J. Korean Math. Soc. **54** (2017), No. 1, pp. 267–280
https://doi.org/10.4134/JKMS.j150720
pISSN: 0304-9914 / eISSN: 2234-3008

ANALYTIC AND GEOMETRIC PROPERTIES OF OPEN DOOR FUNCTIONS

Ming Li and Toshiyuki Sugawa

Abstract. In this paper, we study analytic and geometric properties of the solution $q(z)$ to the differential equation $q(z) + zq'(z)/q(z) = h(z)$ with the initial condition $q(0) = 1$ for a given analytic function $h(z)$ on the unit disk $|z| < 1$ in the complex plane with $h(0) = 1$. In particular, we investigate the possible largest constant $c > 0$ such that the condition $|\mathrm{Im}\,[zf''(z)/f'(z)]| < c$ on $|z| < 1$ implies starlikeness of an analytic function $f(z)$ on $|z| < 1$ with $f(0) = f'(0) - 1 = 0$.

1. Introduction

We denote by $\mathcal{A}$ the class of holomorphic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$. Let $\mathcal{A}_0$ denote the subclass of $\mathcal{A}$ consisting of functions p with $p(0) = 1$. Let $\mathcal{A}_1$ be the class of functions of the form $zp(z)$ for $p \in \mathcal{A}_0$. In other words, $f \in \mathcal{A}_1$ if and only if $f \in \mathcal{A}$ and $f(0) = f'(0) - 1 = 0$. We say that a function $f \in \mathcal{A}$ is *subordinate* to another $g \in \mathcal{A}$ and write $f \prec g$ or $f(z) \prec g(z)$ if $f = g \circ \omega$ for a function $\omega \in \mathcal{A}$ such that $\omega(0) = 0$ and $|\omega| < 1$. When g is univalent, $f \prec g$ precisely when $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

The set of functions $q \in \mathcal{A}_0$ with $\mathrm{Re}\, q > 0$ is called the Carathéodory class and will be denoted by $\mathcal{P}$. It is well recognized that the function $q_*(z) = (1 + z)/(1 - z)$ (or its rotation $q_*(e^{i\theta}z)$) maps the unit disk univalently onto the right half-plane and is extremal in many problems. A function $f \in \mathcal{A}_1$ is called *starlike* if f maps $\mathbb{D}$ univalently onto a starlike domain with respect to the origin. Likewise, a function $f \in \mathcal{A}_1$ is called *convex* if f maps $\mathbb{D}$ univalently onto a convex domain. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the classes of starlike and convex functions, respectively. It is well known that $f \in \mathcal{A}_1$ is starlike precisely when $q(z) = \psi_f(z) := zf'(z)/f(z)$ belongs to $\mathcal{P}$ and that $f \in \mathcal{A}_1$ is convex precisely when $h(z) = \varphi_f(z) := 1 + zf''(z)/f'(z)$ belongs to $\mathcal{P}$ (see, for instance, [1]). Note here the relation $h(z) = q(z) + zq'(z)/q(z)$. We also note that $f(z)$ is convex if and only if $zf'(z)$ is starlike for $f \in \mathcal{A}_1$. For a given $h \in \mathcal{A}_0$, we

Received November 30, 2015.
2010 *Mathematics Subject Classification.* 30C45, 30C80.
Key words and phrases. open door lemma, subordination, convex function, strongly starlike function, dilogarithm function.

always find a function $f \in \mathcal{A}_1$ with $1 + zf''/f' = h$. Indeed, by integrating the relation $(\log f')' = f''/f' = (h-1)/z$, we obtain

$$f'(z) = \exp\left(\int_0^z \frac{h(t)-1}{t}dt\right). \tag{1.1}$$

By integrating the above, we get the desired $f \in \mathcal{A}_1$. Similarly, replacing $f'(z)$ by $f(z)/z$ in (1.1), we obtain a representation of $f \in \mathcal{A}_1$ satisfying $zf'/f = h$.

It is obvious that a convex function is starlike. This means analytically that $h = q + zq'/q \in \mathcal{P}$ implies $q \in \mathcal{P}$ for $q \in \mathcal{A}_0$. In other words, $q + zq'/q \prec q_*$ implies $q \prec q_*$.

One can observe that the function

$$h_*(z) := q_*(z) + \frac{zq_*'(z)}{q_*(z)} = \frac{1+z}{1-z} + \frac{2z}{1-z^2} = \frac{1+4z+z^2}{1-z^2}$$

maps the unit disk onto the complex plane $\mathbb{C}$ slit along the two half-lines $\pm iy,\ y \geq \sqrt{3}$. The following was proved by Mocanu [8] and later extended by Miller and Mocanu [5] (see also [6]).

Theorem A (Open Door Lemma). *Suppose that a function $q \in \mathcal{A}_0$ satisfies the subordination condition*

$$q(z) + \frac{zq'(z)}{q(z)} \prec h_*(z) = q_*(z) + \frac{zq_*'(z)}{q_*(z)}.$$

Then $q(z) \prec q_*(z)$.

In particular, if a function $f \in \mathcal{A}_1$ satisfies the subordination $1 + zf''/f' \prec h_*$, then f is starlike. Since the slit domain $h_*(\mathbb{D})$ contains the parallel strip $|\operatorname{Im} w| < \sqrt{3}$, we obtain the following result as a corollary.

Corollary 1.1. *If a function $f \in \mathcal{A}_1$ satisfies the condition*

$$\left|\operatorname{Im}\left[\frac{zf''(z)}{f'(z)}\right]\right| < \sqrt{3}, \quad z \in \mathbb{D},$$

then f is starlike.

