J. Korean Math. Soc. **54** (2017), No. 1, pp. 249–265
https://doi.org/10.4134/JKMS.j150715
pISSN: 0304-9914 / eISSN: 2234-3008

HOPF'S BOUNDARY TYPE BEHAVIOR FOR AN INTERFACE PROBLEM

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Abstract. Interface problem here refers to a second order elliptic problem with a discontinuous coefficient for the second order derivatives. For the corresponding boundary value problem, the maximum principle still holds but Hopf's boundary point lemma may fail. We will give an optimal power type estimate that replaces Hopf's lemma at those boundary points, where this coefficient jumps.

1. Introduction

Before coming to the interface problem it will be beneficial to recall the maximum principle type results in the classical case. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$$

be a second order elliptic operator, i.e., $\sum_{i,j} a_{ij}\xi_i\xi_j \geq c\,|\xi|^2$ for some $c > 0$ and all $\xi \in \mathbb{R}^n$, and with a_{ij}, b_i sufficiently nice. If u is a twice differentiable solution of the boundary value problem

$$\left\{ \begin{array}{ll} Lu \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

then either $u \equiv 0$ or u is strictly positive in the interior of Ω. For $x_0 \in \partial\Omega$ with $u(x_0) = 0$ and when an interior sphere condition is present, the closely related boundary point lemma by Hopf [9] states that either $u \equiv 0$ or u satisfies

$$-\frac{\partial u}{\partial \nu}(x_0) > 0.$$

Here ν is the outward normal at x_0. Hopf's lemma holds at those boundary points where all coefficients a_{ij} and b_i are continuous, see [5, Section 2.3].

The question that comes up, is, what happens if the coefficients a_{ij} and b_i are not continuous. The maximum principle still holds true if these coefficients

Received November 26, 2015; Revised August 22, 2016.
2010 *Mathematics Subject Classification.* 35J15, 35B51.
Key words and phrases. Hopf's lemma, interface, optimal estimate.

are just bounded (see [6, Theorem 8.1]), but that is not sufficient for Hopf's lemma as we will show.

A boundary value problem such as (1) with discontinuous coefficients appears when studying a so-called interface or transmission problem. See for example [12]. Such a problem is modeled formally by

$$\begin{cases} -\nabla \cdot \sigma \nabla u = \sigma f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where σ is piecewise constant with jumps. Obviously, the solution of (2) can't be considered in the classical sense and instead one considers weak solutions, that is, functions $u \in \mathring{W}^{1,2}(\Omega)$ satisfying

$$\int_\Omega \sigma \left(\nabla u \cdot \nabla \varphi - f\varphi \right) dx = 0 \text{ for all } \varphi \in \mathring{W}^{1,2}(\Omega) \tag{3}$$

with $\mathring{W}^{1,2}(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Notice that (3) is the Euler-Lagrange equation for

$$J(u) = \int_\Omega \sigma \left(\tfrac{1}{2} |\nabla u|^2 - fu \right) dx. \tag{4}$$

The reason, that we put σ not only just for the gradient term, but also for f, is that it simplifies some notations and doesn't alter the problem for $f \in L^2(\Omega)$.

We will study what remains of Hopf's boundary point lemma for the solution of (2) at the boundary points in the case that σ is not continuous but a piecewise constant function with a discontinuity at such a boundary point. We assume that σ is constant on subdomains Ω_i with relatively nice boundaries. See Fig. 1. The precise condition follows.

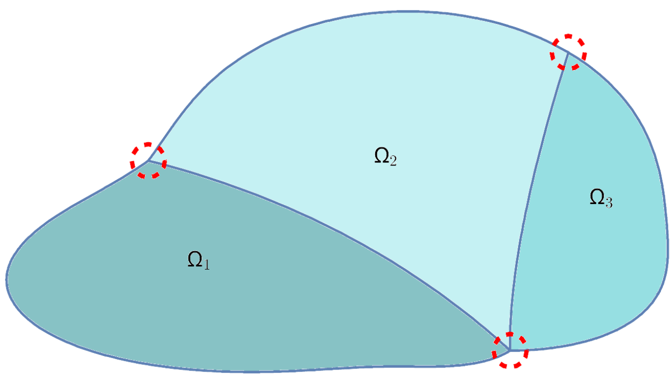


FIGURE 1. *A domain* Ω with *three subdomains and three singular points.*

2. The setting

We consider domains $\Omega \subset \mathbb{R}^2$ that consist of subdomains Ω_i with $i \in \{1, \ldots, k\}$, i.e., $\overline{\Omega} = \bigcup_{i=1}^k \overline{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$. These subdomains, and hence also Ω, have a smooth boundary with the possible exception of finitely many corners. As usual a domain means an open and connected set. The weight function $\sigma : \overline{\Omega} \to \mathbb{R}^+$ is a piecewise constant positive function defined by

$$\sigma(x) = \sigma_i \text{ for } x \in \Omega_i \text{ with } \sigma_i \in \mathbb{R}^+. \tag{5}$$

The existence and uniqueness of the weak solution $u \in \mathring{W}^{1,2}(\Omega)$ satisfying (3) is guaranteed by the Riesz representation theorem. Regularity questions near $\partial\Omega_i \cap \partial\Omega_j$ but away from the boundary $\partial\Omega$ have already been stated in [17]. Assuming that the subdomains meet at $\partial\Omega$ in cone-like way, Nicaise and Sändig [16] could show that $u_i := u_{|\Omega_i}$ can be written as $u_i = \tilde{u}_i + h_i$, where $\tilde{u}_i \in W^{2,2}(\Omega_i)$ and h_i is harmonic on Ω_i. Moreover, if one considers $p_0 \in \partial\Omega \cap \partial\Omega_i \cap \partial\Omega_j$ for some $i \neq j$, that is, a boundary point where at least two subdomains meet, then, although the solution u has a non-smooth behaviour in a neighborhood of p_0, this behaviour is similar as for corners studied by Kondratiev [10]. Indeed, in [16] one finds that for $f \in L^2(\Omega)$ the solution u has the following decomposition near such $p_0 = 0$:

$$u(x) = \tilde{u}(x) + \eta(|x|) \sum_{0<\mu_j<1} c_j |x|^{\sqrt{\mu_j}} \phi_j\left(\tfrac{x}{|x|}\right). \tag{6}$$

Here $\tilde{u}|_{\Omega_i} \in W^{2,2}(\Omega_i)$, η is an appropriate radially symmetric smooth cut-off function equal to 1 in a neighborhood of $p_0 = 0$, the c_j are real constants and (μ_j, ϕ_j) are eigenvalues/eigenfunctions of a weighted Laplace Beltrami operator, with the weight depending on σ, under homogeneous Dirichlet boundary conditions on the red circles in Figure 1 scaled to unity, that is, on

$$\tfrac{1}{\rho}\Omega \cap \partial B_1(0) := \left\{y \in \mathbb{R}^2; |y| = 1 \text{ and } \rho y \in \Omega\right\} \tag{7}$$

for some $\rho > 0$. Indeed, $x \mapsto |x|^{\sqrt{\mu_j}} \phi_j\left(\frac{x}{|x|}\right)$ are singular functions independent of f, which are harmonic on $\Omega_i \cap B_\rho(0)$ for all $i \in \{1, \ldots, k\}$. For polygonal interface problems see also [15].

