J. Korean Math. Soc. **54** (2017), No. 1, pp. 227–248
https://doi.org/10.4134/JKMS.j150713
pISSN: 0304-9914 / eISSN: 2234-3008

EQUIVARIANT VECTOR BUNDLES OVER GRAPHS

Min Kyu Kim

Abstract. In this paper, we reduce the classification problem of equivariant (topological complex) vector bundles over a simple graph to the classification problem of their isotropy representations at vertices and midpoints of edges. Then, we solve the reduced problem in the case when the simple graph is homeomorphic to a circle. So, the paper could be considered as a generalization of [3].

1. Introduction

In topology, complex vector bundles can be classified by their Chern classes under some mild conditions, see [9, Theorem 3.2]. But, there is no such general result on equivariant (topological complex) vector bundles even over an arbitrary closed two-surface. Recently, equivariant vector bundles over a two-sphere have been classified in [7]. In [7], we endow the two-sphere with the structure of an equivariant simplicial complex, and we can observe that the data of an equivariant vector bundle except its (inequivariant, i.e., not considering a group action) Chern class is concentrated on the equivariant vector bundle restricted to the one-skeleton (i.e., a graph) of the barycentric subdivision of the simplicial complex, see Example 1.6 and Figure 1.2. In fact, the concentration of the data is observed in equivariant vector bundles over an arbitrary closed two-surface. So, the author believes that equivariant vector bundles over a graph should be understood before we investigate equivariant vector bundles over an arbitrary closed two-surface. This is the motivation of the paper.

To state our results, we introduce some terminologies and notations. For some terminologies, we postpone their definitions to Section 2 to avoid lengthy Introduction.

Definition 1.1 (Simple graph)**.** Let $\mathcal{V}$ be a finite set, and $\mathcal{E}$ be a set of subsets of $\mathcal{V}$ with two distinct elements. Then, an ordered pair $\Gamma = (\mathcal{V}, \mathcal{E})$ is called a *simple graph*. Two sets $\mathcal{V}$ and $\mathcal{E}$ are called the *vertex set* and *edge set* of Γ,

Received November 25, 2015; Revised December 7, 2015.
2010 *Mathematics Subject Classification.* Primary 57S25; Secondary 20C99.
Key words and phrases. equivariant vector bundle, representation.

respectively. And, an element of $\mathcal{V}$ and $\mathcal{E}$ is called a *vertex* and an *edge* of Γ, respectively.

Two distinct vertices v_1 and v_2 of a simple graph are *adjacent* if they are contained in an edge. We define a geometric realization of a simple graph.

Definition 1.2 (Geometric realization of a simple graph)**.** For a simple graph Γ, we consider a one-dimensional space in $\mathbb{R}^d$ for some natural number d which satisfies the following:

(1) vertices of Γ are embedded in $\mathbb{R}^d$,
(2) embedded two vertices are connected by a line segment if and only if two vertices are adjacent,
(3) two distinct line segments can intersect in at most one embedded vertex.

Such a space endowed with the subspace topology induced from $\mathbb{R}^d$ is called a *geometric realization* of Γ, and denoted by $|\Gamma|$. The point in $|\Gamma|$ corresponding to a vertex $v \in \Gamma$ is also called a *vertex*, and denoted by the same notation v. The line segment of $|\Gamma|$ corresponding to an edge e of Γ is also called an *edge*, and denoted by $|e|$.

The line segment connecting the midpoint and a vertex of an edge of $|\Gamma|$ is called a *bisegment*.

Notation 1.3 (Bisegment)**.** Denote by $\mu(e)$ the midpoint of an edge $|e|$ of $|\Gamma|$. The bisegment connecting $\mu(e)$ and a vertex v of an edge $|e|$ of $|\Gamma|$ is denoted by $[v, \mu(e)]$ or $[\mu(e), v]$.

Now, we introduce a graphical action.

Definition 1.4 (Graphical action)**.** A compact Lie group G *acts graphically* on a simple graph $\Gamma = (\mathcal{V}, \mathcal{E})$ if G acts continuously on $\mathcal{V}$ and $\mathcal{E}$ in a compatible way, i.e.,

$$g \cdot \{v_1, v_2\} = \{g \cdot v_1,\, g \cdot v_2\} \qquad \text{for any edge } \{v_1, v_2\} \in \mathcal{E}.$$

Then, the graphical action on Γ induces a unique group action on embedded vertices of $|\Gamma|$, so it induces a unique action on $|\Gamma|$, also called a *graphical action*, defined by

$$g \cdot \Big(t\, v_1 + (1-t)\, v_2 \Big) \;:=\; t\, g \cdot v_1 + (1-t)\, g \cdot v_2$$

for any $t \in [0,1]$ and adjacent vertices $v_1,\, v_2 \in \mathcal{V}$.

We list two examples of graphical actions.

Example 1.5 (Group actions on a circle)**.** For a natural number $n \geq 3$, let $\mathcal{V}$ be the subset $\{e^{2\pi k\sqrt{-1}/n} \,|\, k \in \mathbb{Z}_n\}$ of $\mathbb{C}$, and $\mathcal{E}$ be the set

$$\left\{ \left\{ e^{2\pi k\sqrt{-1}/n}, e^{2\pi (k+1)\sqrt{-1}/n} \right\} \;\middle|\; k \in \mathbb{Z}_n \right\},$$

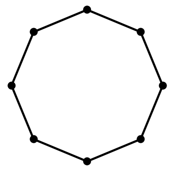


FIGURE 1.1. Example 1.5

see Figure 1.1 for the case when $n = 8$. For an integer j_1 coprime to n and an integer j_2,

(1) let the cyclic group $\mathbb{Z}_n = \langle\, a \mid a^n = \mathrm{id}\, \rangle$ act transitively on $\mathcal{V}$ by

$$a \cdot e^{2\pi k\sqrt{-1}/n} := e^{2\pi(k+j_1)\sqrt{-1}/n} \quad \text{for any } k \in \mathbb{Z}_n,$$

(2) let the dihedral group

$$\mathrm{D}_n = \left\langle a, b \,\middle|\, a^n = \mathrm{id},\ b^2 = \mathrm{id},\ bab = a^{-1} \right\rangle$$

act transitively on $\mathcal{V}$ by

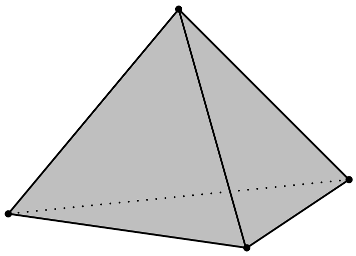
$$a \cdot e^{2\pi k\sqrt{-1}/n} := e^{2\pi(k+j_1)\sqrt{-1}/n} \quad \text{and}$$
$$b \cdot e^{2\pi k\sqrt{-1}/n} := e^{2\pi(-k+j_2)\sqrt{-1}/n} \quad \text{for any } k \in \mathbb{Z}_n.$$

Then, both $\mathbb{Z}_n$- and D_n-actions induce graphical actions on Γ and its geometric realization $|\Gamma|$ which is homeomorphic to a circle. The stabilizer subgroup of $\mathbb{Z}_n$ at each point of $|\Gamma|$ is trivial. Also, the stabilizer subgroup of D_n at each vertex and the midpoint of each edge of $|\Gamma|$ is isomorphic to $\mathbb{Z}_2$, and the stabilizer subgroups of D_n at the other points of $|\Gamma|$ are trivial.