We recall a notion of strong starlikeness. A function $f \in \mathcal{A}_1$ is called *strongly starlike of order* α for an $0 < \alpha$ if $|\arg[zf'(z)/f(z)]| < \pi\alpha/2$ for $z \in \mathbb{D}$. We denote by $\mathcal{S}_\alpha^*$ the class of strongly starlike functions in $\mathcal{A}_1$ of order α. Obviously, we have $\mathcal{S}_1^* = \mathcal{S}^*$. For geometric characterizations of strongly starlike functions, see [10] and references therein.

In the present paper, we try to find or estimate the best possible constant $\gamma > 0$ such that the condition $|\operatorname{Im}[zf''(z)/f'(z)]| < \gamma$ implies $f \in \mathcal{S}_\alpha^*$. More precisely, the number is defined as $\gamma(\mathcal{S}_\alpha^*)$, where

$$\gamma(\mathcal{F}) = \sup\left\{\gamma \geq 0 : \varphi_f(\mathbb{D}) \subset W_\gamma \text{ implies } f \in \mathcal{F} \text{ for } f \in \mathcal{A}_1\right\}$$

for a subset $\mathcal{F}$ of $\mathcal{A}_1$ and

$$W_\gamma = \{w \in \mathbb{C} : |\operatorname{Im} w| < \gamma\}$$

is a parallel strip of width 2γ. We recall that $\varphi_f = 1 + zf''/f'$.

We will show the following estimates of $\gamma(\mathcal{S}^*_\alpha)$. See also Figure 3 below for the graphs of $\gamma(\mathcal{S}^*_\alpha)$ the upper and lower bounds.

Theorem 1.2. *Let* $0 < \alpha < 1$. *Then*

$$\sqrt{3}\alpha < \frac{\alpha + (1+\alpha)\sin(\pi\alpha/2)}{\sqrt{1+2\sin(\pi\alpha/2)}} < \gamma(\mathcal{S}^*_\alpha) < \frac{\sqrt{3}\pi\alpha}{\sqrt{3+\alpha}}.$$

We remark that a similar (but not better) result can be found at [2, Theorem 1.6]. Mocanu [7, Corollary 1.1] showed that $\gamma(\mathcal{S}^*_{\pi/4}) \geq 1$. Our estimate gives $\gamma(\mathcal{S}^*_{\pi/4}) > (2+3\sqrt{2})/4\sqrt{1+\sqrt{2}} = 1.0044319\cdots$. Note that the lower bound in this theorem tends to $\sqrt{3}$ as $\alpha \to 1$, which agrees with Corollary 1.1. When $\alpha = 1$, we can slightly improve the upper bound in the last theorem.

Theorem 1.3. $\sqrt{3} \leq \gamma(\mathcal{S}^*) < 2.5$.

Though it is difficult to compute the exact value of $\gamma(\mathcal{S}^*)$, the next result gives us a way to compute it numerically.

Theorem 1.4. *Let* $\theta_c = 2\arctan(e^{2/c}) \in (\pi/2, \pi)$ *for* $c > 0$. *Let* $F(c) = v(1)$ *for* $c \geq 0$, *where* $v(t)$ *is the solution to the initial value problem of ordinary differential equation*

$$v(t) + \frac{tv'(t)}{v(t)} = 1 + \frac{c}{2}\log\frac{1+te^{i\theta_c}}{1-te^{i\theta_c}}, \quad v(0) = 1.$$

Then

$$\gamma(\mathcal{S}^*) = \pi c_0/4,$$

where c_0 *is the smallest positive number such that* $\operatorname{Re} F(c_0) = 0$.

We remark that $F(c) = q_c(e^{i\theta_c})$, where q_c is given in Section 3.

The organization of the present paper is as follows. In Section 2, we investigate geometric properties of the solution q to the differential equation $q + zq'/q = h$ for a given $h \in \mathcal{A}_0$. We believe that our observation will be helpful for other kinds of problems concerning the subordination of ψ_f and φ_f. In order to estimate the quantity $\gamma(\mathcal{S}^*_\alpha)$, we study in Section 3 the extremal case when $h = h_c = 1 + c \cdot \operatorname{arctanh}$, which maps $\mathbb{D}$ onto the parallel strip $W_{\pi c/4}$ for $c > 0$. We will show that the solution q_c to the differential equation $q_c + zq_c'/q_c = h_c$ maps $\mathbb{D}$ univalently onto a smooth Jordan domain if c is not very large (Theorem 3.1). Lemma 3.2 will describe $\gamma(\mathcal{S}^*_\alpha)$ in terms of the above solutions q_c. Section 4 is devoted to the proof of the main theorems. The last section gives concluding remarks on numerical experiments. By using Mathematica, we can generate a graph of the function $\alpha \mapsto \gamma(\mathcal{S}^*_\alpha)$ and some approximation of the value $\gamma(\mathcal{S}^*)$ though there is no rigorous error estimate for the present experiments.

2. Some observations of Open Door Lemma

Throughout the present paper, for simplicity, we consider only functions in $\mathcal{A}_0$ for the Open Door Lemma. For the most general version of Open Door Lemma, the reader should consult the monograph [6] by Miller and Mocanu.

The following result is contained in Theorems 3.2i and 3.4b of [6, p. 97, p. 124].

Lemma 2.1. *Let $h \in \mathcal{A}_0$ map $\mathbb{D}$ univalently onto a convex domain. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{q(z)} = h(z) \tag{2.2}$$

has an analytic solution q with $\operatorname{Re} q > 0$. Then q is univalent and $q \prec h$. If $p \in \mathcal{A}_0$ satisfies

$$p(z) + \frac{zp'(z)}{p(z)} \prec h(z), \tag{2.3}$$

then $p \prec q$ and q is the best dominant.

The lemma immediately yields the following corollary.

