect transversally. However, we realize that the incidence relations of isometric spheres are not conjugacy invariants in dimension 4. The isometric sphere has

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been used to construct the Ford fundamental domains for the action of discrete groups of isometries in the classical dimensions 2 and 3 [7, 9, 11]. Thus, these new phenomena need to be understood well for the study of discrete groups of isometries acting on hyperbolic 4-space. We here offer the first step in the direction.

An isometry of hyperbolic space $\mathbb{H}^{n+1}$ can be represented as a 2×2 matrix whose entries are the Clifford numbers C_{n-1} satisfying some conditions. The action of the 2×2 matrix is the usual action of Möbius transformations. This is a natural generalization of the classical settings, $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$, via identifying the real numbers $\mathbb{R}$ with the Clifford algebra C_0 and the complex numbers $\mathbb{C}$ with the Clifford algebra C_1. Note that $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ are the group of isometries of hyperbolic 2-space and 3-space respectively. For the isometries of hyperbolic 4-space, we use C_2 which is also quaternion numbers for the matrix representations. One advantage of these representations is that it gives us an automatic extension from m-dimensional representations to $(m+1)$-dimensional representations.

On the boundary at infinity, the isometric sphere of the Möbius transformation is the set of points where a Möbius transformation acts as a Euclidean isometry. More precisely, for a Möbius transformation f of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$, the *isometric sphere* I_f of f is the set of points where $|f'(x)| = 1$. Then the Möbius transformation f can be written in the form $\psi \circ \tau \circ \sigma$, where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. In fact, $\psi(x) = TAT^{-1}(x),\ A \in O(3)$ and $T(x) = x + f(\infty)$ for any $x \in \mathbb{R}^3$ [12]. We call this *the isometric sphere decomposition*. If f is of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a, b, c and d are quaternions satisfying some conditions (Definition 2.1), the isometric spheres I_f and $I_{f^{-1}}$ are 2-dimensional spheres of the same radius $|c|^{-1}$ centered at $f^{-1}(\infty) = -c^{-1}d$ and $f(\infty) = ac^{-1}$ respectively in $\mathbb{R}^3$ [4].

For any Möbius transformation $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{C})$ with $c \neq 0$, the trace $a + d$ has information about the incidence relation between isometric spheres $I_f = S(-\frac{d}{c}, \frac{1}{|c|})$ and $I_{f^{-1}} = S(\frac{a}{c}, \frac{1}{|c|})$. The incidence relation is enough information to determine the type of a given hyperbolic isometry, that is, f is loxodromic if the two isometric spheres I_f and $I_{f^{-1}}$ are disjoint, parabolic if they are tangent, and elliptic if they intersect transversally. The trace is a conjugacy invariant and hence the incidence relation between isometric spheres is also a conjugacy invariant.

However, since Clifford numbers do not commute in general, the trace of a Clifford matrix is not a conjugacy invariant. Furthermore, for a Möbius transformation of $\hat{\mathbb{R}}^3$, the incidence relations of isometric spheres are not conjugacy invariants. For example, $f = \left(\begin{smallmatrix} e_1 & 0 \\ e_1 + e_1 e_2 & -e_1 \end{smallmatrix}\right)$ is a parabolic isometry fixing 0. The centers of two isometric spheres I_f and $I_{f^{-1}}$ are $f^{-1}(\infty) = \frac{1}{2}(1 - e_2)$ and

$f(\infty) = \frac{1}{2}(1+e_2)$. The distance between two centers is $d(f^{-1}(\infty), f(\infty)) = 1$, but the radius of the isometric sphere is $\frac{1}{\sqrt{2}}$. Hence, two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally. This is a contrast to lower dimensions where the two isometric spheres of a parabolic isometry can only intersect tangentially. On the other hand, f is conjugate to $g = \left(\begin{smallmatrix} e_1 & 0 \\ e_1e_2 & -e_1 \end{smallmatrix}\right)$ by $h = \left(\begin{smallmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{smallmatrix}\right)$, whose two isometric spheres $I_g = S(-e_2, 1)$ and $I_{g^{-1}} = S(e_2, 1)$ are tangent to each other. It is still true that parabolic or elliptic fixed points are in the intersection of two isometric spheres.

We summarize our results as follows: Let $f = \psi\tau\sigma$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where $\psi\tau\sigma$ be the isometric sphere decomposition. When the Euclidean isometry ψ is the identity map, the type of f depends on the incidence relation of the two isometric spheres I_f and $I_{f^{-1}}$.

Theorem 3.2 *Let $f = \tau\sigma \in \text{Möb}(\hat{\mathbb{R}}^3)$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$. Then f is loxodromic if the two isometric spheres I_f and $I_{f^{-1}}$ are disjoint; parabolic if I_f and $I_{f^{-1}}$ are tangent; elliptic if I_f and $I_{f^{-1}}$ intersect transversally.*

If ψ is not the identity map, then ψ fixes a point p in the reflection plane of τ (p is possibly ∞). We classify f as a non-boundary elliptic if p is in the interior of the isometric sphere I_f (Proposition 3.12); a parabolic if p is on the isometric sphere I_f (Proposition 3.5, Proposition 3.6); a loxodromic if p is in the exterior of the isometric sphere I_f (Proposition 3.4, Proposition 3.10). We can find the statement in Theorem 3.13.

We summarize the complete classification in Table 1. Finally, we have the following characterization of parabolic isometries.