For σ as in (5) the problem of finding a minimizer $u \in \mathring{W}^{1,2}(\Omega)$ for the energy functional (4) with given $f \in L^2(\Omega)$ leads to the following set of equations:

$$\left\{\begin{array}{cl} -\Delta u_i = f & \text{in } \Omega_i, \\ \left.\begin{array}{c} u_i = u_j \\ \sigma_i \frac{\partial u_i}{\partial \nu_i} = -\sigma_j \frac{\partial u_j}{\partial \nu_j} \end{array}\right\} & \text{on } \partial\Omega_i \cap \partial\Omega_j, \\ u_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega, \\ \text{for } i, j \in \{1, \ldots, k\} & \text{with } i \neq j, \end{array}\right. \tag{8}$$

where $u_i = u|_{\Omega_i}$ and ν_i is the outward normal with respect to Ω_i. We refer to Appendix C in order to see that $\tilde{u}$ from (6) satisfies (8). The power type part in (6) satisfies the conditions on $\partial\Omega_i \cap \partial\Omega_j$ in (8) by construction.

We will restrict ourselves mainly to the 2-dimensional case. Regularity for the 2-dimensional case was also focused upon by Mercier in [14]. Moreover, the problem was studied in [3], but it seems that this paper did not consider the appropriate power type functions in the decomposition as in (6).

When considering two-dimensional domains with multiple subdomains and such that $\partial\Omega_i$ and $\partial\Omega_j$ meet at $p_0 \in \partial\Omega$, it seems quite natural to assume that near such a point p_0 the subdomains look like sectors. Since this simplifies the arguments we will indeed make such an assumption, that is, after translation and rotation, we assume the subdomains to be as follows.

Condition 1. *Let $0 = \theta_0 < \theta_1 < \cdots < \theta_k < 2\pi$. The domain $\Omega \subset \mathbb{R}^2$ is such that for some $\rho > 0$ (and $\rho < 1$ for technical reasons)*

$$\left(\tfrac{1}{2\rho}\Omega\right) \cap B_1(0) = C := \{(r,\theta); 0 < r < 1,\ 0 < \theta < \theta_k\}, \tag{9}$$

with the subdomains Ω_i, $i = 1, 2, \ldots, k$ of Ω such that

$$\left(\tfrac{1}{2\rho}\Omega_i\right) \cap B_1(0) = C_i := \{(r,\theta); 0 < r < 1,\ \theta_{i-1} < \theta < \theta_i\}. \tag{10}$$

We write

$$\Gamma_i = \{(r,\theta); 0 < r < 1,\ \theta = \theta_i\}. \tag{11}$$

Like in (7) we set $\frac{1}{2\rho}\Omega_{(i)} := \left\{y \in \mathbb{R}^2; 2\rho y \in \Omega_{(i)}\right\}$. Condition 1 is illustrated in Fig. 2.

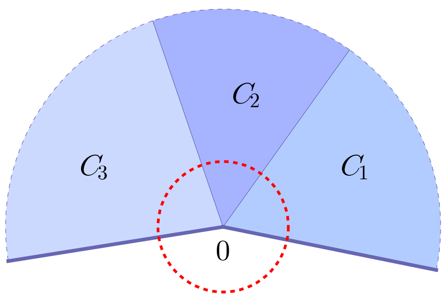


FIGURE 2. *The domain Ω and its subdomains are shaped like cones near the vertex $p_0 = 0$.*

Remark 1.1. A domain Ω will in general have several points where interfaces meet at the boundary and we will call these $\{p_0 = 0, p_1, \ldots, p_\ell\}$. Since our result is mainly based on a local analysis, it is sufficient to consider only the behaviour near $p_0 = 0$. The remaining p_i with $i \in \{1, \ldots, \ell\}$ may even lie in the interior of Ω.

Assuming that interfaces meet at $p_0 = 0 \in \partial\Omega$, a rescaling of the problem in (8) leads to the following boundary value on a sector C as in (9):

$$\left\{\begin{array}{cl} -\Delta u_i = f_i := f|_{C_i} & \text{in } C_i, \quad i = 1, \ldots, k, \\ u_1 = 0 & \text{on } \Gamma_0, \\ \left.\begin{array}{c} u_i = u_{i+1} \\ \sigma_i \frac{\partial u_i}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{i+1}}{\partial \theta} \end{array}\right\} & \text{on } \Gamma_i, i = 1, \ldots, k-1, \\ u_k = 0 & \text{on } \Gamma_k, \\ u_i = w & \text{on } \partial C \cap \partial B_1(0), \end{array}\right. \tag{12}$$

where w is some given nonnegative function. The fourth line in (12) displays the jump conditions.

The problem in (12) is closely related to the study of elliptic equations near corners as can be found in [7], [8], [10], [11], [13]. In [2, Theorem 6] a Hopf's type estimate near a corner for the solution of a Poisson problem can be found. The present proof follows similar steps but since additional technicalities appear, we will give the details.

3. Main result

For the sake of simple statements we will use the following notation.

Notation 2. Let $u, v : A \mapsto \mathbb{R}^+$ be two positive functions. We write '$v(x) \preceq u(x)$ *for* $x \in A$', if there exists a constant $c > 0$ such that $v(x) \leq cu(x)$ for all $x \in A$. If $v(x) \preceq u(x)$ and $u(x) \preceq v(x)$ for $x \in A$, we write '$v(x) \simeq u(x)$ *for* $x \in A$'.

Moreover, we will use the function $d : \Omega \to \mathbb{R}^+$ that denotes the distance to the boundary:

$$d(x) = d(x, \partial\Omega) := \inf\{|x - x^*|; x^* \in \partial\Omega\}.$$

Assuming Condition 1 and defining $\tilde{\sigma}(\theta) = \sigma(\rho\theta)$, a crucial role will be played by

$$\mu_1 = \inf_{\phi \in \mathring{W}^{1,2}(0,\theta_k)} \frac{\int_0^{\theta_k} \tilde{\sigma}(\theta) \ \phi'(\theta)^2 \ d\theta}{\int_0^{\theta_k} \tilde{\sigma}(\theta) \ \phi(\theta)^2 \ d\theta}. \tag{13}$$

The number μ_1 is the first eigenvalue of a weighted Laplace-Beltrami operator on $\partial C \cap \partial B_1(0)$ under Dirichlet boundary conditions and is strictly positive. See Appendix A.