Corollary 2.2. *Let $h \in \mathcal{A}_0$ be a univalent function with convex image containing the parallel strip W_γ. If* (2.2) *has an analytic solution q with* $\operatorname{Re} q > 0$, *then $\gamma(\mathcal{S}^*) \geq \gamma$.*

It is, in general, not easy to analyse the solution to the differential equation (2.2) for a given h. Therefore, practically, we start from a function q with $\operatorname{Re} q > 0$ and look at the image of $\mathbb{D}$ under the function h defined by (2.2). If it is a convex domain containing W_γ, then the last corollary implies $\gamma(\mathcal{S}^*) \geq \gamma$. Therefore, to make a suitable choice of q, it is helpful to observe the boundary behaviour of the solution to the equation (2.2) for the targeted h.

For $q \in \mathcal{A}_0$ and $\theta \in \mathbb{R}$, we define $\beta(\theta) = \beta(\theta; q)$ (modulo 2π rigorously speaking) by

$$\beta(\theta) = \lim_{z \to e^{i\theta}} \left[\arg q'(z) + \theta + \frac{\pi}{2} \right]$$

if it exists. When q and q' extend continuously to $\{z \in \overline{\mathbb{D}} : |z - e^{i\theta}| < \delta\}$ for some $\delta > 0$ and $q'(e^{i\theta}) \neq 0$, one has

$$\lim_{t \to \theta} \arg \frac{d}{dt} q(e^{it}) = \lim_{t \to \theta} \left[\arg q'(e^{it}) + t + \frac{\pi}{2} \right] = \beta(\theta).$$

Thus, $\beta(\theta)$ means the argument of a tangent vector of the boundary curve $q(e^{it})$ at $t = \theta$. Even if the above limit does not exist, the following limits may exist:

$$\beta_\pm(\theta) = \lim_{t \to \theta\pm} \arg \frac{d}{dt} q(e^{it}) = \lim_{t \to \theta\pm} \left[\arg q'(e^{it}) + t + \frac{\pi}{2} \right]$$

for each signature. Assume that q maps $\mathbb{D}$ univalently onto a Jordan domain Ω. Ω is said to be *smooth* if β can be chosen as a continuous function on $\mathbb{R}$.

This means that Ω has continuously varying tangent at each boundary point. For further properties of $\beta(\theta)$, see [9, §3.2].

We summarise properties of the solutions to (2.2).

Proposition 2.3. *Let* $h \in \mathcal{A}_0$ *and take* $f \in \mathcal{A}_1$ *so that* (1.1) *is fulfilled. Then a unique meromorphic solution* q *to the differential equation* (2.2) *with* $q(0) \neq 0$ *is given by* $q(z) = zf'(z)/f(z)$. *If the solution* q *has a pole at* $z = z_0$, *then its order is* 1 *and its residue is* z_0.

Suppose that $h(z) = O(|z - \zeta|^{-\alpha})$ *as* z *tends to a boundary point* $\zeta = e^{i\theta} \in \partial\mathbb{D}$ *in* $\mathbb{D}$ *for a constant* $\alpha < 1$. *Then the limits* $\lim_{z\to\zeta} f'(z) =: f'(\zeta)$ *and* $\lim_{z\to\zeta} f(z) =: f(\zeta)$ *exist and* $f'(\zeta) \neq 0$. *If, in addition,* $f(\zeta) \neq 0$, *then* $\lim_{z\to\zeta} q(z) =: q(\zeta)$ *exists and* $q(\zeta) \neq 0$. *Moreover the following hold.*

(i) *Suppose that the finite limit* $\lim_{z\to\zeta} h(z) =: h(\zeta)$ *exists and* $f(\zeta) \neq 0$. *Then,* $q(z) = q(\zeta)[1 + \bar{\zeta}(h(\zeta) - q(\zeta) + o(1))(z - \zeta)]$ *as* $z \to \zeta$. *Moreover, if* $q(\zeta) \neq h(\zeta)$, *then*

$$\beta(\theta) = \arg q(\zeta) + \arg\,[h(\zeta) - q(\zeta)] + \frac{\pi}{2}.$$

(ii) *If* $h(z) = (A + o(1))(\zeta - z)^{-\alpha}$ *as* $z \to \zeta$ *in* $\mathbb{D}$ *for constants* $A \neq 0$ *and* $0 < \alpha < 1$, *then* $q(z) = q(\zeta)[1 - \frac{\bar{\zeta}A}{1-\alpha}(\zeta - z)^{1-\alpha}]$ *as* $z \to \zeta$.

$$\beta_\pm(\theta) = \arg q(\zeta) + \arg A - \alpha\theta + (1 \pm \alpha)\frac{\pi}{2}.$$

(iii) *If* $h(z) = -(A + o(1))\log(\zeta - z)$ *as* $z \to \zeta$ *in* $\mathbb{D}$ *for a constant* $A \neq 0$, *then* $q(z) = q(\zeta)[1 + (\bar{\zeta}A + o(1))(z - \zeta)\log(z - \zeta)]$ *and*

$$\beta(\theta) = \arg q(\zeta) + \arg A + \frac{\pi}{2}.$$

Remark. In the limit above, $z \to \zeta$ means that z approaches ζ in $\mathbb{D}$ without any restriction such as radial or non-tangential limits.

Proof. Observe first that any analytic function q with $q(0) \neq 0$ satisfying (2.2) on $\mathbb{D}$ must have the initial value $q(0) = 1$ because of $h(0) = 1$. We now show that (2.2) has a meromorphic solution q on $\mathbb{D}$. Since $\mathbb{D}$ is simply connected, an analytic function f on $\mathbb{D}$ can be defined uniquely by (1.1) and the condition $f(0) = 0$. Note that $f'(z) \neq 0$ for $z \in \mathbb{D}$. Therefore, $q(z) = zf'(z)/f(z)$ is a meromorphic solution to (2.2) on $\mathbb{D}$ with at most simple poles.

It is easy to show existence of the limits when $z \to \zeta$ in $\mathbb{D}$. (See the proof of (ii) to get basic ideas to do that.) Hence, we have shown the first assertion in the proposition.