Theorem 3.14. *Let $f = \psi\tau\sigma \in \text{Möb}(\hat{\mathbb{R}}^3)$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. Then f is parabolic if and only if ψ fixes a point in $I_f \bigcap I_{f^{-1}}$.*

Section 2 will cover more Clifford representations. Section 3 will classify isometries of hyperbolic 4-space using the isometric sphere decomposition. For the basics on hyperbolic geometry the reader is referred to Beardon [5], Maskit [11], Ratcliffe [12] and Wilker [17]; for a Clifford matrix representation, to Ahlfors [2, 3, 4], Cao-Waterman [6], Hersonsky [8], Tan-Wong-Zhang [13], Wada [15] and Waterman [16]. The paper represents a portion of my Ph.D. Thesis [10] completed at the Graduate Center, City University of New York. I am

Table 1. Classification of $f \in \mathrm{Isom}(\mathbb{H}^4)$ with $f(\infty) \neq \infty$.

$f = \psi\tau\sigma$, $\psi = id$	
$I_f \bigcap I_{f^{-1}} = \emptyset$	hyperbolic
$I_f \bigcap I_{f^{-1}} =$ one point	strictly parabolic
$I_f \pitchfork I_{f^{-1}} =$ a circle	boundary elliptic

$f = \psi\tau\sigma$, $\psi \neq id$ and $A_\psi \cap \tau = \{p\}$		
$I_f \bigcap I_{f^{-1}} = \emptyset$		loxodromic
$I_f \bigcap I_{f^{-1}} = \{q\}$	$p = q$	screw parabolic
$I_f \bigcap I_{f^{-1}} = \{q\}$	$p \neq q \Leftrightarrow p \in Ext(I_f)$	loxodromic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in C$	screw parabolic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in Ext(I_f)$	loxodromic
$I_f \pitchfork I_{f^{-1}} = C$	$p \in Int(I_f)$	non-boundary elliptic

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2. Möbius transformations and Clifford representations

We classify all isometries of hyperbolic space into three types with respect to their fixed points. If it has a fixed point in $\mathbb{H}^{n+1}$, then it is *elliptic*. If it is not elliptic and has exactly one fixed point on the boundary at infinity $\hat{\mathbb{R}}^n$, then it is *parabolic*; otherwise it is *loxodromic*. An elliptic isometry is called *boundary elliptic* if it has a fixed point on $\hat{\mathbb{R}}^n$; otherwise it is *non-boundary elliptic*.

On the boundary at infinity $\hat{\mathbb{R}}^n$, a parabolic isometry is conjugate to $x \mapsto Ax + a$ with $A \in \mathrm{SO}(n)$, $a \in \mathbb{R}^n \setminus \{0\}$ by a Möbius transformation. If $A = I$, then it is called *strictly parabolic*; otherwise it is *screw parabolic*. Hyperbolic 4-space is the lowest dimension hyperbolic space where screw parabolic isometries appear.

The Clifford algebra C_{n-1} is the associative algebra over the real numbers generated by $e_1, e_2, \ldots, e_{n-1}$ subject to the relations ${e_i}^2 = -1$ $(i = 1, \ldots, n-1)$ and $e_ie_j = -e_je_i$ $(i \neq j)$ and no others. The null product of generators is the real number 1. A *Clifford number* $a \in C_{n-1}$ is of the form $\sum a_I I$ where the sum is over all products $I = e_{v_1}e_{v_2}\cdots e_{v_p}$ with $1 \leq v_1 < v_2 < \cdots < v_p \leq n-1$ and $a_I \in \mathbb{R}$. The Euclidean *norm* $|a|$ of $a = \sum a_I I \in C_{n-1}$ is given by $|a|^2 = \sum a_I^2$. We denote by $(a)_{\mathbb{R}}$ the real part of a. We can identify C_0 with the real numbers $\mathbb{R}$, C_1 with the complex numbers $\mathbb{C}$ and C_2 with the quaternion numbers. Here are the three involutions in the Clifford algebra C_{n-1}:

(1) The main involution $a \mapsto a'$ is obtained by replacing each e_i with $-e_i$. Thus, $(ab)' = a'b'$ and $(a+b)' = a' + b'$.
(2) Reversion $a \mapsto a^*$ is obtained by replacing each $e_{v_1}e_{v_2}\cdots e_{v_p}$ with $e_{v_p}e_{v_{p-1}}\cdots e_{v_1}$. Therefore, $(ab)^* = b^*a^*$ and $(a+b)^* = a^* + b^*$.

(3) Conjugation $a \mapsto \overline{a}$ is the composition of the main involution and reversion, i.e., $\overline{a} = (a')^* = (a^*)'$.

A *vector* is a Clifford number of the form $x = x_0 + x_1e_1 + \cdots + x_{n-1}e_{n-1} \in C_{n-1}$ where the x_i's are real numbers. The set of all vectors forms an n-dimensional subspace which we identify with $\mathbb{R}^n$. For any vector x, $x^* = x$ and $\overline{x} = x'$. Every non-zero vector x is invertible with $x^{-1} = \frac{\overline{x}}{|x|^2}$. A *Clifford group* Γ_{n-1} is a multiplicative group generated by all non-zero vectors of C_{n-1}. We note that $\Gamma_{n-1} = C_{n-1} - \{0\}$ is true for only $n = 1, 2, 3$.

Definition 2.1. A matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is said to be a *Clifford matrix* if the following conditions are satisfied:

(1) $a, b, c, d \in \Gamma_{n-1} \cup \{0\}$.
(2) $ad^* - bc^* = 1$.
(3) $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^n$.

A Clifford matrix A has its multiplicative inverse Clifford matrix $A^{-1} = \left(\begin{smallmatrix} d^* & -b^* \\ -c^* & a^* \end{smallmatrix}\right)$. Hence, all Clifford matrices form a group, denoted by $\mathrm{SL}(\Gamma_{n-1})$.