Theorem 3. *Suppose that* $\Omega \subseteq \mathbb{R}^2$ *is as in Condition 1 and take* C *and* C_i *from there. Assume that* $u \in \mathring{W}^{1,2}(\Omega)$ *satisfies the boundary value problem* (8) *and* $0 \lneqq f \in W^{-1,2}(\Omega) \cap C(\bar{\Omega} \setminus \{0\})$. *Let* μ_1 *be as in* (13). *Then the following results hold.*

a) *For all* $x \in \Omega \cap B_\rho(0)$ *one finds*

$$|x|^{\sqrt{\mu_1} - 1} d(x) \preceq u(x). \tag{14}$$

b) *Moreover, let* $m > -2$ *and suppose that*

$$f(x) \preceq |x|^m \text{ for } x \in \Omega, \tag{15}$$

and for $\Omega' = \{(r\cos\theta, r\sin\theta); 0 < r < r_0, \theta_a < \theta < \theta_b\} \subset \Omega$, *with some* $r_0 > 0$ *and* $0 \leq \theta_a < \theta_b \leq \theta_k$,

$$|x|^m \preceq f(x) \text{ for } x \in \Omega'. \tag{16}$$

Then we find:

(1) *if* $m + 2 < \sqrt{\mu_1}$, *then*

$$u(x) \simeq |x|^{m+1} d(x) \text{ for } x \in \Omega \cap B_\rho(0), \tag{17}$$

(2) *if* $m + 2 = \sqrt{\mu_1}$, *then*

$$u(x) \simeq |x|^{\sqrt{\mu_1}-1} \ln\left(\tfrac{1}{|x|}\right) d(x) \text{ for } x \in \Omega \cap B_\rho(0). \tag{18}$$

(3) *if* $m + 2 > \sqrt{\mu_1}$, *then*

$$u(x) \simeq |x|^{\sqrt{\mu_1}-1} d(x) \text{ for } x \in \Omega \cap B_\rho(0). \tag{19}$$

Remark 3.1. The items (17)–(19) contain both estimates from below and from above. In fact these estimates are independent and only combined in one equivalence relation in order to show the sharpness of the estimate. From the proof one might see, that (15) yields the estimates from above and (16) the ones from below.

Remark 3.2. The functions $x \mapsto |x|^s$, $x \mapsto |x|^{s-1} d(x)$ and $x \mapsto |x|^{s-1} \ln\left(\frac{1}{|x|}\right) d(x)$ lie in $W^{1,2}(\Omega)$ for $s > 0$. Since $\mu_1 > 0$ and $m + 2 > 0$, the right hand sides in (17-19) indeed are in $W^{1,2}(\Omega)$.

Remark 3.3. If Ω consists near 0 of just two subdomains Ω_1 and Ω_2 such that after a rotation we find that

$$\Omega_1 \cap B_\rho(0) = \{(x_1, x_2); x_1 > 0 \text{ and } x_2 > 0\} \cap B_\rho(0),$$
$$\Omega_2 \cap B_\rho(0) = \{(x_1, x_2); x_1 < 0 \text{ and } x_2 > 0\} \cap B_\rho(0),$$

i.e., $\partial\Omega$ is straight with Γ_1 perpendicular, then $\mu_1 = 1$ and (14) gives us the classical Hopf lemma even if σ_1 and σ_2 are different. For any other angle there is in general no linear growth near the boundary point. See Example 1 in Appendix D.

Proof. First let us remark that a maximum principle like Theorem 5 (see Appendix B) implies that $u \geq 0$ on Ω. Since $u_i \in W^{2,p}\big(\Omega_i \setminus \bigcup_{j=0}^{\ell} B_\varepsilon(p_j)\big)$ for all $p < \infty$, these u_i are C^1 away from the p_j's. The strong maximum principle implies that on each Ω_i one either has $u_i \equiv 0$ or $u_i > 0$. The jump condition for u at interior layer points shows

$$\sigma_i \frac{\partial}{\partial \nu_i} u_i + \sigma_j \frac{\partial}{\partial \nu_j} u_j = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j \cap \Omega, \tag{20}$$

which means that for $\tilde{x} \in \partial\Omega_i \cap \partial\Omega_j \cap \Omega$ we find $\frac{\partial}{\partial\nu_i} u_i(\tilde{x}) \geq 0$ or $\frac{\partial}{\partial\nu_j} u_j(\tilde{x}) \geq 0$. If $\frac{\partial}{\partial\nu_i} u_i(\tilde{x}) \geq 0$ and $u_i(\tilde{x}) = 0$ holds for some $\tilde{x} \in \partial\Omega_i \cap \partial\Omega_j \cap \Omega$, then one obtains by Hopf's boundary point lemma that $u_i \equiv 0$ on Ω_i. Moving from Ω_i to a neighbouring Ω_j the condition in (20) implies that $u_j \equiv 0$ on Ω_j and hence by continuation $u \equiv 0$ on Ω, a contradiction. So we obtain $u > 0$ in Ω.

With the classical Hopf's boundary point lemma at $x \in \partial\Omega \setminus \{p_0, \ldots, p_\ell\}$ we find that for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$u(x) \geq c_\varepsilon d(x) \text{ for } x \in \Omega \setminus \bigcup_{j=0}^{\ell} B_\varepsilon(p_j).$$

By regularity results we find the reverse inequality on $\Omega \setminus \bigcup_{j=0}^{\ell} B_\varepsilon(p_j)$ for each $\varepsilon > 0$. Note that the constants in the estimate do depend on $\varepsilon > 0$ and might blow up when taking $\varepsilon \downarrow 0$.

We are left with proving the estimates near p_j and to do so we restrict ourselves, as stated in the theorem, to the neighborhood of the singular point at 0, where after a scaling the problem appears as in (12) and where $w(x)$ on $\partial C \cap \partial B_1(0)$ is a function equivalent the tangential distance along $\partial B_1(0)$ to $\rho^{-1}\partial\Omega$.

In a similar way as in [2], we construct upper and lower barrier functions for the solution of (12) with the right hand side $f \simeq |x|^m$. The maximum principle is used to show that the specially tailored barrier functions will give the estimates. The maximum principle that we use, is for functions as in (6). Such functions are sufficiently regular to have a well-defined trace on $\partial\Omega$ and $\partial\Omega_i$. Since $\tilde{u}|_{C_i} \in W^{2,2}(C_i)$ holds, the power type solutions are $C^1_{\text{piecewise}}$ as a function of θ. Hence one may integrate by parts and use a maximum principle as in Theorem 5 (see Appendix B).