We show now assertion (i). As for the formula of $\beta(\theta)$, one needs only to take the argument of both sides of the identity

$$zq'(z) = q(z)(h(z) - q(z)) \tag{2.4}$$

and put $z = \zeta = e^{i\theta}$.

We next show assertion (ii). Suppose that $h(z) = (A + o(1))(\zeta - z)^{-\alpha}$ as $z \to \zeta$ in $\mathbb{D}$ for $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$. (We will surpress the description "in $\mathbb{D}$" in the rest of the proof for brevity.) Since $0 < \alpha < 1$, we have

$$\frac{h(z)-1}{z} = \left(\frac{A}{\zeta} + o(1)\right)(\zeta - z)^{-\alpha}, \quad z \to \zeta,$$

and the limit

$$C = \lim_{z\to\zeta} \int_0^z \frac{h(t)-1}{t}dt = \int_0^\zeta \frac{h(t)-1}{t}dt$$

exists. Thus, in view of (1.1), one obtains

$$\begin{aligned} e^{-C} f'(z) &= \exp\left[\int_\zeta^z \frac{h(t)-1}{t}dt\right] \\ &= \exp\left[(-K + o(1))(\zeta - z)^{1-\alpha}\right] \\ &= 1 + (-K + o(1))(\zeta - z)^{1-\alpha} \end{aligned}$$

as $z \to \zeta$, where $K = \bar{\zeta}A/(1-\alpha)$. In particular, the limits $\lim_{z\to\zeta} f'(z) = e^C =: f'(\zeta)$, $\lim_{z\to\zeta} f(z) =: f(\zeta)$ exist and

$$f(z) = f(\zeta) + f'(\zeta)\left[(z-\zeta) + (\tfrac{K}{2-\alpha} + o(1))(\zeta - z)^{2-\alpha}\right], \quad z \to \zeta.$$

If $f(\zeta) \neq 0$, then $f(z)/f(\zeta) = 1 + q(\zeta)(z-\zeta)/\zeta + O(|z-\zeta|^{2-\alpha})$ so that

$$\frac{q(z)}{q(\zeta)} = 1 + (-K + o(1))(\zeta - z)^{1-\alpha} + \frac{1-q(\zeta)}{\zeta}(z-\zeta)$$

as $z \to \zeta$. Thus the first part of (ii) has been shown.

To show the relation for $\beta_\pm(\theta)$ in (ii), we note that $\arg\left[(\zeta-z)^{-1}\right] \to -\theta \pm \pi/2$ as $t \to \theta\pm$ for $\zeta = e^{i\theta}$. Since $|h(z)| \to +\infty$ as $z \to \zeta$ in this case, (2.4) yields

$$\begin{aligned} \beta_\pm(\theta) &= \arg q(\zeta) + \lim_{t\to\theta\pm} \arg\left[h(e^{it}) - q(e^{it})\right] + \frac{\pi}{2} \\ &= \arg q(\zeta) + \lim_{t\to\theta\pm} \arg h(e^{it}) + \frac{\pi}{2} \\ &= \arg q(\zeta) + \arg A + \alpha\left(-\theta \pm \frac{\pi}{2}\right) + \frac{\pi}{2}. \end{aligned}$$

Finally, we show assertion (iii). We can compute in the same way as in (ii) except for the integrals:

$$\int_\zeta^z \log(t-\zeta)dt = (z-\zeta)[\log(z-\zeta) - 1] = (1+o(1))(z-\zeta)\log(z-\zeta) = o(1)$$

and

$$\int_\zeta^z (t-\zeta)\log(t-\zeta)dt = \frac{(z-\zeta)^2}{2}[\log(z-\zeta) - 1/2] = O\left((z-\zeta)^2\log(z-\zeta)\right)$$

as $z \to \zeta$. Thus the conclusion follows. □

In the last proposition, the condition in (ii) means roughly that $z = \zeta$ corresponds via the function $h(z)$ to the tip at infinity of a sector with opening angle $\pi\alpha$. It should also be noted that assertion (iii) can be regarded as a limiting case of assertion (ii).

3. An extremal case

For $c > 0$, we define

$$h_c(z) = 1 + c \operatorname{arctanh} z = 1 + \frac{c}{2} \log \frac{1+z}{1-z} = 1 + c \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}, \quad z \in \mathbb{D},$$

and let q_c be the solution to the initial value problem of the ODE:

$$q_c(z) + \frac{z q_c'(z)}{q_c(z)} = h_c(z), \quad z \in \mathbb{D}, \quad q_c(0) = 1. \tag{3.5}$$

Note that h_c maps the unit disk $\mathbb{D}$ onto the parallel strip $W_{\pi c/4} = \{w : |\operatorname{Im} w| < \pi c/4\}$. Let $f_c \in \mathcal{A}_1$ be the solution to the equation $\varphi_{f_c} = h_c$. Namely, $f = f_c$ can be determined by (1.1) with $h = h_c$. Then, $q_c(z) = z f_c'(z)/f_c(z)$. We compute

$$\int_0^z \frac{h_c(t) - 1}{t} dt = c\chi_2(z),$$

where

$$\chi_2(z) = \frac{1}{2} \left[\operatorname{Li}_2(z) - \operatorname{Li}_2(-z) \right] = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^2}$$

is called Legendre's chi-function (see [3, §1.8]) and $\operatorname{Li}_2(z)$ is the dilogarithm function. Note that

$$\chi_2(1) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8} < +\infty.$$

Therefore, by (1.1), f_c is expressed by

$$f_c(z) = \int_0^z \exp\left[c\chi_2(t)\right] dt = z \int_0^1 \exp\left[c\chi_2(tz)\right] dt.$$

Hence,

$$\frac{1}{q_c(z)} = \frac{f_c(z)}{z f_c'(z)} = \int_0^1 \exp c\left[\chi_2(tz) - \chi_2(z)\right] dt. \tag{3.6}$$

We define two numbers c_1 and c_* as the largest possible ones with the properties

$$0 < c < c_1 \quad \Rightarrow \quad \operatorname{Re} q_c > 0 \text{ on } \mathbb{D},$$
$$0 < c < c_* \quad \Rightarrow \quad q_c \prec h_c.$$

By Corollary 1.1 and Lemma 2.1, we observe that the following inequalities hold:

$$\frac{4\sqrt{3}}{\pi} \leq c_1 \leq c_*.$$

It is also easy to show that $\operatorname{Re} q_{c_1} > 0$ and $q_{c_*} \prec h_{c_*}$. We are now able to show the following.