A Clifford matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(\Gamma_{n-1})$ induces a Möbius transformation of $\hat{\mathbb{R}}^n$ by $Ax = (ax+b)(cx+d)^{-1}$ for any vector $x = x_0+x_1e_1+\cdots+x_{n-1}e_{n-1} \in \mathbb{R}^n$, and $\infty \mapsto \infty$ if $c = 0$ and $\infty \mapsto ac^{-1}$, $-c^{-1}d \mapsto \infty$ if $c \neq 0$. Moreover, any orientation-preserving Möbius transformation can be presented as a Clifford matrix. Replacing x with $x + x_ne_n$, we can automatically extend the action of A to a Möbius transformation $x+x_ne_n \mapsto (a(x+x_ne_n)+b)(c(x+x_ne_n)+d)^{-1}$ of $\hat{\mathbb{R}}^{n+1}$. The coefficient of the last generator e_n of image is $\frac{x_n}{|cx+d|^2}$. This shows that the extension keeps the upper half-space $\mathbb{H}^{n+1}$ invariant. In fact, the quotient group of Clifford matrices $\mathrm{SL}(\Gamma_{n-1})$ by modulo $\pm\mathrm{I}$ is isomorphic to a group of Möbius transformations $\mathrm{Möb}(\hat{\mathbb{R}}^n)$.

Lemma 2.2 ([2])**.** *For $a, b \in \Gamma_2$, $ab^{-1} \in \mathbb{R}^3$ if and only if $a^*b \in \mathbb{R}^3$.*

Proposition 2.3. (1) *For any vector $x \in \mathbb{R}^3$ and any quaternion $a \in C_2$, $ax - xa' \in \mathbb{R}^3$.*
(2) *For any vector $x \in \mathbb{R}^3$, $xe_1e_2x = |x|^2e_1e_2$.*

Proof. Let $x = x_0 + x_1e_1 + x_2e_2 \in \mathbb{R}^3$ and $a = a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2 \in C_2$.

Item 1 comes from the following: The coefficient of e_1e_2-component of ax is

$$[(a_0 + a_1e_1 + a_2e_2 + a_3e_1e_2)(x_0 + x_1e_1 + x_2e_2)]_{e_1e_2} = a_1x_2 - a_2x_1 + a_3x_0$$

which is the same as the coefficient of e_1e_2-component of xa'.

To show item 2, we compute

$$\begin{aligned} xe_1e_2x &= xe_1e_2(x_0 + x_1e_1 + x_2e_2) = x(x_0e_1e_2 - x_1e_1e_1e_2 - x_2e_2e_1e_2) \\ &= x(x_0 - x_1e_1 - x_2e_2)e_1e_2 = |x|^2e_1e_2. \end{aligned}$$

□

Lemma 2.4 ([6])**.** *If $\lambda = \cos\theta + \sin\theta\xi e_1e_2 \in \Gamma_2$ with $\xi \in \mathbb{R}^3$ and $|\xi| = 1$, then $\rho_\lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right)$ is a rotation around ξ by 2θ and hence $\rho_\lambda \in \mathrm{SO}(3)$.*

If $A \in \mathrm{SL}(\Gamma_2)$ fixes ∞, then A is of the form $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix}$ for some quaternions λ and μ.

Theorem 2.5 ([6]). *A Möbius transformation* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ *is loxodromic if and only if* $|\lambda| \neq 1$.

If λ is a real number, μ is a vector because $\lambda\mu^* \in \mathbb{R}^3$ by Definition 2.1. Hence, we have the following mutually exclusive three cases.

Theorem 2.6 ([6]). *A Möbius transformation* $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \mathrm{SL}(\Gamma_2)$ *with* $|\lambda| = 1$ *is strictly parabolic if* $\lambda \in \mathbb{R}$, *screw parabolic if* $\mu \notin \mathbb{R}^3$, *elliptic otherwise.*

3. Classification of isometries

Throughout this section, we assume that $f \in \mathrm{Möb}(\hat{\mathbb{R}}^3)$ with $f(\infty) \neq \infty$, $f(\infty) \neq f^{-1}(\infty)$ and $f = \psi \circ \tau \circ \sigma$ is the isometric sphere decomposition. We will use the convention that a reflection in a plane (or a sphere) is also its reflection plane (or a sphere).

Lemma 3.1 ([5]). *Let D be an open ball in $\mathbb{R}^n$ and f be a Möbius transformation acting on $\hat{\mathbb{R}}^n$. If $f(\overline{D}) \subset D$, then f is a loxodromic element and has a fixed point in $f(\overline{D})$.*

Suppose that two isometric spheres I_f and $I_{f^{-1}}$ are disjoint. Then $f(\overline{D}) = \psi \circ \tau \circ \sigma(\overline{D}) \subsetneq D$, where D is a ball bounded by $I_{f^{-1}}$. By Lemma 3.1, f is a loxodromic. Hence, it suffices to see which type f is when two isometric spheres I_f and $I_{f^{-1}}$ intersect each other.

First, we consider the case that the Euclidean isometry ψ of the isometric sphere decomposition of f is the identity map as follows.

Theorem 3.2. *Let $f = \tau\sigma \in \mathrm{Möb}(\hat{\mathbb{R}}^3)$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$. Then f is loxodromic if the two isometric spheres I_f and $I_{f^{-1}}$ are disjoint*; *parabolic if I_f and $I_{f^{-1}}$ are tangent*; *elliptic if I_f and $I_{f^{-1}}$ intersect transversally.*

Proof. We note that $f = \tau\sigma$ fixes every point in the intersection of the two isometric spheres I_f and $I_{f^{-1}}$. The intersection of I_f and $I_{f^{-1}}$ is either a circle or one point. When the intersection is a circle, f is boundary elliptic. If the intersection is exactly one point, we conjugate f by a Möbius transformation m which sends the intersection point to a point ∞. Then mfm^{-1} is the composition of two reflections on two parallel Euclidean planes $m(\tau)$ and $m(\sigma)$. Hence mfm^{-1} is a Euclidean translation, in other words strictly parabolic and so is f. $\square$

From now on, we assume that ψ is not the identity map and the two isometric spheres I_f and $I_{f^{-1}}$ intersect each other. Then the intersection of the two isometric spheres can be again either one point or a circle. In the following two Propositions 3.4, 3.5, we will show that if the intersection is exactly one point, f can be either screw parabolic or loxodromic depending on the action of Euclidean isometry ψ.