Let $\phi_{1,\sigma}$ be the function in Lemma 4 (see Appendix A) normalised by

$$\max\{\phi_{1,\sigma}(\theta); 0 < \theta < \theta_k\} = 1.$$

Defining $\Phi : C \to \mathbb{R}$ by

$$\Phi(r\cos\theta, r\sin\theta) = r^{\sqrt{\mu_1}}\phi_{1,\sigma}(\theta)$$

and writing $\Phi_i = \Phi|_{C_i}$, we find that Φ satisfies

$$(21) \qquad \begin{cases} -\Delta\Phi_i = 0 & \text{in } C_i \text{ with } i = 1, \ldots, k, \\ \Phi_1 = 0 & \text{on } \Gamma_0, \\ \left.\begin{array}{c} \Phi_i = \Phi_{i+1} \\ \sigma_i \frac{\partial}{\partial\theta}\Phi_i = \sigma_{i+1}\frac{\partial}{\partial\theta}\Phi_{i+1} \end{array}\right\} & \text{on } \Gamma_i \text{ with } i = 1, \ldots, k-1, \\ \Phi_k = 0 & \text{on } \Gamma_k, \\ \Phi_i = \phi_{1,\sigma} & \text{on } \partial C \cap \partial B_1(0). \end{cases}$$

Since $\phi_{1,\sigma}$ satisfies (27) and since $d\left(x,\partial\Omega\right) = |x|\, d\left(\frac{x}{|x|},\partial\Omega\right)$ for $x \in \Omega \cap B_\rho\left(0\right)$, one finds that

$$\Phi\left(x\right) \simeq |x|^{\sqrt{\mu_1}}\, d\left(\frac{x}{|x|},\Gamma_0 \cup \Gamma_k\right) \simeq |x|^{\sqrt{\mu_1}}\, d\left(\frac{x}{|x|},\frac{1}{\rho}\partial\Omega\right) \simeq |x|^{\sqrt{\mu_1}-1}\, d\left(x,\partial\Omega\right).$$

Indeed, the equivalences follow from $\left(\frac{1}{\rho}\Omega\right) \cap B_2\left(0\right) = 2C$ and by scaling. In the remainder the Maximum Principle as in Theorem 5 is used. In the following we will use auxiliary functions $u_{\ell a}$ and $u_{\ell b}$ with $\ell \in \{1,2,3\}$, which all are in $W^{1,2}\left(C\right)$.

(1) Let $m+2 < \sqrt{\mu_1}$.

- **Estimate from above:** Set υ_κ the solution of

$$\left\{\begin{array}{ll} -\upsilon''_{\kappa,i}(\theta) + \kappa\upsilon_{\kappa,i}(\theta) = 1 & \text{for } \theta \in [\theta_{i-1},\theta_i] \text{ and } i \in \{1,\ldots,k\}, \\ \upsilon_{\kappa,i}(\theta_i) = \upsilon_{\kappa,i+1}(\theta_i) & i = 1,\ldots,k-1, \\ \sigma_i\upsilon'_{\kappa,i}(\theta_i) = \sigma_{i+1}\upsilon'_{\kappa,i+1}(\theta_{i+1}) & i = 1,\ldots,k-1, \\ \upsilon_\kappa(0) = \upsilon_\kappa(\theta_k) = 0, & \end{array}\right. \tag{22}$$

with $\kappa = -(m+2)^2$ and the same σ_i as in (12). Since $\kappa < \mu_1$, one finds that such a solution υ_κ exists uniquely, is positive and furthermore, we find

$$\upsilon_\kappa \simeq \phi_{1,\sigma}.$$

Taking $u_{1a} := |x|^{m+2}\upsilon_\kappa\left(\frac{x}{|x|}\right)$ we observe that u_{1a} satisfies the following boundary value problem:

$$\left\{\begin{array}{ll} -\Delta u_{1a}|_{C_i} = |x|^m & \text{in } C_i,\ i = 1,\ldots,k, \\ u_{1a,i} = u_{1a,i+1} & \text{on } \Gamma_i,\ i = 1,\ldots,k-1, \\ \sigma_i\frac{\partial u_{1a,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{1a,i+1}}{\partial\theta} & \text{on } \Gamma_i, i = 1,\ldots,k-1, \\ u_{1a} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{1a,1} = 0 & \text{on } \Gamma_0, \\ u_{1a,k} = 0 & \text{on } \Gamma_k. \end{array}\right.$$

Since $f \preceq |x|^m$ on Ω, it follows by the maximum principle that $u \preceq u_{1a} \simeq |x|^{m+2}\phi_{1,\sigma}\left(\frac{x}{|x|}\right)$.

- **Estimate from below:** We take κ as before and we let ω_κ be the solution of

$$\left\{\begin{array}{ll} -\omega''_{\kappa,i}(\theta) + \kappa\omega_{\kappa,i}(\theta) = \chi_{(\theta_a,\theta_b)}(\theta) & \text{for } \theta \in [\theta_{i-1},\theta_i] \text{ and } i \in \{1,\ldots,k\}, \\ \omega_{\kappa,i}(\theta_i) = \omega_{\kappa,i+1}(\theta_i) & i = 1,\ldots,k-1, \\ \sigma_i\omega'_{\kappa,i}(\theta_i) = \sigma_{i+1}\omega'_{\kappa,i+1}(\theta_{i+1}) & i = 1,\ldots,k-1, \\ \omega_\kappa(0) = \omega_\kappa(\theta_k) = 0. & \end{array}\right. \tag{23}$$

Here χ_A is the characteristic function for a set A. Similarly as in the previous case, we find $0 \le \omega_\kappa \simeq \phi_{1,\sigma}$ in $(0, \theta_k)$. By setting

$$u_{1b} := |x|^{m+2} \omega_\kappa \left(\frac{x}{|x|} \right),$$

one finds that u_{1b} satisfies

$$\begin{cases} -\Delta u_{1b}|_{C_i} = |x|^m \chi_{(\theta_a, \theta_b)} \left(\frac{x}{|x|} \right) & \text{in } C_i, \ i = 1, \dots, k, \\ u_{1b,i} = u_{1b,i+1} & \text{on } \Gamma_i, \ i = 1, \dots, k-1, \\ \sigma_i \frac{\partial u_{1b,i}}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{1b,i+1}}{\partial \theta} & \text{on } \Gamma_i, i = 1, \dots, k-1, \\ u_{1b} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{1b,1} = 0 & \text{on } \Gamma_0, \\ u_{1b,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

Thus, by the maximum principle we find $|x|^{m+2} \phi_{1,\sigma} \left(\frac{x}{|x|} \right) \simeq u_{1b} \preceq u$.