Theorem 3.1. *Let $0 < c \leq c_*$. Then the solution q_c to (3.5) is a non-vanishing analytic function on the unit disk $\mathbb{D}$ and satisfies the relation $\overline{q_c(z)} = q_c(\bar{z})$ and the inequalities*

$$0 < q_c(-1) < |q_c(z)| < q_c(1) < +\infty, \quad z \in \mathbb{D}.$$

If, in addition, $c \leq c_1$, then q_c is univalent on $\mathbb{D}$ and the image $q_c(\mathbb{D})$ is a smooth Jordan domain in the sense that its boundary has continuously varying tangent.

Proof. By Proposition 2.3, we first see that q_c extends meromorphically to $\overline{\mathbb{D}}\backslash\{1,-1\}$ and that $q_c(z)$ has finite limits $q_c(\pm 1)$ as $z \to \pm 1$ in $\mathbb{D}$. The symmetry property in the real axis is immediate from uniqueness of the initial value problem for ODE.

We now prove the inequalities in the assertion. In view of the expression (3.6), the reciprocal $1/q_c$ is analytic on $\mathbb{D}$. We now look at the function $p(x) = 1/q_c(x)$ for $-1 < x < 1$. Since

$$\chi_2'(x) - t\chi_2'(tx) = \sum_{n=0}^{\infty} \frac{1-t^{2n+1}}{2n+1} x^{2n} > 0$$

for $-1 < x < 1$ and $0 < t < 1$, one obtains $p'(x) < 0$, which implies that $q_c(x)$ is increasing in $-1 < x < 1$.

Let $u(\theta) = R(\theta)e^{i\Theta(\theta)} = q_c(e^{i\theta})$ with $\Theta(0) = 0$ for $0 \leq \theta \leq \pi$. Then, we deduce from (3.5) that

$$\begin{aligned} q_c(e^{i\theta}) + \frac{e^{i\theta} q_c'(e^{i\theta})}{q_c(e^{i\theta})} &= u(\theta) + \frac{u'(\theta)}{iu(\theta)} = R(\theta)e^{i\Theta(\theta)} + \frac{R'(\theta)}{iR(\theta)} + \Theta'(\theta) \\ &= \frac{c}{2} \log \frac{1+e^{i\theta}}{1-e^{i\theta}} + 1 = \frac{c}{2}\left(\log\cot\frac{\theta}{2} + \frac{\pi i}{2}\right) + 1 \end{aligned}$$

for $0 < \theta < \pi$. Taking the real and the imaginary parts of the above formula, we get

$$R(\theta)\cos\Theta(\theta) + \Theta'(\theta) = \frac{c}{2}\log\cot\frac{\theta}{2} + 1, \tag{3.7}$$

$$R(\theta)\sin\Theta(\theta) - \frac{R'(\theta)}{R(\theta)} = \frac{c\pi}{4}. \tag{3.8}$$

Since $q_c \prec h_c$, we note that $\operatorname{Im} q_c(e^{i\theta}) = R(\theta)\sin\Theta(\theta) \leq c\pi/4$. Hence, $R'(\theta) \leq 0$, which means that $R(\theta)$ is non-increasing in $0 < \theta < \pi$. In particular, $q_c(-1) = R(\pi) \leq R(\theta) \leq R(0) = q_c(1)$. The same is true for $-\pi < \theta < 0$

by the symmetry. The maximum modulus principle now implies the desired inequalities. In particular, we note that q_c is bounded on $\mathbb{D}$.

Now we assume that $0 < c \leq c_1$. Then Lemma 2.1 implies that q_c is univalent. By Proposition 2.3, we see that q_c meromorphically continues to $\overline{\mathbb{D}} \setminus \{1, -1\}$. Since q_c is bounded, there is no pole of q_c on $\overline{\mathbb{D}}$. Hence, q_c is analytic on $\overline{\mathbb{D}} \setminus \{1, -1\}$. We next show that $R(\theta)$ is strictly decreasing in $0 < \theta < \pi$. If not, since $R(\theta)$ is non-increasing, there is an interval $I = (a, b)$ with $0 < a < b < \pi$ such that $R(\theta)$ is constant, say R_0, on I. Then (3.8) yields that $R_0 \sin \Theta(\theta) = c\pi/4$ for $\theta \in I$, which implies that Θ is also constant on I. By the identity theorem, q_c must be constant, which is a contradiction. We have proved that $R(\theta)$ is strictly decreasing in $0 < \theta < \pi$. By symmetry, the same is true for $R(-\theta)$ with $0 < \theta < \pi$.