Lemma 3.3. *Let* $f = \left(\begin{smallmatrix} \alpha' & \alpha'-\alpha \\ \alpha' & \alpha' \end{smallmatrix}\right) \in \mathrm{SL}(\Gamma_2) \leq \mathrm{SL}(\Gamma_3)$ *where* $\alpha \in \Gamma_2$ *is a unit quaternion satisfying* $\alpha \neq \pm 1$ *and* $\alpha' - \alpha \neq 0$. *Then* f *has no fixed point in the upper half-space* $\mathbb{H}^4$.

Proof. Suppose that f has a fixed point $v = x + te_3 \in \mathbb{H}^4$ where $x = x_0 + x_1e_1 + x_2e_2 \in \mathbb{R}^3$, t is a positive real number and e_1, e_2 and e_3 generate the Clifford algebra C_3.

$$\begin{aligned}
&f(v) = v \\
\Leftrightarrow\ &\alpha'(x+te_3) + \alpha' - \alpha = (x+te_3)\alpha'(x+te_3) + (x+te_3)\alpha' \\
\Leftrightarrow\ &\alpha' x + t\alpha' e_3 + \alpha' - \alpha = x\alpha' x + tx\alpha' e_3 + t\alpha x' e_3 + t^2 e_3\alpha' e_3 + x\alpha' + t\alpha e_3.
\end{aligned}$$

Since $t^2 e_3 \alpha' e_3 = t^2 \alpha {e_3}^2 = -t^2\alpha$, we have

$$\alpha' x + \alpha' - \alpha = x\alpha' x - t^2\alpha + x\alpha' \tag{1}$$

and

$$\begin{aligned}
t\alpha' e_3 &= t(x\alpha' + \alpha x' + \alpha)e_3 \\
\Rightarrow \alpha' &= x\alpha' + \alpha x' + \alpha \quad \text{since } t \neq 0 \\
\Rightarrow \alpha' - \alpha &= x\alpha' + \alpha x'.
\end{aligned} \tag{2}$$

From (2), we have $x \neq 0$ since $\alpha' - \alpha \neq 0$. From (2),

$$\alpha' - x\alpha' = \alpha + \alpha x' \text{ and hence } |1 - x| = |x' + 1|.$$

Therefore, $x_0 = 0$ and hence $x' = -x$. We replace $\alpha' - \alpha$ of equation (1) with $x\alpha' + \alpha x'$ to have the following

$$\begin{aligned}
\alpha' x + x\alpha' + \alpha x' &= x\alpha' x - t^2\alpha + x\alpha' \\
\Leftrightarrow \alpha' x - \alpha x &= (x\alpha' - t^2\alpha x^{-1})x
\end{aligned}$$

which implies

$$\begin{aligned}
\alpha' - \alpha &= x\alpha' - t^2\alpha x^{-1} \text{ since } x \text{ is invertible} \\
&= x\alpha' - \frac{t^2}{|x|^2}\alpha x' \text{ since } x \neq 0.
\end{aligned} \tag{3}$$

From (2) and (3),

$$\begin{aligned}
x\alpha' + \alpha x' &= x\alpha' - \frac{t^2}{|x|^2}\alpha x' \\
\Leftrightarrow (1 + \frac{t^2}{|x|^2})\alpha x' &= 0.
\end{aligned}$$

Since $1+\frac{t^2}{|x|^2}\neq 0$ and $\alpha\neq 0$, $x=0$ which is a contradiction. Therefore, f has no fixed point in $\mathbb{H}^4$. $\square$

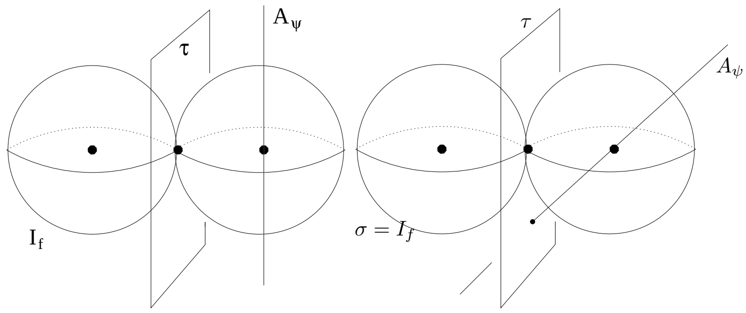

FIGURE 1. $I_f \bigcap I_{f^{-1}}$ = one point : loxodromic

Proposition 3.4. *Suppose that I_f and $I_{f^{-1}}$ meet tangentially at a point q in $\mathbb{R}^3$. If ψ does not fix the intersection point q, then f is loxodromic (see Figure 1).*