(2) Let $m + 2 = \sqrt{\mu_1}$.

- **Estimate from above:** We set υ_0 the solution of

$$\begin{cases} -\upsilon_{0,i}''(\theta) = 1 & \text{for } \theta \in [\theta_{i-1}, \theta_i] \text{ and } i \in \{1, \dots, k\}, \\ \upsilon_{0,i}(\theta_i) = \upsilon_{0,i+1}(\theta_i) & i = 1, \dots, k-1, \\ \sigma_i \upsilon_{0,i}'(\theta_i) = \sigma_{i+1} \upsilon_{0,i+1}'(\theta_{i+1}) & i = 1, \dots, k-1, \\ \upsilon_0(0) = \upsilon_0(\theta_k) = 0, \end{cases} \tag{24}$$

which is simply the solution of (22) with $\kappa = 0$. Since $\upsilon_0 \simeq \phi_{1,\sigma}$ in $(0, \theta_k)$, we can choose a positive constant γ such that

$$\gamma(m+2)\upsilon_0(\theta) \le 2\phi_{1,\sigma}(\theta) \ \text{ for all } \theta \in (0, \theta_k).$$

Then by taking

$$u_{2a}(x) := |x|^{m+2} \ln \left(\frac{1}{|x|} \right) \phi_{1,\sigma} \left(\frac{x}{|x|} \right) + \gamma |x|^{m+2} \upsilon_0 \left(\frac{x}{|x|} \right)$$

one finds that u_{2a} satisfies

$$-\Delta u_{2a} = |x|^m \left(2(m+2)\phi_{1,\sigma} \left(\frac{x}{|x|} \right) - \gamma(m+2)^2 \upsilon_0 \left(\frac{x}{|x|} \right) + \gamma \right),$$

which implies

$$\begin{cases} -\Delta u_{2a}|_{C_i} \simeq |x|^m & \text{in } C_i, \ i = 1, \dots, k, \\ u_{2a,i} = u_{2a,i+1} & \text{on } \Gamma_i, \ i = 1, \dots, k-1, \\ \sigma_i \frac{\partial u_{2a,i}}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{2a,i+1}}{\partial \theta} & \text{on } \Gamma_i, \ i = 1, \dots, k-1, \\ u_{2a} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{2a,1} = 0 & \text{on } \Gamma_0, \\ u_{2a,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

We observe that $-\Delta u = f(x) \le -\Delta u_{2a}$ in Ω and $u \simeq u_{2a}$ on $\partial\Omega$. By the maximum principle we get the following estimate from above:

$$u \preceq u_{2a} \simeq |x|^{\sqrt{\mu_1}} \ln\left(\tfrac{1}{|x|}\right)\ \phi_{1,\sigma}\left(\tfrac{x}{|x|}\right).$$

- **Estimate from below:** For getting a lower barrier for u in this case, we set

$$u_{2b} := \zeta |x|^{m+2} \ln\left(\frac{1}{|x|}\right) \phi_{1,\sigma}\left(\frac{x}{|x|}\right) + |x|^{m+2}\omega_0\left(\frac{x}{|x|}\right),$$

where ω_0 is the solution of (23) with $\kappa = 0$ and where $\zeta > 0$ is such that

$$2\zeta\phi_{1,\sigma}(\theta) \leq (m+2)\omega_0(\theta) \ \text{ for all } \theta \in (0, \theta_k).$$

Then u_{2b} satisfies the following equation for all $x \in C_i$

$$-\Delta u_{2b}|_{C_i} = |x|^m \chi_{(\theta_a,\theta_b)}\left(\frac{x}{|x|}\right) + 2(m+2)\zeta|x|^m\phi_{1,\sigma}\left(\frac{x}{|x|}\right) - (m+2)^2|x|^m\omega_0\left(\frac{x}{|x|}\right).$$

Hence u_{2b} is a bound from below since

$$\begin{cases} -\Delta u_{2b}|_{C_i} \preceq |x|^m \chi_{(\theta_a,\theta_b)}\left(\frac{x}{|x|}\right) & \text{in } C_i,\ i = 1,\ldots,k, \\ u_{2b,i} = u_{2b,i+1} & \text{on } \Gamma_i,\ i = 1,\ldots,k-1, \\ \sigma_i \frac{\partial u_{2b,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{2b,i+1}}{\partial\theta} & \text{on } \Gamma_i,\ i = 1,\ldots,k-1, \\ u_{2b} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{2b,1} = 0 & \text{on } \Gamma_0, \\ u_{2b,k} = 0 & \text{on } \Gamma_k, \end{cases}$$

which implies by the maximum principle that $u_{2b} \preceq u$.

(3) Let $m + 2 > \sqrt{\mu_1}$.

- **Estimate from above:** An upper barrier function for u in this case will be

$$u_{3a} := \left(|x|^{\sqrt{\mu_1}} - |x|^{m+2}\right)\phi_{1,\sigma}\left(\frac{x}{|x|}\right) + \gamma|x|^{m+2}\upsilon_0\left(\frac{x}{|x|}\right),$$

where υ_0 is the solution of (24) and $\gamma > 0$ satisfies

$$\gamma(m+2)^2\upsilon_0(\theta) \leq \left((m+2)^2 - \mu_1\right)\phi_{1,\sigma}(\theta).$$

Then u_{3a} satisfies the following equation:

$$-\Delta u_{3a} = |x|^m\left((m+2)^2 - \mu_1\right)\phi_{1,\sigma}\left(\frac{x}{|x|}\right) - |x|^m\gamma(m+2)^2\upsilon_0\left(\frac{x}{|x|}\right) + \gamma|x|^m$$

for $x \in \Omega$. So one finds that

$$\begin{cases} -\Delta u_{3a}|_{C_i} \simeq |x|^m & \text{in } C_i,\ i = 1,\ldots,k, \\ u_{3a,i} = u_{3a,i+1} & \text{on } \Gamma_i,\ i = 1,\ldots,k-1, \\ \sigma_i\frac{\partial u_{3a,i}}{\partial\theta} = \sigma_{i+1}\frac{\partial u_{3a,i+1}}{\partial\theta} & \text{on } \Gamma_i,\ i = 1,\ldots,k-1, \\ u_{3a} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{3a,1} = 0 & \text{on } \Gamma_0, \\ u_{3a,k} = 0 & \text{on } \Gamma_k, \end{cases}$$

and this implies that $u \preceq u_{3a} \simeq |x|^{\sqrt{\mu_1}} \phi_{1,\sigma}\left(\frac{x}{|x|}\right)$.