We next show that $q_c'(e^{i\theta}) \neq 0$ for $0 < \theta < \pi$. If not, $q_c'(\zeta_0) = 0$ for some $\zeta_0 = e^{i\theta_0}$, $0 < \theta_0 < \pi$, which leads to $q(\zeta_0) = h(\zeta_0)$ by (3.5). Since $q_c(z) = q_c(\zeta_0) + \psi(z)^k$ near $z = \zeta_0$ for an analytic function $\psi(z)$ with $\psi(\zeta_0) = 0$, $\psi'(\zeta_0) \neq 0$ and an integer $k \geq 2$, we can see that $q_c(\mathbb{D})$ covers a small sector with a tip at $q_c(\zeta_0)$ and opening angle is nearly $k\pi \geq 2\pi$. This, however, contradicts the fact that $q_c(\mathbb{D})$ is contained in $h_c(\mathbb{D})$ whose boundary contains a line passing through $h_c(\zeta_0) = q_c(\zeta_0)$. By symmetry, we now see that $q_c'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus \{1, -1\}$.

Since q_c maps the real interval $(-1, 1)$ into the positive real axis, the upper (lower) half-disk is mapped into the upper (lower) half-plane by q_c. Therefore, we conclude that $q_c(\mathbb{D})$ is a Jordan domain and $q_c(\partial\mathbb{D} \setminus \{1, -1\})$ is real analytically smooth. By Proposition 2.3(iii), the curve $q_c(\partial\mathbb{D})$ has continuously varying tangent at $\zeta = \pm 1$ as well. Thus we get the last conclusion. □

We define c_α for $0 < \alpha \leq 1$ as the largest possible number so that

$$0 < c < c_\alpha \quad \Rightarrow \quad |\arg q_c| < \frac{\pi\alpha}{2} \text{ on } \mathbb{D}.$$

Obviously, when $\alpha = 1$ this number agrees with c_1 defined before Theorem 3.1. The following result reduces the computation of $\gamma(\mathcal{S}_\alpha^*)$ to the investigation of mapping properties of the function q_c.

Lemma 3.2. *For $0 < \alpha \leq 1$, the relation $\gamma(\mathcal{S}_\alpha^*) = \pi c_\alpha/4$ holds.*

Proof. Let $\gamma = \gamma(\mathcal{S}_\alpha^*)$. Then $|\operatorname{Im}[zf_c''/f_c']| = |\operatorname{Im} h_c| < \pi c/4 \leq \gamma$ for $c \leq 4\gamma/\pi$. By the definition of the number $\gamma(\mathcal{S}_\alpha^*)$, we obtain $f_c \in \mathcal{S}_\alpha^*$, which means that $|\arg q_c| < \pi\alpha/2$. Therefore, we have $c_\alpha \geq 4\gamma(\mathcal{S}_\alpha^*)/\pi$.

Next assume that $|\operatorname{Im}[zf''/f']| < \pi c_\alpha/4$. Then $\varphi_f = 1 + zf''/f' \prec h_{c_\alpha}$. We note that $c_\alpha \leq c_1 \leq c_*$. By Theorem 3.1 together with Lemma 2.1, we see that $q_f = zf'/f \prec q_{c_\alpha}$. Since $|\arg q_{c_\alpha}| < \pi\alpha/2$, we have $f \in \mathcal{S}_\alpha^*$. Hence, $\gamma(\mathcal{S}_\alpha^*) \geq \pi c_\alpha/4$. We now conclude that $\gamma(\mathcal{S}_\alpha^*) = \pi c_\alpha/4$. □

4. Proof of main results

In order to obtain upper bounds for c_α, we use the Carathéodory-Toeplitz theorem (see, for instance, [11, Theorem IV.22]).

Lemma 4.1. *Let $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ be a formal power series. Then, p represents an analytic function on $\mathbb{D}$ with $\operatorname{Re} p > 0$ if and only if*

$$\Delta_n(p) := \begin{vmatrix} 2 & b_1 & b_2 & \cdots & b_n \\ \overline{b_1} & 2 & b_1 & \cdots & b_{n-1} \\ \overline{b_2} & \overline{b_1} & 2 & \cdots & b_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{b_n} & \overline{b_{n-1}} & \overline{b_{n-2}} & \cdots & 2 \end{vmatrix} \geq 0$$

for all $n \geq 1$.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Expand the function $q_c(z)$ in the form

$$q_c(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = 1 + b_1 z + b_2 z^2 + \cdots .$$

Then, by comparing the series expansion of the both sides of (3.5) (or, alternatively, via the formula (1.1), by using the relation $q_c(z) = z f_c'(z)/f_c(z)$), we obtain

$$q_c(z) = 1 + \frac{c}{2} z + \frac{c^2}{12} z^2 + \frac{c}{12} z^3 + \frac{c^2(24 - c^2)}{720} z^4 + \frac{c(5c^2 + 72)}{2160} z^5 + \frac{c^2(c^4 - 24c^2 + 522)}{30240} z^6 + \frac{c(1620 + 189c^2 - 7c^4)}{90720} z^7 + \cdots .$$

By using computer algebra, we get

$$\Delta_6(q_c) = 2^{-21} \cdot 3^{-14} \cdot 5^{-6} \cdot 7^{-2} \cdot \big(9c^{12} - 3168c^{10} + 117032c^8 - 5676096c^6 + 456371280c^4 - 10334615040c^2 + 62705664000\big) \cdot \big(9c^{12} - 2328c^{10} + 209872c^8 - 9890976c^6 + 266580720c^4 - 3412575360c^2 + 15676416000\big)$$

and find that its minimal positive root ρ_6 is approximately 3.1735. Hence, by Lemma 3.2, $\gamma(\mathcal{S}^*) = \pi c_1/4 < \pi \rho_6/4 < 2.4925 < 2.5$.

The first inequality in Theorem 1.3 follows from Corollary 1.1. □

We may increase the number $n = 6$ when applying Lemma 4.1 to obtain a better upper bound. See the last section for such attempts.