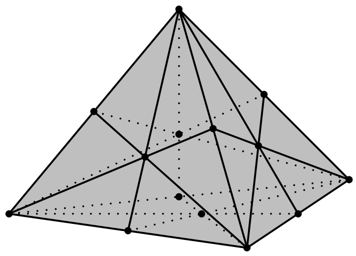
Example 1.6 (Group action on a regular tetrahedron)**.** Let the tetrahedral group T of order 24 act naturally on a regular tetrahedron endowed with the structure of a usual simplicial complex $\mathcal{K}$, i.e., four faces, six edges, and four vertices, which is invariant under the T-action, see Figure 1.2. The T-action induces a T-action on the barycentric subdivision $\mathrm{Sd}\,\mathcal{K}$ of $\mathcal{K}$. Then, the induced T-action on the one-skeleton $(\mathrm{Sd}\,\mathcal{K})^{(1)}$ of $\mathrm{Sd}\,\mathcal{K}$ can be considered as a graphical G-action.

Hereafter, we assume that a compact Lie group G acts (not necessarily effectively) graphically on a simple graph Γ and its geometric realization $|\Gamma|$. Now, we introduce a useful notation on restriction of an equivariant vector bundle.

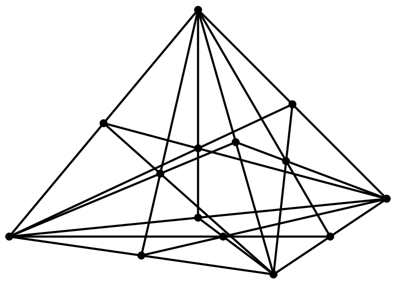
Notation 1.7 (Restriction of an equivariant vector bundle)**.** Let a compact Lie group G act continuously on a topological space X. For a subset A of X, let G_A be the maximal subgroup of G preserving A. For an equivariant vector bundle E over X, the restriction $E|_A$ of E over A is preserved by the G_A-action.



(a) $\mathcal{K}$



(b) Sd $\mathcal{K}$



(c) (Sd $\mathcal{K}$)$^{(1)}$

FIGURE 1.2. Example 1.6

So, we can endow the bundle $E|_A$ with the structure of a G_A-vector bundle, and denote it by E_A. Especially for a one point subset $\{x\} \subset X$, we denote $E_{\{x\}}$ simply by E_x, which is just a G_x-representation and called the *isotropy representation* of E at x.

The notation on the restriction of an equivariant vector bundle is not conventional, but we use it persistently for its convenience. If a reader comes across a strange or unfamiliar notation in the paper, it would probably be the notation.

Before we go further, we state a basic lemma on two properties of isotropy representations of an equivariant vector bundle. For a proof, see Section 2.

Lemma 1.8. *Let a compact Lie group G act continuously on a topological space X, and let E be an equivariant vector bundle over X.*

(1) *If a closed subgroup H of G fixes a path-connected subset A of X, then*

$$\operatorname{res}_H^{G_x} E_x \;\cong\; \operatorname{res}_H^{G_y} E_y \qquad \text{for any } x, y \in A.$$

(2) *For any* $x \in X$ *and* $g \in G$,

$$E_{g \cdot x} \;\cong\; {}^{g}E_x,$$

where the superscript ^{g} *means a conjugate representation.*

Let $\mathcal{Z}$ be the subset of $|\Gamma|$ consisting of vertices and midpoints of edges of $|\Gamma|$, which is endowed with the G-action induced from the G-action on $|\Gamma|$. Then, each bisegment of $|\Gamma|$ can be expressed as $[x, x']$ for some $x, x' \in \mathcal{Z}$ by Notation 1.3. We will show that the data of an equivariant vector bundle over $|\Gamma|$ is concentrated on its restriction over $\mathcal{Z}$. For this, we define the following restriction map:

$$\mathrm{res}_{\mathcal{Z}} : \mathrm{Vect}_G(|\Gamma|) \longrightarrow \mathrm{Vect}_G(\mathcal{Z}), \quad E \longmapsto E_{\mathcal{Z}}.$$

Proposition 1.9 (Uniqueness)**.** *The map* $\mathrm{res}_{\mathcal{Z}}$ *is injective.*

That is, if two bundles in $\mathrm{Vect}_G(|\Gamma|)$ coincide over $\mathcal{Z}$, then they are equivariantly isomorphic. By this uniqueness, we can prove a result on equivariant triviality of an equivariant vector bundle.

Definition 1.10 (Equivariant triviality)**.** Let a compact Lie group G act continuously on a topological space X. An equivariant vector bundle E over X is called *equivariantly trivial* if E is equivariantly isomorphic to the equivariant vector bundle $X \times W$ over X for some G-representation W.

Corollary 1.11 (Triviality)**.** *A bundle* E *in* $\mathrm{Vect}_G(|\Gamma|)$ *is equivariantly trivial, i.e.,* $E \cong |\Gamma| \times W$ *for some* G*-representation* W*, if and only if there exists a* G*-representation* W *such that* $\mathrm{res}^G_{G_x} W \cong E_x$ *for each* $x \in \mathcal{Z}$.

To describe the image of $\mathrm{res}_{\mathcal{Z}}$, we define the following set:

Definition 1.12. We denote by $\mathrm{IR}_G(|\Gamma|)$ the set of elements $(W_x)_{x \in \mathcal{Z}}$ in the product $\prod_{x \in \mathcal{Z}} \mathrm{Rep}(G_x)$ satisfying the following two conditions:

(1) for any bisegment $[x, x']$ of $|\Gamma|$,

$$\mathrm{res}^{G_x}_{G_{[x,x']}} W_x \;\cong\; \mathrm{res}^{G_{x'}}_{G_{[x,x']}} W_{x'},$$

(2) for any $g \in G$ and $x \in \mathcal{Z}$,

$$W_{g \cdot x} \;\cong\; {}^{g}W_x,$$

where IR is the initial of isotropy representation. And, we define the following map:

$$\mathrm{ir}_{\mathcal{Z}} : \mathrm{Vect}_G(|\Gamma|) \longrightarrow \mathrm{IR}_G(|\Gamma|), \quad E \longmapsto (E_x)_{x \in \mathcal{Z}}.$$

This map is well-defined by Lemma 1.8.