- **Estimate from below:** The estimate from below one directly finds by the harmonic function

$$u_{3b} := |x|^{\sqrt{\mu_1}} \phi_{1,\sigma}\left(\frac{x}{|x|}\right),$$

which satisfies

$$\begin{cases} -\Delta u_{3b}|_{C_i} = 0 & \text{in } C_i,\ i = 1, \ldots, k, \\ u_{3b,i} = u_{3b,i+1} & \text{on } \Gamma_i,\ i = 1, \ldots, k-1, \\ \sigma_i \frac{\partial u_{3b,i}}{\partial \theta} = \sigma_{i+1} \frac{\partial u_{3b,i+1}}{\partial \theta} & \text{on } \Gamma_i, i = 1, \ldots, k-1, \\ u_{3b} \simeq \phi_{1,\sigma} & \text{on } \ \partial C \cap \partial B_1(0), \\ u_{3b,1} = 0 & \text{on } \Gamma_0, \\ u_{3b,k} = 0 & \text{on } \Gamma_k. \end{cases}$$

Again the maximum principle implies $u_{3b} \preceq u$ in Ω.

Comparing with the results in [2, Theorem 6], we observe that the solution of the problem (12) has the same form as the solution of the Poisson problem near a conical point but with a different type of regularity. □

Appendix A. Existence and positivity of the first eigenfunction

Concerning the first eigenfunction one may have look at [4]. For positivity see the result in [4, Volume I, Chapter VI, §6 (page 452)], which became known as Courant's Nodal Domain Theorem. In [4] however one hardly finds the precise conditions for the coefficients. A place with sufficiently general conditions on the coefficients, which allow our piecewise constant σ, is [6, Section 8.12]. The eigenfunction, that we are interested in, is the first eigenfunction ϕ for the weighted Laplace-Beltrami operator on the intersection of Ω and the unit sphere. This function is used in the power type functions $|x|^a \phi\,(x/\,|x|)$, which are defined near a point, where boundary and interface meet.

In our 2-dimensional case the Rayleigh quotient, for which the first eigenfunction is a minimizer, is as follows. With $0 = \theta_0 < \theta_1 < \cdots < \theta_k < 2\pi$ the Rayleigh quotient R_σ is defined on $\mathring{W}^{1,2}(0, \theta_k) \setminus \{0\}$ by

$$R_\sigma(\phi) = \frac{\int_0^{\theta_k} \tilde{\sigma}(\theta)\ \phi'(\theta)^2\ d\theta}{\int_0^{\theta_k} \tilde{\sigma}(\theta)\ \phi(\theta)^2\ d\theta}, \tag{25}$$

where $\tilde{\sigma}(\theta) = \sigma_i \in \mathbb{R}^+$ for $\theta \in (\theta_{i-1}, \theta_i)$.

Lemma 4. *Let R_σ be defined in* (25). *Then the following holds.*

(1) *R_σ attains its infimum μ_1 for some $\phi_{1,\sigma} \in \mathring{W}^{1,2}(0, \theta_k) \setminus \{0\}$ and $\mu_1 \geq \frac{1}{4} \frac{\min \sigma_i}{\max \sigma_i} > 0$.*

(2) *The minimizing function $\phi_{1,\sigma}$ is unique up to multiplication, has a fixed sign and, after normalizing by*

$$\max\{\phi_{1,\sigma}(\theta)\,; 0 < \theta < \theta_k\} = 1, \tag{26}$$

satisfies for some $C_\sigma, c_\sigma > 0$;

$$c_\sigma \sin\left(\frac{\pi}{\theta_k}\theta\right) \le \phi_{1,\sigma}(\theta) \le C_\sigma \sin\left(\frac{\pi}{\theta_k}\theta\right) \text{ for all } \theta \in [0,\theta_k]. \tag{27}$$

(3) $\phi_{1,\sigma}$ *is the unique first eigenfunction, in the sense that*

$$\phi_{1,\sigma,i} := \phi_{1,\sigma}|_{[\theta_{i-1},\theta_i]} \in C^2[\theta_{i-1},\theta_i]$$

satisfies

$$\begin{cases} -\phi''_{1,\sigma,i} = \mu_1\phi_{1,\sigma,i} \quad \text{on } (\theta_{i-1},\theta_i) \text{ with } i \in \{1,\dots,k\}, \\ \phi_{1,\sigma,1}(0) = 0, \\ \left.\begin{array}{l} \phi_{1,\sigma,i}(\theta_i) = \phi_{1,\sigma,i+1}(\theta_i) \\ \sigma_i\phi'_{1,\sigma,i}(\theta_i) = \sigma_{i+1}\phi'_{1,\sigma,i+1}(\theta_i) \end{array}\right\} \text{ for } i \in \{1,\dots,k-1\}, \\ \phi_{1,\sigma,k}(\theta_k) = 0, \end{cases} \tag{28}$$

and there is no other, independent, eigenfunction for $\mu \le \mu_1$.

Proof. By [6, Section 8.12] one finds that the minimizer $\phi_{1,\sigma} \in \mathring{W}^{1,2}(0,\theta_k)$ of (25), that we may normalize by (26), exists, is unique and is of fixed sign. Let μ_1 be the minimum value of (25). Since for $\phi \not\equiv 0$ one has

$$\frac{\int_0^{\theta_k}\tilde\sigma(\theta)\ \phi'(\theta)^2\,d\theta}{\int_0^{\theta_k}\tilde\sigma(\theta)\ \phi(\theta)^2\,d\theta} \ge \frac{\min\sigma_i}{\max\sigma_i}\frac{\int_0^{\theta_k}\phi'(\theta)^2\,d\theta}{\int_0^{\theta_k}\phi(\theta)^2\,d\theta} \ge \frac{1}{4}\frac{\min\sigma_i}{\max\sigma_i}, \tag{29}$$

one finds $\mu_1 \ge \frac{1}{4}\frac{\min\sigma_i}{\max\sigma_i}$. In the last step of (29) we used the optimal Poincaré constant, which is $(\pi/\theta_k)^2$ and which can be estimated from below by $\frac14$, since $\theta_k \le 2\pi$ holds. The function $\phi_{1,\sigma}$ satisfies the weak Euler-Lagrange equation

$$\int_0^{\theta_k}\sigma\left(\phi'_{1,\sigma}\phi' - \mu_1\phi_{1,\sigma}\phi\right)dx = 0 \text{ for all } \phi \in \mathring{W}^{1,2}(0,\theta_k).$$