Proof of Theorem 1.2. Let $0 < \alpha < 1$. For the function $q(z) = [(1+z)/(1-z)]^\alpha$, Mocanu [7] considered the corresponding open door function $h(z) = q(z) + zq'(z)/q(z)$. The authors showed in [4, Lemma 3.3] that the image $h(\mathbb{D})$ contains a parallel strip of the form W_γ for some $\gamma > g(\alpha)$, where

$$g(\alpha) = \frac{1}{2}\left[(1+\alpha)\sqrt{1 + 2\sin(\pi\alpha/2)} - \frac{1-\alpha}{\sqrt{1 + 2\sin(\pi\alpha/2)}}\right]$$

$$= \frac{\alpha + (1+\alpha)\sin(\pi\alpha/2)}{\sqrt{1+2\sin(\pi\alpha/2)}}.$$

The Mocanu theorem [7, Theorem 2] (see also [4]), which is a version of the Open Door Lemma, implies that $\gamma(\mathcal{S}^*_\alpha) \geq \gamma > g(\alpha)$.

We next show that $g(\alpha) > \sqrt{3}\alpha$. Since

$$\begin{aligned} g''(\alpha) &= \pi \frac{8\cos(\alpha\pi/2)\sin(\alpha\pi/2) - (1+\alpha)\pi\sin^2(\alpha\pi/2) - 2\pi\sin(\alpha\pi/2) - (1+\alpha)\pi}{8(1+\sin(\alpha\pi/2))^{3/2}} \\ &\quad - (1-\alpha)\frac{3\pi^2\cos^2(\alpha\pi/2)}{8(1+2\sin(\alpha\pi/2))^{5/2}} \\ &\leq \frac{\pi(8\cos(\alpha\pi/2) - 3\pi)\sin(\alpha\pi/2)}{8(1+\sin(\alpha\pi/2))^{3/2}} - (1-\alpha)\frac{3\pi^2\cos^2(\alpha\pi/2)}{8(1+2\sin(\alpha\pi/2))^{5/2}}, \end{aligned}$$

it is easy to see that $g''(\alpha) < 0$ for $0 < \alpha < 1$, in other words, $g(x)$ is strictly concave. Hence, we have the inequality $g(\alpha) > g(0) + (g(1) - g(0))\alpha = \sqrt{3}\alpha$ for $0 < \alpha < 1$.

Finally, we consider the upper estimate. Let $c \leq c_\alpha$. Then

$$p(z) = [q_c(z)]^{1/\alpha} = 1 + \frac{c}{2\alpha}z + \frac{(3-\alpha)c^2}{24\alpha^2}z^2 + \frac{4\alpha^2 c + (1-\alpha)c^3}{48a^3}z^3 + \cdots$$

has positive real part on $\mathbb{D}$. By Lemma 4.1, $\Delta_2(p) \geq 0$ is necessary for $p \in \mathcal{P}$. A straightforward computation yields

$$\begin{aligned} \Delta_2(p) &= \frac{(9-\alpha^2)c^4 - 288\alpha^2c^2 + 2304\alpha^4}{288\alpha^4} \\ &= \frac{\{(3+\alpha)c^2 - 48\alpha^2\}\{(3-\alpha)c^2 - 48\alpha^2\}}{288\alpha^4}. \end{aligned}$$

By solving the inequality $\Delta_2(p) \geq 0$, we obtain $c_\alpha \leq 4\sqrt{3}\alpha/\sqrt{3+\alpha}$. Now Lemma 3.2 gives the desired upper bound. □

Proof of Theorem 1.4. Recall that c_1 is the largest possible number such that $\operatorname{Re} q_c > 0$ on $\mathbb{D}$ for $0 < c < c_1$. Theorem 3.1 tells us that q_{c_1} is a bounded univalent function on $\mathbb{D}$ and that the boundary of $D = q_{c_1}(\mathbb{D})$ does not touch the origin. Then the argument function $\Theta(\theta)$ defined in the proof of Theorem 3.1 with $c = c_1$ satisfies that $\Theta(0) = \Theta(\pi) = 0$ and $0 < \Theta(\theta) \leq \pi/2$ for $0 < \theta < \pi$. By maximality, $\Theta(\theta_0) = \pi/2$ for some $0 < \theta_0 < \pi$. Since $\pi/2$ is the possible largest value of $\Theta(\theta)$, we have $\Theta'(\theta_0) = 0$. Then, we substitute these into (3.7) to get $(c_1/2)\log\cot(\theta_0/2) + 1 = 0$, equivalently, $\theta_0 = 2\arctan(e^{2/c_1}) = \theta_{c_1}$. Hence, $\operatorname{Re} F(c_1) = \operatorname{Re} q_{c_1}(e^{i\theta_0}) = \operatorname{Re}[R(\theta_0)e^{i\Theta(\theta_0)}] = 0$. For $0 < c < c_1$, we have $\operatorname{Re} F(c) = \operatorname{Re} q_c(e^{i\theta_c}) > 0$. Thus we conclude that c_1 is the smallest positive number such that $\operatorname{Re} F(c_1) = 0$, that is to say, $c_1 = c_0$. □

5. Numerical experiments

By using Mathematica Ver. 10, we can evaluate the right-hand side of (3.6). In this way, we can compute the values of $q_c(z)$ numerically. In Figure 2, we exhibit the graph of the function $c \mapsto \operatorname{Re} F(c)$, where $F(c)$ is given in Theorem 1.4. Numerical experiments give us $\operatorname{Re} F(3.02756) \approx 1.06 \times 10^{-6}$ and $\operatorname{Re} F(3.02757) \approx -2.80 \times 10^{-6}$. Thus, if the numerical computations were correct, we would have $c_0 = c_1 \approx 3.0276$. The image of $\mathbb{D}$ under the mapping q_{c_1} is generated in this way (see Figure 1).