Now, we can state our main result.

Theorem 1.13. *The map* $\mathrm{ir}_{\mathcal{Z}}$ *is bijective.*

Theorem 1.13 reduces the classification problem of equivariant vector bundles over a simple graph to calculation of representations. As an example, we will show how to classify equivariant vector bundles over a circle which is equivariantly homeomorphic to a graphical action in Section 5. So, our main result could be considered as a generalization of [3].

The paper is organized as follows: we introduce some preliminaries in Section 2, and prove Proposition 1.9 and Theorem 1.13 in Section 3. In Section 4, we introduce the terminology representation extension and some results about it. By using those results, we classify equivariant vector bundles over a circle which is equivariantly homeomorphic to a graphical action in Section 5.

2. Preliminary

In this section, we introduce some preliminaries. We start with group action.

Definition 2.1 (Group action)**.** A compact Lie group G *acts continuously* on a topological space X if there exists a continuous function

$$\Phi : G \times X \longrightarrow X, \quad (g, x) \longmapsto \Phi(g, x)$$

satisfying the following:

- $\Phi\Big(g, \Phi(h, x)\Big) = \Phi(gh, x)$ for any $g, h \in G$ and $x \in X$,
- $\Phi(\mathrm{id}, x) = x$ for any $x \in X$ and the identity element id of G.

We denote $\Phi(g, x)$ simply as $g \cdot x$ or gx. A topological space X is called a *G-space* if G acts continuously on X.

In this section, we assume that a compact Lie group G acts continuously on a topological space X. For a subgroup H of G and a subset A of X, the subgroup H *fixes* A or *preserves* A if

$$h \cdot a = a \quad \text{or} \quad h \cdot a \in A \qquad \text{for each } h \in H \text{ and } a \in A,$$

respectively. For any point $x \in X$, the *stabilizer subgroup* of G at x is

$$G_x := \big\{ g \in G \,|\, gx = x \big\}.$$

The *G-orbit* of $x \in X$ is

$$G \cdot x := \big\{ g \cdot x \in X \,|\, g \in G \big\}.$$

If a G-orbit is equal to the whole space X, then the G-action is called *transitive*. The G-action on X is called *faithful* or *effective* if the kernel of the G-action is trivial, i.e., $\langle \mathrm{id} \rangle$, where the *kernel* of the G-action on X is the maximal subgroup of G fixing the whole space X. In this section, we do not assume that the G-action on X is effective. For G-spaces X and Y, a continuous map $\eta : X \longrightarrow Y$ is called *equivariant* if

$$\eta(g \cdot x) = g \cdot \eta(x) \qquad \text{for each } g \in G \text{ and } x \in X.$$

Group representation is a special type of group actions on vector spaces.

Definition 2.2 (Group representation)**.** A continuous action of a compact Lie group G on a finite-dimensional complex vector space W is called a *G-representation* if each element of G acts on W through a linear map.

A G-representation W is called *irreducible* if there is no subspace invariant under the G-action other than $\{0\}$ and W itself. For a G-representation W and a closed subgroup H of G, the representation W with the restricted H-action is denoted by $\operatorname{res}^G_H W$. For two G-representations W and V, a linear map $\eta : W \longrightarrow V$ is called an *equivariant isomorphism* or *G-isomorphism* if it is a linear isomorphism and also equivariant, and in this case we use the notation $W \cong V$. Denote by $\operatorname{iso}_G(W, V)$ the set of equivariant isomorphisms from W to V. Especially, we denote $\operatorname{iso}_G(W, W)$ simply by $\operatorname{iso}_G(W)$. Let $\operatorname{Rep}(G)$ be the set of equivariant isomorphism classes of G-representations. For a G-representation W, the function $\chi : G \longrightarrow \mathbb{C}$ sending each $g \in G$ to the trace of the linear map from W to itself determined by the action of g is called the *character* of W. It is well-known that two G-representations are equivariantly isomorphic if and only if their characters are the same.

Next, we introduce equivariant vector bundle.

Definition 2.3 (Equivariant vector bundle)**.** For a compact Lie group G and a G-space X, a (topological complex) vector bundle $p : E \longrightarrow X$ over X is called an *equivariant vector bundle* or a *G-vector bundle* over X if the following hold:

- G acts continuously on E so that the projection p is equivariant,
- the group action is fiberwise linear, i.e., the map

  $$p^{-1}(x) \longrightarrow p^{-1}(g \cdot x), \quad u \longmapsto g \cdot u \quad \text{for } u \in p^{-1}(x)$$

  is linear for any $x \in X$ and $g \in G$.

Now, we can obtain a proof of Lemma 1.8(1) by the following lemma:

Lemma 2.4. *Let E be an equivariant vector bundle over a G-space X. If X is path-connected and G acts trivially on X, i.e., G fixes the whole space X, then any isotropy representations of E are G-isomorphic to each other.*

Proof. See [5, p. 68]. □

Before we prove Lemma 1.8(2), we define a conjugate representation.

Definition 2.5 (Conjugate representation)**.** Let H be a closed (not necessarily normal) subgroup of a compact Lie group G. For a given element $g \in G$ and an H-representation W, the gHg^{-1}-representation ${}^gW$ is defined to be the vector space W with the new gHg^{-1}-action

$$gHg^{-1} \times W \longrightarrow W, \quad (k, u) \longmapsto (g^{-1}kg) \cdot u$$

for $k \in gHg^{-1}$ and $u \in W$.

Now, we prove Lemma 1.8(2).

Lemma 2.6. *Let E be an equivariant vector bundle over a G-space X. For any $x \in X$ and $g \in G$, the isotropy representation $E_{g\cdot x}$ is equivariantly isomorphic to ${}^gE_x$.*

Proof. We note that the stabilizer subgroup $G_{g\cdot x}$ is equal to gG_xg^{-1}, so $E_{g\cdot x}$ is a gG_xg^{-1}-representation. To avoid confusion between the G-action on E and the gG_xg^{-1}-action on ${}^gE_x$, we denote $k \cdot u$ for $k \in gG_xg^{-1}$ and $u \in {}^gE_x$ by $k \star u$. We define the following map:

$$\eta : {}^gE_x \longrightarrow E_{g\cdot x}, \quad u \longmapsto g \cdot u \qquad \text{for } u \in E_x.$$

Then,

$$\begin{aligned}\eta\Big((ghg^{-1}) \star u\Big) &= \eta(hu) \\ &= ghu \\ &= ghg^{-1}gu \\ &= ghg^{-1}\eta(u)\end{aligned}$$

for any $u \in E_x$ and $h \in G_x$, so η is an equivariant isomorphism. □

We introduce some terminologies on maps between equivariant vector bundles.