Taking testfunctions with support in (θ_{i-1},θ_i) one finds that

$$\phi_{1,\sigma,i} := \phi_{1,\sigma}|_{[\theta_{i-1},\theta_i]} \in W^{2,2}(\theta_{i-1},\theta_i) \tag{30}$$

satisfies $-\phi''_{1,\sigma,i} = \mu_1\phi_{1,\sigma,i}$ on (θ_{i-1},θ_i) and even that $\phi_{1,\sigma,i} \in C^\infty[\theta_{i-1},\theta_i]$. Since $\phi_{1,\sigma} \in \mathring{W}^{1,2}(0,\theta_k)$ holds, the functions $\phi_{1,\sigma,i}$ satisfy the continuity equation $\phi_{1,\sigma,i}(\theta_i) = \phi_{1,\sigma,i+1}(\theta_i)$ and the boundary conditions $\phi_{1,\sigma,1}(0) = \phi_{1,\sigma,k}(\theta_k) = 0$. The jump condition $\sigma_i\phi'_{1,\sigma,i}(\theta_i) = \sigma_{i+1}\phi'_{1,\sigma,i+1}(\theta_i)$ follows by taking testfunctions in the weak Euler-Lagrange equation with support near θ_i. Assuming $\phi_{1,\sigma} \ge 0$ holds, the strict positivity, with $\phi'_{1,\sigma}(0) > 0$ and $\phi'_{1,\sigma}(\theta_k) < 0$, follows from the unique continuation. Indeed, if $\phi'_{1,\sigma,i}(\theta^*) = 0 = \phi_{1,\sigma,i}(\theta^*)$ for some i and some $\theta^* \in [\theta_{i-1},\theta_i]$, then $\phi_{1,\sigma} \equiv 0$. The estimate in (27) is a direct consequence. □

Appendix B. A maximum principle for weak solutions

The following result is quite direct but we nevertheless state it for easy reference.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\sigma \in L^\infty(\Omega)$ with $\sigma \geq \sigma_0 > 0$. Suppose that $u \in W^{1,2}(\Omega)$ is such that $\min(u,0) \in \mathring{W}^{1,2}(\Omega)$ and satisfies*

$$\int_\Omega \sigma \ \nabla u \cdot \nabla \varphi \ dx \geq 0 \ \ \textit{for all } \varphi \in \mathring{W}^{1,2}(\Omega) \ \ \textit{with } \varphi \geq 0.$$

Then one finds $u \geq 0$ in Ω.

Proof. With $\varphi = -\min(u,0)$, which lies in $\mathring{W}^{1,2}(\Omega)$ and is nonnegative, we find

$$0 \leq \int_\Omega \sigma \ \nabla u \cdot \nabla \varphi \ dx = -\int_\Omega \sigma \ \left|\nabla \varphi\right|^2 \ dx \leq 0.$$

Hence $\nabla \varphi = 0$, which implies $\varphi = 0$ and hence $u \geq 0$. □

Appendix C. Equivalent solutions

Lemma 6. *Suppose that the domain Ω is the union of subdomains Ω_i with $i = 1, \ldots, k$, that is $\overline{\Omega} = \bigcup_{i=1}^k \overline{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$, and is such that $\partial\Omega, \Omega \cap \partial\Omega_i \cap \partial\Omega_j \in C^2$. Suppose also that the weight function $\sigma : \overline{\Omega} \to \mathbb{R}$ is a piecewise constant positive function defined by*

$$\sigma(x) = \sigma_i \ \textit{for } x \in \Omega_i \ \textit{with } \sigma_i \in \mathbb{R}^+.$$

Suppose that $u \in W^{1,2}(\Omega)$ and $u_i := u|_{\Omega_i} \in W^{2,2}(\Omega_i)$. Then the following two statements are equivalent:

(1) *u is such that*

$$\begin{cases} -\Delta u_i = f & \textit{in } \Omega_i, \\ \left.\begin{array}{c} u_i = u_j \\ \sigma_i \frac{\partial u_i}{\partial \nu_i} = -\sigma_j \frac{\partial u_j}{\partial \nu_j} \end{array}\right\} & \textit{as traces on } \Omega \cap \partial\Omega_i \cap \partial\Omega_j. \end{cases} \tag{31}$$

(2) *u satisfies*

$$\int_\Omega \sigma \ \left(\nabla u \cdot \nabla \varphi - f\varphi\right) dx = 0 \ \ \textit{for all } \varphi \in \mathring{W}^{1,2}(\Omega). \tag{32}$$

Proof. One directly finds for $\varphi \in \mathring{W}^{1,2}(\Omega)$ that

$$\int_\Omega \sigma \ \left(\nabla u \cdot \nabla \varphi - f\varphi\right) dx = \sum_{i=1}^k \int_{\Omega_i} \sigma_i \ \left(\nabla u_i \cdot \nabla \varphi - f\varphi\right) dx$$

$$= \sum_{i=1}^k \left(\int_{\partial\Omega_i} \sigma_i \ \frac{\partial u_i}{\partial \nu_i} \ \varphi \ dx + \int_{\Omega_i} \sigma_i \ \left(-\Delta u_i - f\right) \varphi dx \right)$$

(33)

$$= \sum_{i,j=1}^{k} \left(\int_{\partial\Omega_i \cap \partial\Omega_j} \left(\sigma_i \frac{\partial u_i}{\partial \nu_i} + \sigma_j \frac{\partial u_j}{\partial \nu_j} \right) \varphi \, dx + \int_{\Omega_i} \sigma_i \ \left(-\Delta u_i - f \right) \varphi dx \right).$$

By the assumption that $u_i \in W^{2,2}(\Omega_i)$ these integrals are well-defined. Note that the boundary integral over $\partial\Omega$ drops out since $\varphi = 0$ as trace on $\partial\Omega$.

So (31) implies (32).

Assuming (32) and testing with $\varphi \in C_0^\infty(\Omega_i)$ gives $-\Delta u_i - f = 0$. The condition $u_i = u_j$ on $\partial\Omega_i \cap \partial\Omega_j$ follows from $u \in W^{1,2}(\Omega)$. By taking testfunctions with support intersecting $\partial\Omega_i \cap \partial\Omega_j$ one establishes the jump condition in the normal derivatives. □

Appendix D. Some examples

Example 1. We first consider the simplest case, namely we take Ω locally flat with $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$ and

$$\Omega_1 \cap B_\rho(0) = \{(r\cos\theta, r\sin\theta); 0 < r < \rho \text{ and } \theta \in (0, \theta_1)\},$$
$$\Omega_2 \cap B_\rho(0) = \{(r\cos\theta, r\sin\theta); 0 < r < \rho \text{ and } \theta \in (\theta_1, \pi)\}.$$

Then by a direct computation one shows that the first eigenfunction of (28) is given by

$$\phi_1(\theta) = \begin{cases} \sin(\lambda\theta) & \text{for } \theta \in [0, \theta_1], \\ \frac{\sin(\lambda\theta_1)}{\sin(\alpha_1(\lambda))} \sin(\lambda(\theta - \theta_1) + \alpha_1(\lambda)) & \text{for } \theta \in (\theta_1, \pi], \end{cases}$$

where $\alpha_1(\lambda) = \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot(\lambda\theta_1)\right)$ and where λ is the smallest positive number such that $\phi_1(\pi) = 0$.