Proof. We may assume that the tangential intersection point q is 0. Then $\tau\sigma$ is strictly parabolic with a fixed point 0. So, it is of the form $\left(\begin{smallmatrix}1 & 0\\ c & 1\end{smallmatrix}\right)$ for a non-zero vector $c\in\mathbb{R}^3$. We may conjugate $f=\psi\tau\sigma$ by a dilation so that $c=1$. Then, the center $f^{-1}(\infty)$ of the isometric sphere I_f is -1 and the center $f(\infty)$ of the isometric sphere $I_{f^{-1}}$ is 1. Since $\psi(x)=TAT^{-1}(x)$ for some $A\in O(3)$ and $T(x)=x+f(\infty)$ for any $x\in\mathbb{R}^3$ in the decomposition of f, we have

$$\psi=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & \alpha'\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\alpha & \alpha'-\alpha\\ 0 & \alpha'\end{pmatrix}$$

for a unit quaternion $\alpha\in\Gamma_2$ with $\alpha\neq\pm 1$. Since the Euclidean isometry ψ does not fix 0 which is the tangential intersection point of the two isometric spheres, $\alpha'-\alpha\neq 0$.

$$f=\psi\tau\sigma=\begin{pmatrix}\alpha' & \alpha'-\alpha\\ \alpha' & \alpha'\end{pmatrix}.$$

By Lemma 3.3, f has no fixed point in $\mathbb{H}^4$ and hence it has a fixed point $u\in\hat{\mathbb{R}}^3$, i.e., $\alpha'u+\alpha'-\alpha=u(\alpha'u+\alpha')$. The fixed point u cannot be 0 since ψ does not fix 0. We conjugate f by a Möbius transformation which sends u to ∞ to obtain $\widetilde{f}$.

$$\widetilde{f}=\begin{pmatrix}0 & 1\\ -1 & u\end{pmatrix}\begin{pmatrix}\alpha' & \alpha'-\alpha\\ \alpha' & \alpha'\end{pmatrix}\begin{pmatrix}u & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}\alpha'u+\alpha' & -\alpha'\\ 0 & \alpha'-u\alpha'\end{pmatrix}.$$

Since $\widetilde{f}\in \mathrm{SL}(\Gamma_2)$, $(\alpha'u+\alpha')(\alpha'-u\alpha')^*=1$. This implies that $|\alpha'u+\alpha'|=|\alpha'||u+1|=|u+1|$ and $|\alpha'-u\alpha'|=|1-u||\alpha'|=|1-u|$ are simultaneously either

1 or not. However, they cannot be simultaneously 1 since $u \neq 0$. Therefore, $\widetilde{f}$ is loxodromic and so is f (see Lemma 2.4). $\square$

Proposition 3.5. *Suppose that I_f and $I_{f^{-1}}$ meet tangentially at a point q in $\mathbb{R}^3$. If ψ fixes the intersection point q, then f is screw parabolic.*

Proof. In the proof of Proposition 3.4, suppose that ψ fixes the tangential intersection point of two isometric spheres, i.e., $\psi(0) = 0$. Then $\psi = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha' \end{smallmatrix}\right)$ for a unit quaternion $\alpha = \cos\theta + \sin\theta e_1 e_2$ and hence

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \alpha' c & \alpha' \end{pmatrix}.$$

Since $\alpha' c$ is not a vector, f is a screw parabolic element (see Theorem 2.6). $\square$

Now, suppose that two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$. Then the rotational axis A_ψ of ψ intersects τ at a point, say p (p might be ∞). If the intersection point p belongs to the circle C, then f fixes p (See Figure 2). In Proposition 3.6. We will show that in this case f is screw parabolic When the intersection point p does not belong to the circle C (See Figure 3), f is loxodromic or elliptic. That will be the last case.

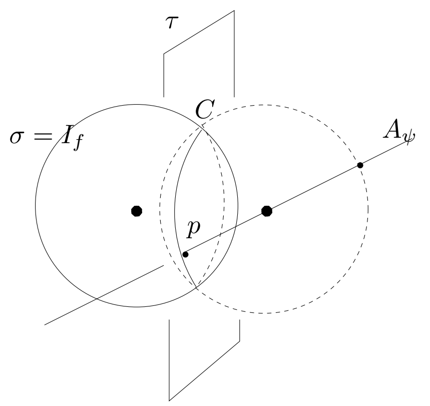

FIGURE 2. $I_f \pitchfork I_{f^{-1}} = C$ and $\tau \cap A_\psi \in C$: Screw parabolic

Proposition 3.6. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subseteq \tau \subseteq \mathbb{R}^3$. If ψ fixes a point p in C, then f is screw parabolic* (*See Figure 2*).

Proof. We may assume that $p = 0$. Then $\tau\sigma$ is a boundary elliptic element which fixes every point in C. In particular, $\tau\sigma$ fixes 0, but does not fix ∞.

If we conjugate $\tau\sigma$ by $h = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) : 0 \mapsto \infty$, then the conjugation will be a rotation about a line $h(C)$ which does not pass 0. Hence,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \begin{pmatrix} 1 & -\eta \\ 0 & 1 \end{pmatrix}$$

for a non-zero vector η in $\mathbb{R}^3$ and $\lambda = \cos\theta_\lambda + \sin\theta_\lambda \xi e_1 e_2 \notin \mathbb{R}^3$ for a unit vector ξ in $\mathbb{R}^3$. So we have

$$\tau\sigma = \begin{pmatrix} \lambda' & 0 \\ \lambda\eta - \eta\lambda' & \lambda \end{pmatrix}. \tag{4}$$

Note that $\lambda\eta - \eta\lambda'$ is a non-zero vector since $\tau\sigma$ fixes C. If the real part $(\lambda)_{\mathbb{R}} = \cos\theta_\lambda$ of λ is 0, then $\tau\sigma$ has order 2 and $I_{f^{-1}} = I_f$ which means $f(\infty) = f^{-1}(\infty)$. Therefore, $(\lambda)_{\mathbb{R}} \neq 0$.