Definition 2.7 (Fiberwise isomorphism and isomorphism)**.** For G-spaces X and Y, a continuous map η from a G-vector bundle E over X to a G-vector bundle F over Y is called a *fiberwise isomorphism* if the restriction of η to each fiber of E is an (inequivariant) linear isomorphism to a fiber of F. Moreover, a fiberwise isomorphism η is called an *isomorphism* if $X = Y$ and the restriction of η to each fiber is an (inequivariant) linear isomorphism to itself.

Since E and F are equivariant vector bundles, an *equivariant fiberwise isomorphism* and an *equivariant isomorphism* (or a *G-isomorphism*) from E to F are also defined.

For G-vector bundles E and F over X, we use the notation $E \cong F$ if there is an equivariant isomorphism between them. We denote by $\mathrm{iso}_G(E, F)$ the set of equivariant isomorphisms from E to F, and denote $\mathrm{iso}_G(E, E)$ simply by $\mathrm{iso}_G(E)$. We denote by $\mathrm{Vect}_G(X)$ the set of equivariant isomorphism classes of equivariant vector bundles over a G-space X, and the set $\mathrm{Vect}_G(X)$ is an abelian semigroup under the Whitney sum.

Now, we state a well-known fact on $\mathrm{Vect}_G(X)$.

Lemma 2.8. *Let a compact Lie group G act continuously on compact Hausdorff spaces X and Y. If $\eta : X \longrightarrow Y$ is an equivariant homotopy equivalence, then the following map is bijective*:

$$\eta^* : \mathrm{Vect}_G(Y) \longrightarrow \mathrm{Vect}_G(X), \qquad E \longmapsto \eta^*E,$$

where η^*E is the pullback vector bundle of E by η.

Proof. This is an equivariant version of [1, Lemma 1.4.4(1)]. Proof for the lemma with respect to a finite group G can be found in [1, p. 40]. To prove the lemma with respect to a compact Lie group G, we only have to generalize averaging operator, which is defined with respect to a finite group in [1], to a compact Lie group. This is possible by invariant integration (or invariant Haar measure). For invariant integration, see [6, Section 2.7] for example. Therefore, we obtain a proof. □

3. Proof

In this section, we prove Proposition 1.9 and Theorem 1.13. To prove Proposition 1.9, we need the following lemma:

Lemma 3.1. *Let a compact group G act trivially on the interval* $[0, 1]$. *Then, any equivariant vector bundle E over* $[0, 1]$ *is equivariantly trivial. For any* $\eta_0 \in \mathrm{iso}_G(E_0)$ *and* $\eta_1 \in \mathrm{iso}_G(E_1)$, *there exists a G-isomorphism η of E such that* $\eta|_{E_x} = \eta_x$ *for* $x = 0, 1$, *where E_x is the isotropy representation of E at x.*

Proof. Since the interval $[0, 1]$ is equivariant homotopy equivalent to a one point set, the bundle E is equivariant trivial by Lemma 2.8. So, we may assume that $E = [0, 1] \times W$ for some G-representation W. Since $G_x = G$ for any $x \in [0, 1]$, the set $\mathrm{iso}_G(E)$ can be considered as the set of continuous maps from $[0, 1]$ to $\mathrm{iso}_G(W)$. So, we only have to prove that $\mathrm{iso}_G(W)$ is path-connected to obtain a proof of the lemma. By Schur's lemma, $\mathrm{iso}_G(W)$ is homeomorphic to a product of complex general linear groups, see [2, Theorem II.1.10 and Exercise II.1.16.9 of p. 72.] for Schur's lemma. More precisely, if W is G-isomorphic to

$$l_1 U_1 \oplus \cdots \oplus l_m U_m$$

for natural numbers l_i's and irreducible G-representations U_i's satisfying $U_i \not\cong U_j$ for $i \neq j$, then $\mathrm{iso}_G(W)$ is homeomorphic to

$$\mathrm{GL}(l_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(l_m, \mathbb{C}).$$

And, it is well-known that any complex general linear group is path-connected. So, $\mathrm{iso}_G(W)$ is path-connected. Therefore, we obtain a proof. □

Proof of Proposition 1.9. Let E and E' be two equivariant vector bundles over $|\Gamma|$ such that $\mathrm{res}_{\mathcal{Z}}(E) \cong \mathrm{res}_{\mathcal{Z}}(E')$. We will construct an equivariant isomorphism $\eta : E \longrightarrow E'$ to obtain a proof.

First, we construct an equivariant isomorphism from $E_{\mathcal{Z}}$ to $E'_{\mathcal{Z}}$. We only have to do it over an arbitrary G-orbit of $\mathcal{Z}$. Pick a point x in an arbitrary G-orbit of $\mathcal{Z}$. By assumption, there exists an equivariant isomorphism $\eta_x : E_x \longrightarrow E'_x$. By using this, we can construct an inequivariant isomorphism

$\eta_{gx} : E_{gx} \longrightarrow E'_{gx}$ for any $g \in G$ satisfying the following commuting diagram:

$$\begin{array}{ccc} E_x & \xrightarrow{\eta_x} & E'_x \\ {\scriptstyle g\cdot}\downarrow & & \downarrow{\scriptstyle g\cdot} \\ E_{gx} & \xrightarrow{\eta_{gx}} & E'_{gx}. \end{array}$$

It can be checked that (1) η_{gx} is well-defined, i.e., if $gx = g'x$ for some $g, g' \in G$, then $\eta_{gx} = \eta_{g'x}$, and (2) η_{gx} is equivariant. Then, we can define an inequivariant isomorphism from $E_{G\cdot x}$ to $E'_{G\cdot x}$ as follows:

$$\eta_{G\cdot x} : E_{G\cdot x} \longrightarrow E'_{G\cdot x}, \quad u \longmapsto \eta_y(u) \qquad \text{if } u \in E_y \text{ for some } y \in G \cdot x.$$

And, we can check that $\eta_{G\cdot x}$ is equivariant by definition. Therefore, we obtain an equivariant isomorphism $\eta_{\mathcal{Z}} : E_{\mathcal{Z}} \longrightarrow E'_{\mathcal{Z}}$.