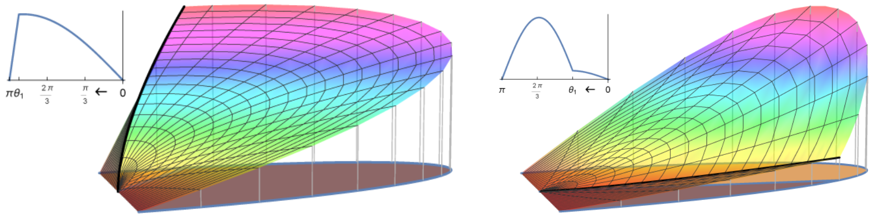
The behaviour at 0 as in (14) and (19) is given by $r^\lambda \phi_1(\theta)$ with $\lambda = \sqrt{\mu_1}$. Assuming that $\sigma_1 > \sigma_2$ and letting $\frac{\sigma_1}{\sigma_2} \to \infty$, one finds the 'extreme' cases for $\theta_1 \uparrow \pi$ and for $\theta_1 = \frac{1}{3}\pi$. These cases correspond with $\lambda \downarrow \frac{1}{2}$ respectively $\lambda \uparrow \frac{3}{2}$. Sketches with nearby values can be found in Fig. 3. One may show that for all $\sigma_1, \sigma_2 \in \mathbb{R}^+$ and all $0 = \theta_0 < \theta_1 < \theta_2 = \pi$ it holds that

$$\frac{1}{2} < \sqrt{\mu_1} < \frac{3}{2}.$$

Note that for $\theta_1 = \frac{1}{2}\pi$ one finds $\varphi_1(\theta) = \sin\theta$ and $\mu_1 = 1$ independently of σ_1 or σ_2.

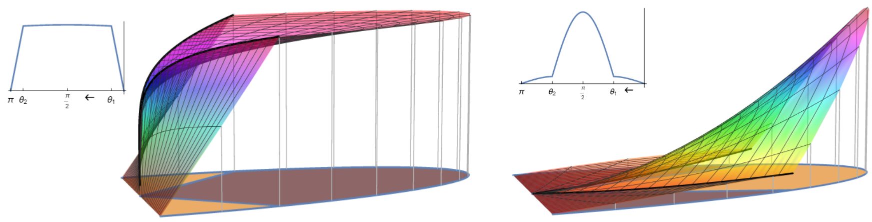
Example 2. Also in the next case Ω is flat, but now it has three subdomains, such that

$$\Omega_1 \cap B_\rho(0) = \{(r\cos\theta, r\sin\theta); 0 < r < \rho \text{ and } \theta \in (0, \theta_1)\},$$
$$\Omega_2 \cap B_\rho(0) = \{(r\cos\theta, r\sin\theta); 0 < r < \rho \text{ and } \theta \in (\theta_1, \theta_2)\},$$
$$\Omega_3 \cap B_\rho(0) = \{(r\cos\theta, r\sin\theta); 0 < r < \rho \text{ and } \theta \in (\theta_2, \pi)\},$$



$\sigma_1 = 100\sigma_2,\ \theta_1 = \frac{11}{12}\pi,\ \sqrt{\mu_1} = 0.568\cdots$ $\sigma_1 = 100\sigma_2,\ \theta_1 = \frac{1}{3}\pi,\ \sqrt{\mu_1} = 1.43\cdots$

FIGURE 3. *Plots of $r^\lambda \phi_1(\theta)$, which show the typical behaviour of u near a boundary point where $\partial\Omega$ is smooth and σ has one jump. The inset displays the eigenfunction ϕ_1. For the sake of easy comparison the positive direction of the inset is to the left.*



$\sigma_1 = \sigma_3 = 0.01\sigma_2,\ \theta_1 = \frac{1}{9}\pi,\ \theta_2 = \frac{8}{9}\pi,\ \sqrt{\mu_1} = 0.152\cdots$ $\sigma_1 = \sigma_3 = 100\sigma_2,\ \theta_1 = \frac{1}{4}\pi,\ \theta_2 = \frac{3}{4}\pi,\ \sqrt{\mu_1} = 1.87\cdots$

FIGURE 4. *Plots of $r^\lambda \phi_1(\theta)$ are showing the typical behaviour of u near a boundary point where σ has two jumps. The inset displays the corresponding eigenfunction ϕ_1. Again the inset has the positive direction to the left.*

and $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2 \cup \Omega_3}$. Then

$$\phi_1(\theta) = \begin{cases} \sin(\lambda\theta) & \text{for } \theta \in [0, \theta_1], \\ \dfrac{\phi_1(\theta_1)}{\sin(\alpha_1(\lambda))} \sin(\lambda(\theta - \theta_1) + \alpha_1(\lambda)) & \text{for } \theta \in (\theta_1, \theta_2], \\ \dfrac{\phi_1(\theta_2)}{\sin(\alpha_2(\lambda))} \sin(\lambda(\theta - \theta_2) + \alpha_2(\lambda)) & \text{for } \theta \in (\theta_2, \pi], \end{cases}$$

with

$$\alpha_1(\lambda) = \operatorname{arccot}\left(\frac{\sigma_1}{\sigma_2}\cot(\lambda\theta_1)\right) \text{ and}$$
$$\alpha_2(\lambda) = \operatorname{arccot}\left(\frac{\sigma_2}{\sigma_3}\cot(\lambda(\theta_2 - \theta_1)) + \alpha_1(\lambda)\right).$$

Again λ is the smallest positive number such that $\phi_1(\pi) = 0$.

Again the behaviour at 0 as in (14) and (19) is given by $r^{\lambda}\phi_1(\theta)$ with $\lambda = \sqrt{\mu_1}$. If $\sigma_1 = \sigma_3 > \sigma_2$ one finds the extreme cases when $\frac{\sigma_2}{\sigma_1} \to 0$ for $\theta_1 = \frac{1}{4}\pi$, $\theta_2 = \frac{3}{4}\pi$. For $\sigma_1 = \sigma_3 < \sigma_2$ and $\frac{\sigma_2}{\sigma_1} \to \infty$ the 'extreme' case appears for $\theta_1 = \pi - \theta_2 \downarrow 0$. See Fig. 4.

For three subdomains as above one may show that for all $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}^+$ and all $0 = \theta_0 < \theta_1 < \theta_2 < \theta_3 = \pi$ it holds that

$$0 < \sqrt{\mu_1} < 2.$$

Acknowledgement. The present paper is part of the Ph.D.-Thesis [1] of M. Beygmohammadi, which has been successfully defended on October 19, 2015.

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