Let v be a vector $\tau\sigma(\infty) = \lambda'(\lambda\eta - \eta\lambda')^{-1} \in \mathbb{R}^3$ which is the center of $I_{f^{-1}}$. Since ψ fixes 0 and v, it is of the form $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha' \end{smallmatrix}\right)$, for $\alpha = \cos\theta + \sin\theta \frac{v}{|v|} e_1 e_2$ with $\theta \in (0, 2\pi)$. Therefore, we have

$$f = \psi\tau\sigma = \begin{pmatrix} \alpha\lambda' & 0 \\ \alpha'(\lambda\eta - \eta\lambda') & \alpha'\lambda \end{pmatrix}$$

and we can conjugate f to

$$\begin{pmatrix} \alpha'\lambda & \alpha'(\eta\lambda' - \lambda\eta) \\ 0 & \alpha\lambda' \end{pmatrix} \text{ by } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $|\alpha'\lambda| = 1$ since $|\alpha| = 1 = |\lambda|$. So we know that it is either parabolic or elliptic by Theorem 2.4. The $(e_1e_2)^{th}$-coefficient of $\alpha'(\eta\lambda' - \lambda\eta)$ is

$$\begin{aligned} &\left[\left(\cos\theta + \sin\theta \frac{v'}{|v|}(e_1e_2)'\right)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)} \\ &= \left[\cos\theta(\eta\lambda' - \lambda\eta) - \frac{\sin\theta}{|v|}\lambda(\eta\lambda' - \lambda\eta)'^{-1}(e_1e_2)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)} \end{aligned}$$

since $\eta\lambda' - \lambda\eta$ is a non-zero vector,

$$\begin{aligned} &= \left[-\frac{\sin\theta}{|v||\eta\lambda' - \lambda\eta|^2}\lambda(\eta\lambda' - \lambda\eta)^*(e_1e_2)(\eta\lambda' - \lambda\eta)\right]_{(e_1e_2)} \\ &= \left[-\frac{\sin\theta}{|v|}\lambda(e_1e_2)\right]_{(e_1e_2)} \end{aligned}$$

since $xe_1e_2x = |x|^2 e_1e_2$ for any vector x (See Proposition 2.3.),

$$= \frac{\sin\theta}{|v|}(\lambda)_{\mathbb{R}} \neq 0.$$

Therefore, f is screw parabolic. □

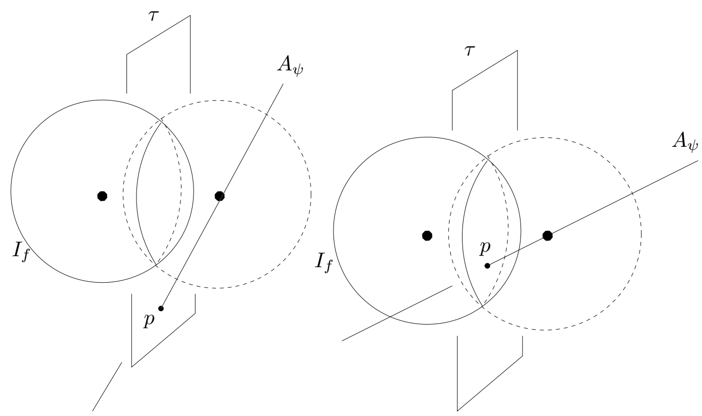

FIGURE 3. Loxodromic or non-boundary elliptic

Remark 3.7. The key of the above proof is $(\lambda)_{\mathbb{R}} \neq 0$. It is also true for higher dimensions. Hence, we can generalize the lemma: in any dimensions $n \geq 3$, if ψ fixes exactly one point in the intersection of the two isometric spheres, then f is screw parabolic.

Ahlfors shows that f is parabolic if and only if f is Möbius conjugate to a matrix of the form $\left(\begin{smallmatrix} vc & 0 \\ c & cv \end{smallmatrix}\right) \in \mathrm{SL}(\Gamma_2)$ with $v \in \mathbb{R}^3$ [4]. Then the two isometric spheres are $S(-v, \frac{1}{|c|})$ and $S(v, \frac{1}{|c|})$. The distance between the centers of spheres is $2|v| = \frac{2}{|c|}$ since $|vc| = 1$. Therefore, the two isometric spheres are tangent. We have seen that every parabolic element has a normalized parabolic element in its conjugacy class whose two isometric spheres are tangent. We also have the following corollary.

Corollary 3.8. *Let f be a parabolic element. Then f is strictly parabolic if and only if the position of the pair of isometric spheres is tangential for any element of* $\mathrm{Möb}(\hat{\mathbb{R}}^3)$ *conjugate to f.*

The last case is when the intersection point of τ and A_ψ does not belong to the intersection circle of two isometric spheres (see Figure 3). In this case, f does not fix any points of the intersection $I_f \cap I_{f^{-1}}$. Hence, f is loxodromic or non-boundary elliptic since a parabolic or elliptic fixed point can only be in the intersection $I_f \cap I_{f^{-1}}$.

Before this case, we will see a characterization of a Kleinian group generated by two Möbius transformations. This characterization will not be generalized into higher dimensions because of the presence of screw parabolic elements.

Lemma 3.9 ([1])**.** *Suppose that α, β and $\alpha\beta$ are parabolic, hyperbolic or elliptic Möbius transformations acting on $\mathbb{H}^3$. If α and β do not share a fixed point, then they preserve a common hyperbolic plane in $\mathbb{H}^3$ and the group $\langle \alpha, \beta \rangle$*

generated by α and β consists of parabolic, hyperbolic or elliptic elements which preserve this plane.

Let p be the intersection point of τ and A_ψ. In particular, suppose that the fixed point p is a point ∞. Then, there is a 2-dimensional f-invariant subspace P in $\mathbb{R}^3$ which is perpendicular to A_ψ and passes through the two centers of isometric spheres since the axis A_ψ is parallel to τ. We can think of the restriction of f on P as an element of $\text{Möb}(\hat{\mathbb{R}}^2)$. Hence, it is loxodromic because the two isometric circles intersect at two points and it has a non-trivial rotation. This idea can be generalized so that f is loxodromic if p belongs to the exterior of the isometric sphere $Ext(I_f)$.