Next, we extend $\eta_{\mathcal{Z}}$ over the whole $|\Gamma|$. We only have to extend it over the G-orbit of an arbitrary bisegment of $|\Gamma|$. Pick an arbitrary bisegment $[x, x']$ of $|\Gamma|$. First, we have $E_{[x,x']} \cong E'_{[x,x']}$ because

$$\begin{aligned} E_{[x,x']} &\cong [x, x'] \times \mathrm{res}^{G_z}_{G_{[x,x']}} E_z && \text{for } z \in \{x, x'\} \text{ by Lemma 3.1} \\ &\cong [x, x'] \times \mathrm{res}^{G_z}_{G_{[x,x']}} E'_z && \text{by } E_z \cong E'_z \\ &\cong E'_{[x,x']} && \text{by Lemma 3.1.} \end{aligned}$$

Then, there exists a $G_{[x,x']}$-isomorphism

$$\eta_{[x,x']} : E_{[x,x']} \longrightarrow E'_{[x,x']}$$

such that $\eta_{[x,x']} = \eta_{\mathcal{Z}}$ on E_x and $E_{x'}$ by Lemma 3.1. By using this, we can define a $G_{g[x,x']}$-isomorphism

$$\eta_{g[x,x']} : E_{g[x,x']} \longrightarrow E'_{g[x,x']}$$

for any $g \in G$ such that $\eta_{g[x,x']} = \eta_{\mathcal{Z}}$ on E_{gx} and $E_{gx'}$ in the same way we defined η_{gx} from η_x. So, we can define a G-isomorphism from $E_{G\cdot[x,x']}$ to $E'_{G\cdot[x,x']}$ as follows:

$$\eta_{G\cdot[x,x']} : E_{G\cdot[x,x']} \longrightarrow E'_{G\cdot[x,x']}, \ \ u \longmapsto \eta_{g[x,x']}(u) \ \ \text{if } u \in E_{g[x,x']} \text{ for some } g \in G$$

such that $\eta_{G\cdot[x,x']} = \eta_{\mathcal{Z}}$ over $G \cdot [x, x'] \cap \mathcal{Z}$. Since we have extended $\eta_{\mathcal{Z}}$ over the G-orbit of an arbitrary bisegment of $|\Gamma|$, we obtain a proof. □

To prove Theorem 1.13, we need two lemmas.

Lemma 3.2. *For each element in $(W_x)_{x\in\mathcal{Z}} \in \mathrm{IR}_G(|\Gamma|)$, there exists a unique equivariant vector bundle F over $\mathcal{Z}$ such that $F_x \cong W_x$ for each $x \in \mathcal{Z}$.*

Proof. For existence, it suffices to construct an equivariant vector bundle F over an arbitrary G-orbit of $\mathcal{Z}$ such that $F_x \cong W_x$ for each x in the G-orbit.

Pick an arbitrary point x_0 in $\mathcal{Z}$. We consider the equivariant vector bundle

$$G \times_{G_{x_0}} W_{x_0} \longrightarrow G \cdot x_0, \quad [g, u] \longmapsto gx_0 \qquad \text{for } g \in G \text{ and } u \in W_{x_0}$$

over the orbit $G \cdot x_0$, where the G-action on the bundle is defined by

$$g' \cdot [g, u] := [g'g, u] \quad \text{for any } g, g' \in G \text{ and } u \in W_{x_0}.$$

Call it F. By definition, $F_{x_0} \cong W_{x_0}$. We can check

$$\begin{aligned} F_{gx_0} &\cong {}^gF_{x_0} && \text{by Lemma 2.6} \\ &\cong {}^gW_{x_0} && \text{by } F_{x_0} \cong W_{x_0} \\ &\cong W_{gx_0} && \text{by definition of } \mathrm{IR}_G(|\Gamma|) \end{aligned}$$

for each $g \in G$. Therefore, we obtain a proof for existence.

The proof of uniqueness is omitted because it is exactly the same with the former part of proof of Proposition 1.9. □

The next lemma deals with how to glue an equivariant vector bundle over a finite set so as to obtain another equivariant vector bundle. To state the lemma, we consider a surjective equivariant fiberwise isomorphism $\eta : E \longrightarrow F$ from a G-vector bundle $p : E \longrightarrow X$ over a finite G-space X to a G-vector bundle $q : F \longrightarrow Y$ over a finite G-space Y. Let $f : X \longrightarrow Y$ be the function satisfying $\eta(E_x) = F_{f(x)}$ for each $x \in X$, i.e., the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\eta} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y. \end{array}$$

By using η, we define an equivariant equivalence relation $\sim$ on E as follows:

$$u \sim u' \quad \text{if } \eta(u) = \eta(u') \qquad \text{for } u, u' \in E,$$

where an equivalence relation $\sim'$ on a G-space Z is *equivariant* if

$$x \sim' y \ \text{ for } x, y \in Z \quad \Longrightarrow \quad gx \sim' gy \ \text{ for any } g \in G.$$

Here, we can check that if $\sim'$ is equivariant, the quotient space $Z/\sim'$ is endowed with a unique continuous G-action such that the quotient map $Z \longrightarrow Z/\sim'$ is equivariant. In these setting, we prove the following lemma:

Lemma 3.3. *The quotient space $E/\sim$ is endowed with the structure of a unique G-vector bundle over Y such that the quotient map*

$$P : E \longrightarrow E/\sim$$

is an equivariant fiberwise isomorphism, where the map

$$r : E/\sim \longrightarrow Y, \quad [u] \longmapsto q(\eta(u)) \qquad \textit{for } u \in E$$

is the bundle projection of $E/\sim$. And, the G-vector bundle $E/\sim$ is G-isomorphic to F.

Proof. First, we forget group actions on E and F. Since $u \sim u'$ implies $\eta(u) = \eta(u')$ by definition, the map η is factored into the composition of P and a map ι by the universal property of the quotient as the following commuting diagram:

$$\begin{array}{ccc} E & \xrightarrow{\ \eta\ } & F\,. \\ {\scriptstyle P}\big\downarrow & \nearrow{\scriptstyle \iota} & \\ E/\sim & & \end{array}$$

In the diagram, we note that

$$P(u) = P(u'), \text{ i.e., } u \sim u' \quad \Longleftrightarrow \quad \eta(u) = \eta(u') \qquad \text{for } u, u' \in E$$

by definition of $\sim$ and P. Moreover, since both P and η are surjective, we can check that ι is bijective.

Now, we show that the quotient space $E/\sim$ is endowed with the structure of a unique (inequivariant) vector bundle over Y such that the quotient map P is an (inequivariant) fiberwise isomorphism. If there exists a vector bundle structure on $E/\sim$ such that P is a fiberwise isomorphism, then it is unique because $P|_{E|_x}$ for each $x \in X$ should be an isomorphism between two fibers. Also, we can endow $q \circ \iota : E/\sim \longrightarrow Y$ with a vector bundle structure through ι so that ι becomes an (inequivariant) isomorphism. Then, $q \circ \iota$ is equal to the bundle projection r of the lemma, and P should be a fiberwise isomorphism by the diagram.