In the same way, based on Lemma 3.2, we can draw a graph of the function $\alpha \mapsto \gamma(\mathcal{S}^*_\alpha)$ together with the upper bound $\sqrt{3}\pi\alpha/\sqrt{3+\alpha}$ and the lower bound $g(\alpha)$ given in Theorem 1.2, see Figure 3. The image looks to have a corner at $q_{c_1}(\pm 1)$. However, if we magnify the neighbourhood of these points large enough, it should look smooth according to Theorem 3.1.

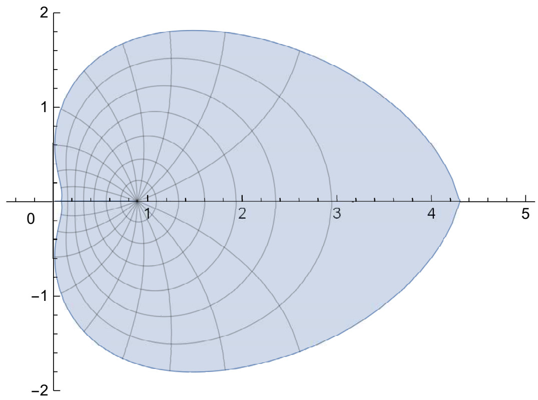

FIGURE 1. Conformal mapping of $\mathbb{D}$ under q_{c_1}.

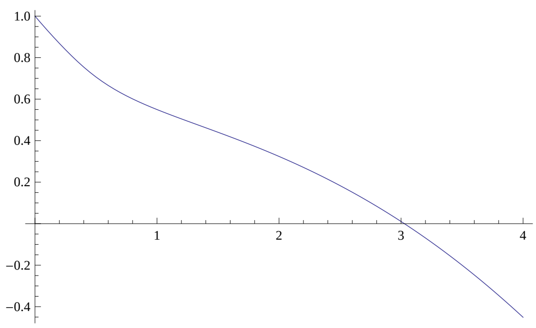

FIGURE 2. The graph of $\operatorname{Re} F(c)$.

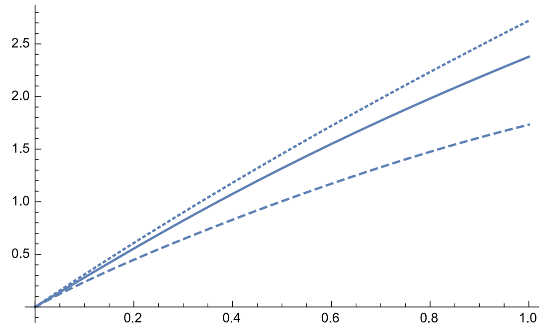

FIGURE 3. The graphs of $\gamma(\mathcal{S}^*_\alpha)$ (solid line) $g(\alpha)$ (dashed line) and $\sqrt{3}\pi\alpha/\sqrt{3+\alpha}$ (dotted line).

In the previous section, we obtained upper bounds for c_1 based on the Carathéodory-Teoplitz theorem. With the help of computer algebra, we can go further. We can compute $\Delta_n(q_c)$ exactly as a polynomial in c with rational coefficients for a small enough n and find numerically the smallest positive root ρ_n of the polynomial $\Delta_n(q_c)$ in c. In this way, Table 1 can be made with the aid of Mathematica. Thus the upper bound 2.5 in Theorem 1.3 can be reduced to some extent. Some results are depicted in Table 1. We see that ρ_{30} is close enough to the expected value $c_1 \approx 3.02756$.

TABLE 1. Approximated values of ρ_n, $\gamma_n = \pi\rho_n/4$ and $\Delta\gamma_n = \gamma_n - \gamma_{n-1}$.

n	ρ_n	γ_n	$-\Delta\gamma_n$
1	4.00000000	3.14159265	
2	3.46410162	2.72069905	0.42089400
3	3.36499696	2.64286243	0.07783660
4	3.33586037	2.61997861	0.02288382
5	3.21295295	2.52344735	0.09653126
6	3.17351296	2.49247125	0.03097610
7	3.17032183	2.48996494	0.00250631
8	3.13275982	2.46046381	0.02950113
9	3.11076636	2.44319018	0.01727363
10	3.10609706	2.43952292	0.00366726
15	3.06686241	2.40870810	0.00907899
20	3.04388463	2.39066140	0.00107182
30	3.04026630	2.38781957	0.00014363

References

[1] P. L. Duren, *Univalent Functions*, Springer-Verlag, 1983.
[2] Y. C. Kim, S. Ponnusamy, and T. Sugawa, *Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives*, J. Math. Anal. Appl. **299** (2004), no. 2, 433–447.
[3] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland Publishing Co., New York-Amsterdam, 1981.
[4] M. Li and T. Sugawa, *Some extentions of the Open Door Lemma*, Studia Univ. Babeş-Bolyai Math. **60** (2015), 421–430.
[5] S. S. Miller and P. T. Mocanu, *Briot-Bouquet differential equations and differential subordinations*, Complex Variables Theory Appl. **33** (1997), no. 1-4, 217–237.
[6] ———, *Differential Subordinations*: *Theory and Applications*, Marcel Dekker, Inc., New York, 2000.
[7] P. T. Mocanu, *On strongly-starlike and strongly-convex functions*, Studia Univ. Babeş-Bolyai Math. **31** (1986), no. 4, 16–21.
[8] ———, *Some integral operators and starlike functions*, Rev. Roumaine Math. Pures Appl. **31** (1986), no. 3, 231–235.
[9] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, 1992.
[10] T. Sugawa, *A self-duality of strong starlikeness*, Kodai Math J. **28** (2005), no. 2, 382–389.
[11] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

Ming Li
Graduate School of Information Sciences
Tohoku University
Aoba-ku, Sendai 980-8579, Japan
E-mail address: li@ims.is.tohoku.ac.jp

Toshiyuki Sugawa
Graduate School of Information Sciences
Tohoku University
Aoba-ku, Sendai 980-8579, Japan
E-mail address: sugawa@math.is.tohoku.ac.jp