Proposition 3.10. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$ and $\psi\tau$ fixes a point $p \in \mathbb{R}^3$. If $p \in Ext(I_f)$, then f is loxodromic.*

Proof. A Euclidean isometry $\psi\tau$ can be written as a composition of a rotation and a reflection with the same fixed point such that the rotational axis and the reflection plane are orthogonal. Now, without loss of generality we may assume that the axis A_ψ of ψ intersects the plane τ at p in orthogonal.

Since $p \in Ext(I_f)$, there are three possible cases considering the distance from the center of isometric sphere I_f to the plane τ. Let D be the ball bounded by the sphere σ. When τ and σ are disjoint, $\overline{\psi\tau(D)}$ and $\overline{D}$ are disjoint. Hence, $f(\overline{\psi\tau(D)}) \subsetneq \psi\tau(D)$, which means f is loxodromic (Lemma 3.1). When τ meet σ tangentially at a point, say q, $\overline{\psi\tau(D)}$ and $\overline{D}$ are disjoint again because ψ does not fixed the point q. Therefore, f is loxodromic as above.

In the last, suppose that τ intersects I_f transversally (see Figure 4). When τ passes the center of I_f, then f keeps τ invariant and the two half spaces divided by τ. So, $f|_\tau$ is conjugate to a Möbius transformation of $\hat{\mathbb{R}}^2$. It has a decomposition whose two isometric sphere have two intersection points and non-trivial rotation, therefore $f|_\tau$ is loxodromic and so is f. If τ does not pass the center, then there is also a unique ball B centered at p, whose boundary sphere ∂B is orthogonal to σ since p is in $Ext(I_f)$ (Figure 4). Then f keeps $\overline{B}$ invariant, and hence f is conjugate to a Möbius transformation of $\hat{\mathbb{R}}^2$. Since $\psi|_B$ and $\tau\sigma|_B$ are elliptic and there is no common disc preserved by $\psi|_B$ and $\tau\sigma|_B$ whose boundary is a circle in ∂B, $f|_B$ is loxodromic by Lemma 3.9. $\square$

Example 3.11. Let C be a unit circle centered at 0 in the $\langle 1, e_1\rangle$-plane. An boundary-elliptic element R which fixes every point on C is of the following form:

$$\begin{aligned} R = m\rho_\lambda m^{-1} &= \begin{pmatrix} \frac{1}{2} & e_1 \\ \frac{1}{2}e_1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & -e_1 \\ -\frac{1}{2}e_1 & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta e_2 \\ -\sin\theta e_2 & \cos\theta \end{pmatrix} \end{aligned} \tag{5}$$

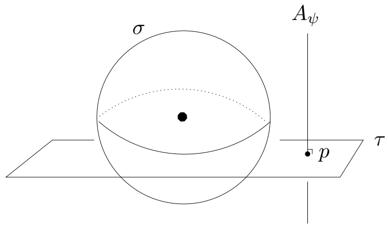

FIGURE 4. $I_f \pitchfork I_{f^{-1}} = C$: Loxodromic

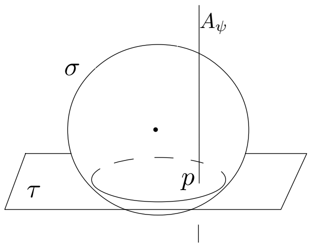

FIGURE 5. $I_f \pitchfork I_{f^{-1}} = C$: Non-boundary elliptic

where $\lambda = \cos\theta + \sin\theta e_1 e_2$, $\theta \in (0, \pi)$ and m is a Möbius transformation with $m(0) = e_1$, $m(2) = 1$ and $m(\infty) = -e_1$.

Proposition 3.12. *Suppose that the two isometric spheres I_f and $I_{f^{-1}}$ intersect transversally in a circle $C \subset \tau \subset \mathbb{R}^3$ and $\psi\tau$ fixes a point $p \in \mathbb{R}^3$. If $p \in Int(I_f)$, then f is non-boundary elliptic.*

Proof. Without loss of generality we may assume that the axis A_ψ of ψ intersects the plane τ at p in orthogonal and τ is the plane generated by 1 and e_1 (see Figure 5). Let C be the circle of the intersection $\tau \cap \sigma$. We may assume that C is the unit circle centered at the origin in the plane $\langle 1, e_1 \rangle$, and p has a coordinate $(t, 0, 0)$ for $t \in (0, 1)$ on the real axis. The angle between τ and σ is in $(0, \frac{\pi}{2}]$, say θ. Then $\tau\sigma$ is an elliptic element whose fixed point set is the circle C and rotation angle is 2θ. Thus, $\tau\sigma$ is of the form (5). Since the axis of ψ is orthogonal to τ and passes $p = (t, 0, 0)$, ψ is of the following form:

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & -2t\sin\tau e_1 \\ 0 & \alpha' \end{pmatrix}, \tag{6}$$

where $\alpha = \cos\tau + \sin\tau e_2(e_1 e_2)$ and $\tau \in (0, \pi)$.

$$\begin{aligned} f = \psi\tau\sigma &= \begin{pmatrix} \alpha & -2t\sin\tau e_1 \\ 0 & \alpha' \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta e_2 \\ -\sin\theta e_2 & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\alpha + 2t\sin\theta\sin\tau e_1 e_2 & -\sin\theta\alpha e_2 - 2t\cos\theta\sin\tau e_1 \\ -\sin\theta\alpha' e_2 & \cos\theta\alpha' \end{pmatrix}, \end{aligned}$$

where $\alpha = \cos\tau + \sin\tau e_1$, $\tau \in (0, \pi)$, $\theta \in (0, \frac{\pi}{2}]$ and $0 < t < 1$.