Second, we consider group actions on E and F. We recall that $E/\sim$ is endowed with a unique G-action such that P is equivariant because $\sim$ is equivariant. Also, we can prove that ι is equivariant with respect to the G-action on $E/\sim$ as follows:

$$\begin{aligned} \iota\big(gP(u)\big) &= \iota\big(P(gu)\big) && \text{by equivariance of } P \\ &= \eta(gu) && \text{by the diagram} \\ &= g\eta(u) && \text{by equivariance of } \eta \\ &= g\iota\big(P(u)\big) && \text{by the diagram} \end{aligned}$$

for any $g \in G$ and $u \in E$, i.e., $\iota(gu') = g\iota(u')$ for any $g \in G$ and $u' \in E/\sim$ because P is surjective.

Now, we prove the first statement. Since ι is equivariant and also an inequivariant isomorphism, $E/\sim$ also becomes an equivariant vector bundle. Moreover, since we have already proved that both the group action and the vector bundle structure on $E/\sim$ such that P is an equivariant fiberwise isomorphism are unique, we obtain a proof for the first statement.

The second statement follows from the first statement because ι is an equivariant isomorphism. $\square$

Remark 3.4. In [8], we investigate the topology of the set consisting of every equivariant equivalence relation $\sim$ on E such that the quotient map $E \longrightarrow$

$E/\sim$ becomes an equivariant fiberwise isomorphism in the case when Y is a one point set.

Now, we are ready to prove Theorem 1.13.

Proof of Theorem 1.13. To begin with, we consider the following commuting diagram:

$$\begin{array}{ccc} \mathrm{Vect}_G(|\Gamma|) & \xrightarrow{\mathrm{res}_{\mathcal{Z}}} & \mathrm{Vect}_G(\mathcal{Z}) \ , \\ {\scriptstyle \mathrm{ir}_{\mathcal{Z}}}\big\downarrow & \nearrow_{i} & \\ \mathrm{IR}_G(|\Gamma|) & & \end{array}$$

where the function i is defined by Lemma 3.2. Since res_Z is injective by Proposition 1.9, ir_Z is also injective by the diagram. So, we only have to show that ir_Z is surjective. For an arbitrary $(W_x)_{x\in\mathcal{Z}} \in \mathrm{IR}_G(|\Gamma|)$, we will construct an equivariant vector bundle E over $|\Gamma|$ such that $\mathrm{ir}_{\mathcal{Z}}(E) = (W_x)_{x\in\mathcal{Z}}$ to obtain a proof of surjectivity.

First, we would consider $|\Gamma|$ as obtained from the disjoint union of edges of $|\Gamma|$ by gluing. Let $\bar{\Gamma} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ be the simple graph of the disjoint union of edges of $\Gamma = (\mathcal{V}, \mathcal{E})$, so the number of vertices of $\bar{\Gamma}$ is twice the number of edges of $\bar{\Gamma}$, see Figure 3.1 for an example. Then, $|\Gamma|$ can be considered as a quotient space of $|\bar{\Gamma}|$, and we denote the quotient map by $\pi : |\bar{\Gamma}| \longrightarrow |\Gamma|$. We can endow the simple graph $\bar{\Gamma}$ with a unique graphical G-action induced from the G-action on Γ so that π is equivariant. Let $\mathcal{M}$ and $\bar{\mathcal{M}}$ be the sets of midpoints of edges of Γ and $\bar{\Gamma}$, which are endowed with G-actions induced from $|\Gamma|$ and $|\bar{\Gamma}|$, respectively.

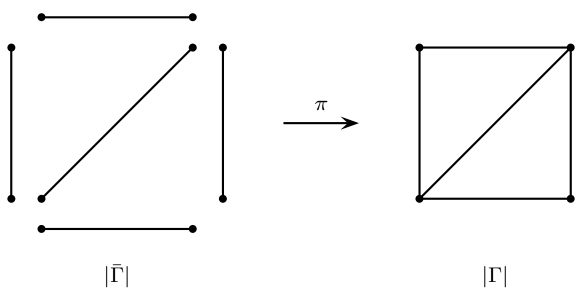


FIGURE 3.1. The quotient map $\pi : |\bar{\Gamma}| \longrightarrow |\Gamma|$

We will obtain a desired bundle E over $|\Gamma|$ from an equivariant vector bundle over $|\bar{\Gamma}|$ by gluing. So, we need first construct an equivariant vector bundle over $|\bar{\Gamma}|$. Since the projection map $p : |\bar{\Gamma}| \longrightarrow \bar{\mathcal{M}}$ is an equivariant homotopy equivalence, the map

$$p^* : \mathrm{Vect}_G(\bar{\mathcal{M}}) \longrightarrow \mathrm{Vect}_G(|\bar{\Gamma}|), \qquad L \longmapsto p^*L$$

is bijective by Lemma 2.8, where we note

$$L_x \cong (p^*L)_x \qquad \text{for each } x \in \bar{\mathcal{M}}. \tag{3.1}$$

So, to construct an equivariant vector bundle over $|\bar{\Gamma}|$, we only have to consider an equivariant vector bundle over $\bar{\mathcal{M}}$. Moreover, since $\pi|_{\bar{\mathcal{M}}} : \bar{\mathcal{M}} \longrightarrow \mathcal{M}$ is an equivariant homeomorphism, the map

$$\left(\pi\big|_{\bar{\mathcal{M}}}\right)^* : \mathrm{Vect}_G\left(\mathcal{M}\right) \longrightarrow \mathrm{Vect}_G(\bar{\mathcal{M}}), \qquad L \longmapsto \left(\pi\big|_{\bar{\mathcal{M}}}\right)^* L$$

is bijective by Lemma 2.8, where we note that

$$\text{(i) } G_x = G_{\pi(x)} \quad \text{and} \quad \text{(ii) } \left(\left(\pi\big|_{\bar{\mathcal{M}}}\right)^* L\right)_x \cong L_{\pi(x)} \qquad \text{for each } x \in \bar{\mathcal{M}}.$$

So, we only have to consider an equivariant vector bundle over $\mathcal{M}$ to construct an equivariant vector bundle over $|\bar{\Gamma}|$. By Lemma 3.2, there exists an equivariant vector bundle F over $\mathcal{Z}$ such that $F_y \cong W_y$ for each $y \in \mathcal{Z}$. Then, the equivariant vector bundle $F_{\mathcal{M}}$ over $\mathcal{M}$ gives the equivariant vector bundle $p^* \circ \left(\pi\big|_{\bar{\mathcal{M}}}\right)^* F_{\mathcal{M}}$ over $|\bar{\Gamma}|$, and we denote it by J. And, let $q : J \longrightarrow |\bar{\Gamma}|$ be the bundle projection of J. The G-vector bundle J satisfies

$$J_x \cong F_{\pi(x)} \qquad \text{for any } x \in \bar{\mathcal{M}} \tag{3.2}$$

by (ii) and (3.1).