Suppose that f fixes a point in $\hat{\mathbb{R}}^3$. That is

$$\begin{aligned}(7)\qquad &(\cos\theta\alpha + 2t\sin\theta\sin\tau e_1e_2)u - \sin\theta\alpha e_2 - 2t\cos\theta\sin\tau e_1\\ &= -\sin\theta u\alpha' e_2 u + \cos\theta u\alpha'\end{aligned}$$

for a vector $u = u_0 + u_1e_1 + u_2e_2 \in \mathbb{R}^3$ (clearly, $u \neq \infty$). The e_1e_2-coefficient of (7) is

$$2t\sin\theta\sin\tau u_0 - \sin\theta\sin\tau - \sin\theta\sin\tau|u|^2 = 0.$$

Since $\sin\theta \neq 0$ and $\sin\tau \neq 0$ for $\theta \in (0, \frac{\pi}{2}]$ and $\tau \in (0, \pi)$,

$$2tu_0 = 1 + |u|^2$$

which implies $u_0 > 0$ and hence,

$$\begin{aligned}2t = \frac{1 + u_0^2 + u_1^2 + u_2^2}{u_0} &\geq\ u_0 + \frac{1 + u_1^2 + u_2^2}{u_0}\\ \geq\ 2\sqrt{1 + u_1^2 + u_2^2}\ &\geq\ 2.\end{aligned}$$

Thus, $t \geq 1$ which is a contradiction. Therefore, f has no fixed points in $\hat{\mathbb{R}}^3$. □

We have seen:

Theorem 3.13. *Let $f = \psi\tau\sigma \in \mathrm{Möb}(\hat{\mathbb{R}}^3)$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. Suppose ψ is not the identity map and fixes a point p in the reflection plane of τ (p is possibly ∞). Then a Möbius transformation f is non-boundary elliptic if p is in the interior of the isometric sphere I_f; parabolic if p is on the isometric sphere I_f; loxodromic if p is in the exterior of the isometric sphere I_f (including when $p = \infty$).*

We also have the following characterization of parabolic isometries:

Theorem 3.14. *Let $f = \psi\tau\sigma \in \mathrm{Möb}(\hat{\mathbb{R}}^3)$ be a Möbius transformation of $\hat{\mathbb{R}}^3$ satisfying $f(\infty) \neq \infty$ and $f(\infty) \neq f^{-1}(\infty)$ where σ is the reflection in the isometric sphere I_f of f, τ is the Euclidean reflection in the perpendicular bisector of the line segment between $f^{-1}(\infty)$ and $f(\infty)$, and ψ is a Euclidean isometry which keeps the isometric sphere $I_{f^{-1}}$ of f^{-1} invariant and fixes $f(\infty)$. Then f is parabolic if and only if ψ fixes a point in $I_f \bigcap I_{f^{-1}}$.*

Corollary 3.15. *Let $f = \psi\tau\sigma$ be elliptic. Then f is boundary elliptic if and only if ψ is the identity map.*

References

[1] C. Adams, J. Haas, and P. Scott, *Simple closed geodesic in hyperbolic 3-manifolds*, Bull. London Math. Soc. **31** (1999), no. 1, 81–86.

[2] L. V. Ahlfors, *Möbius transformations and Clifford numbers*, Differential Geometry and Complex Analysis, H. E. Rauch memorial volume, 65–73, Springer-Verlag, Belin, 1985.

[3] ——, *Old and new in Möbius groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 93–105.

[4] ——, *On the fixed points of Möbius transformation in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. **10** (1985), 15–27.

[5] A. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics, Vol. 91, Springer-Verlag, New York, 1983.

[6] C. Cao and P. L. Waterman, *Conjugacy invariants of Möbius groups*, Quasiconformal Mappings and Analysis, 109–139, Springer, 1995.

[7] T. A. Drumm and J. A. Poritz, *Ford and Dirichlet domains for cyclic subgroups of* $\mathrm{PSL}_2(\mathbb{C})$ *acting on* $\mathbb{H}^3_{\mathbb{R}}$ *and* $\partial\mathbb{H}^3_{\mathbb{R}}$, Conform. Geom. Dyn. **3** (1999), 116–150.

[8] S. Hersonsky, *Covolume estimates for discrete groups of hyperbolic isometries having parabolic elements*, Michigan Math. J. **40** (1993), no. 3, 467–475.

[9] T. Jørgensen, *On cyclic groups of Möbius transformations*, Math. Scand. **33** (1973), 250–260.

[10] Y. Kim, *Rigidity and stability for isometry groups in hyperbolic 4-space*, Thesis, The Graduate Center, City University of New York, 2008.

[11] B. Maskit, *Kleinian Groups*, Springer-Verlag, Berlin, 1988.

[12] J. Ratcliffe, *Foundation of Hyperbolic Manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

[13] S. P. Tan, Y. L. Wong, and Y. Zhang, *Delambre-Gauss formulas for augmented right-angled hexagons in hyperbolic 4-space*, Adv. Math. **230** (2012), no. 3, 927–956.

[14] K. Th. Vahlen, *Über Bewegungen und complexe Zahlen*, Math. Ann. **55** (1902), no. 4, 585–593.

[15] M. Wada, *Conjugacy invariants of Möbius transformations*, Complex Variables Theory Appl. **15** (1990), no. 2, 125–133.

[16] P. L. Waterman, *Möbius transformation in several dimensions*, Adv. Math. **101** (1993), no. 1, 87–113.

[17] J. B. Wilker, *The quaternion formalism for Möbius groups in four or fewer dimensions*, Linear Algebra Appl. **190** (1993), 99–136.

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