Before we go further, we investigate isotropy representation J_v for an arbitrary vertex v of $|\bar{\Gamma}|$. Denote by e the edge of $\bar{\Gamma}$ containing v. Noting that

$$G_v = G_{[v,\mu(e)]} = G_{[\pi(v),\pi(\mu(e))]} \quad \text{and} \quad G_{\mu(e)} = G_{\pi(\mu(e))}, \tag{3.3}$$

we have the following:

$$\begin{aligned} J_v &\cong \mathrm{res}^{G_v}_{G_{[v,\mu(e)]}} J_v && \text{by (3.3)} \\ &\cong \mathrm{res}^{G_{\mu(e)}}_{G_{[v,\mu(e)]}} J_{\mu(e)} && \text{by Lemma 1.8(1)} \\ &\cong \mathrm{res}^{G_{\mu(e)}}_{G_{[v,\mu(e)]}} F_{\pi(\mu(e))} && \text{by (3.2)} \\ &\cong \mathrm{res}^{G_{\pi(\mu(e))}}_{G_{[\pi(v),\pi(\mu(e))]}} F_{\pi(\mu(e))} && \text{by (3.3)} \\ &\cong \mathrm{res}^{G_{\pi(v)}}_{G_{[\pi(v),\pi(\mu(e))]}} F_{\pi(v)} && \text{by Lemma 1.8(1).} \end{aligned}$$

So, there exists a G_v-isomorphism

$$\eta_v : J_v \longrightarrow F_{\pi(v)} \tag{3.4}$$

for an arbitrary vertex v of $|\bar{\Gamma}|$.

In order to obtain an equivariant vector bundle over $|\Gamma|$ from J by gluing, we need construct an equivariant fiberwise isomorphism $\eta : J_{\bar{\mathcal{V}}} \to F_{\mathcal{V}}$ satisfying

$$\eta(J_v) = F_{\pi(v)} \qquad \text{for each } v \in \bar{\mathcal{V}},$$

i.e., satisfying the following commuting diagram:

$$\begin{array}{ccc} J_{\bar{\mathcal{V}}} & \xrightarrow{\ \eta\ } & F_{\mathcal{V}} \\ {\scriptstyle q|_{J_{\bar{\mathcal{V}}}}}\downarrow & & \downarrow{\scriptstyle \text{bundle projection}} \\ \bar{\mathcal{V}} & \xrightarrow{\ \pi|_{\bar{\mathcal{V}}}\ } & \mathcal{V}. \end{array} \tag{3.5}$$

We only have to construct η for fibers over the G-orbit $G \cdot v$ of an arbitrary vertex $v \in \bar{\mathcal{V}}$. We can do it by using η_v of (3.4) as in the proof of Proposition 1.9. So, there is a wanted equivariant fiberwise isomorphism $\eta : J_{\bar{\mathcal{V}}} \to F_{\mathcal{V}}$.

Now, we define an equivariant equivalence relation $\sim$ on J generated by the following relations:

$$u \sim u' \quad \text{if } \eta(u) = \eta(u') \quad \text{for } u, u' \in J_{\bar{\mathcal{V}}},$$

where we note that there are only trivial relations outside $J_{\bar{\mathcal{V}}}$, i.e., $u \sim u$ for $u \in J - J_{\bar{\mathcal{V}}}$. So, if two distinct vectors u, u' in J satisfy $u \sim u'$, then $u, u' \in J_{\bar{\mathcal{V}}}$ and $\eta(u) = \eta(u')$, so we have

$$(\pi \circ q)(u) = (\pi \circ q)(u') \tag{3.6}$$

by the diagram (3.5). Moreover, since (3.6) holds for any $u, u' \in J$ such that $u \sim u'$, the map $\pi \circ q$ is factored into the composition of the quotient map $\Pi : J \longrightarrow J/\sim$ and some map q' by the universal property of the quotient as the following commuting diagram:

$$\begin{array}{ccc} J & \xrightarrow{\ \Pi\ } & J/\sim \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ |\bar{\Gamma}| & \xrightarrow{\ \pi\ } & |\Gamma|. \end{array}$$

Next, we would endow $q' : J/\sim \longrightarrow |\Gamma|$ with the structure of a unique equivariant vector bundle over $|\Gamma|$ such that the map Π is an equivariant fiberwise isomorphism. First, we endow $J/\sim$ with the structure of a unique (inequivariant) vector bundle over $|\Gamma|$ such that the map Π is an (inequivariant) fiberwise isomorphism. This is easy because $J/\sim$ is obtained from vector bundles over intervals (i.e., edges of $|\bar{\Gamma}|$) by gluing through linear maps (i.e., the fiberwise isomorphism). Second, we endow $J/\sim$ with a unique G-action such that the map Π is equivariant. This is possible because $\sim$ is equivariant. Moreover, since J is an equivariant vector bundle and the gluing is done through an equivariant fiberwise isomorphism, the G-action on $J/\sim$ is fiberwise linear, so $J/\sim$ becomes an equivariant vector bundle, call it E.

Now, we investigate isotropy representations of E at vertices of $|\Gamma|$. For this, we consider the equivalence relation $\sim$ restricted to $J_{\bar{\mathcal{V}}}$, call it $\sim_{\bar{\mathcal{V}}}$. Then, $J_{\bar{\mathcal{V}}}/\sim_{\bar{\mathcal{V}}}$ can be endowed with the structure of a unique G-vector bundle such that the quotient map

$$J_{\bar{\mathcal{V}}} \longrightarrow J_{\bar{\mathcal{V}}}/\sim_{\bar{\mathcal{V}}}$$

is an equivariant fiberwise isomorphism and the G-vector bundle $J_{\bar{\mathcal{V}}}/\sim_{\bar{\mathcal{V}}}$ is G-isomorphic to $F_{\mathcal{V}}$ by Lemma 3.3 because $\sim_{\bar{\mathcal{V}}}$ is defined by the equivariant fiberwise isomorphism $\eta : J_{\bar{\mathcal{V}}} \to F_{\mathcal{V}}$. Especially, we have

$$(J_{\bar{\mathcal{V}}}/\sim_{\bar{\mathcal{V}}})_v \cong F_v \quad \text{ for any } v \in \mathcal{V}.$$

Moreover, since $E_{\mathcal{V}}$ is G-isomorphic to $J_{\bar{\mathcal{V}}}/\sim_{\bar{\mathcal{V}}}$ by definition of E, we obtain

$$E_v \cong F_v \qquad \text{for each } v \in \mathcal{V}. \tag{3.7}$$

Last, we check that $E_y \cong W_y$ for any $y \in \mathcal{Z}$. First, we have the following for each $x \in \bar{\mathcal{M}}$:

$$\begin{aligned} E_{\pi(x)} &\cong J_x && \text{by definition of } E \\ &\cong F_{\pi(x)} && \text{by (3.2)} \\ &\cong W_{\pi(x)} && \text{by definition of } F, \end{aligned}$$

i.e., $E_y \cong W_y$ for any $y \in \mathcal{M}$. Second, we show $E_y \cong W_y$ for any $y \in \mathcal{V}$. We already have $E_y \cong F_y$ for each $y \in \mathcal{V}$ by (3.7). Moreover, since $F_y \cong W_y$ for each $y \in \mathcal{Z}$ by definition of F, we have $E_y \cong W_y$ for each $y \in \mathcal{V}$. Therefore, E is a wanted bundle, and we obtain a proof. □

4. Representation extension

In order to apply our main result to the classification problem of equivariant vector bundles in Section 5, we introduce the terminology representation extension and some results about it.

Let G be a compact Lie group, and H be its closed normal subgroup. Then, a G-representation V is called a *representation extension* or a *G-extension* of an H-representation U if $\operatorname{res}_H^G V \cong U$. In the section, we deal with representation extension only in the case when G/H is isomorphic to a cyclic group $\mathbb{Z}_m$ for some natural number m.

To state results on representation extension, we introduce some notations. Let g_0 be a fixed generator of $G/H \cong \mathbb{Z}_m$, and $\Omega(l)$ for $l \in \mathbb{Z}_m$ be the one-dimensional G/H-representation defined by

$$G/H \times \mathbb{C} \longrightarrow \mathbb{C}, \quad (g_0^k, u) \longmapsto e^{2\pi k l \sqrt{-1}/m}\, u \qquad \text{for } k \in \mathbb{Z}_m \text{ and } u \in \mathbb{C}.$$

We also consider $\Omega(l)$ as a G-representation via the projection map $G \longrightarrow G/H$. Let $\operatorname{Irr}(H)$ be the set of characters of irreducible H-representations. The group G acts on $\operatorname{Irr}(H)$ as follows:

$$(g \cdot \chi)(h) := \chi(ghg^{-1}) \qquad \text{for } \chi \in \operatorname{Irr}(H),\ g \in G,\ h \in H.$$

Here, if χ is the character of an irreducible H-representation U, then we can check that $g \cdot \chi$ for $g \in G$ is the character of the conjugate representation ${}^{g^{-1}}U$. From this, we can obtain ${}^{h}U \cong U$ for any $h \in H$ because the trace function is invariant under conjugation, i.e., $h \cdot \chi = \chi$ for any $h \in H$. Sometimes χ may be fixed by the whole G, i.e., ${}^{g}U \cong U$ for any $g \in G$. In fact, if an irreducible

H-representation U has a G-extension V, then the character of U is fixed by G because the character of V is fixed by G. Moreover, the converse also holds as follows:

Theorem 4.1 ([4]). *Let G be a compact Lie group, and H be its closed normal subgroup such that $G/H \cong \mathbb{Z}_m$ for some natural number m. If the character χ of an irreducible H-representation U is fixed by G, then there exists a G-extension V of U. Furthermore, the number of mutually equivariantly non-isomorphic G-extensions of U is m, and they are $V \otimes \Omega(l)$ for $l \in \mathbb{Z}_m$.*

Proof. By [4, Theorem 3.2], U has m mutually equivariantly non-isomorphic G-extensions, call one of them V. By [4, Proposition 3.1] and its proof, each G-extension of U is expressed as $V \otimes \Omega(l)$ for some $l \in \mathbb{Z}_m$. □

Corollary 4.2. *For an irreducible H-representation U and its G-extension V, if W is a G-representation such that $\operatorname{res}_H^G W$ is a multiple of U, then W is a direct sum of $V \otimes \Omega(l)$'s.*

Proof. First, we prove that the induced representation $\operatorname{ind}_H^G U$ is isomorphic to the direct sum $\oplus_{l \in \mathbb{Z}_m} V \otimes \Omega(l)$, see [2, Section III.6] for induced representation. By Frobenius reciprocity, we have

$$\operatorname{Hom}_G\Big(V \otimes \Omega(l),\ \operatorname{ind}_H^G U\Big) \ \cong\ \operatorname{Hom}_H\Big(\operatorname{res}_H^G\ V \otimes \Omega(l),\ U\Big) \tag{4.1}$$

for each $l \in \mathbb{Z}_m$, and the right term of (4.1) is isomorphic (as vector spaces) to $\operatorname{Hom}_H(U, U)$ which is one-dimensional by Schur's lemma, see [2, Proposition III.6.2] for Frobenius reciprocity. So, the left term of (4.1) is also one-dimensional for each $l \in \mathbb{Z}_m$, and this means that there exists an equivariant injective linear map from the direct sum to the induced representation by Schur's lemma because $V \otimes \Omega(l)$'s are mutually equivariantly non-isomorphic by Theorem 4.1. However, since their complex dimensions are the same, i.e., m times the dimension of U, we obtain

$$\operatorname{ind}_H^G U \quad \cong \quad \bigoplus_{l \in \mathbb{Z}_m} V \otimes \Omega(l). \tag{4.2}$$

We may assume that W is nontrivial, i.e., $W \ncong (0)$, and that W is irreducible. Since $\operatorname{res}_H^G W$ is a multiple of U by assumption, we may put $\operatorname{res}_H^G W \cong kU$ for some natural number k because $W \ncong (0)$. Again by Frobenius reciprocity, we have

$$\operatorname{Hom}_G\big(W,\ \operatorname{ind}_H^G U\big) \ \cong\ \operatorname{Hom}_H\big(\operatorname{res}_H^G W,\ U\big). \tag{4.3}$$

Since $\operatorname{res}_H^G W \cong kU$, the right term of (4.3) is k-dimensional by Schur's Lemma, so is the left term of (4.3). And, the left term of (4.3) is isomorphic to the following:

$$\bigoplus_{l \in \mathbb{Z}_m} \operatorname{Hom}_G\Big(W,\ V \otimes \Omega(l)\Big) \tag{4.4}$$

by (4.2). But, since W is irreducible and $V \otimes \Omega(l)$'s are all mutually equivariantly non-isomorphic, (4.4) is at most one-dimensional by Schur's lemma, i.e., $k \leq 1$. Since k is a natural number, we have $k = 1$, i.e., W is a G-extension of U, so $W \cong V \otimes \Omega(l)$ for some $l \in \mathbb{Z}_m$ by Theorem 4.1. Therefore, we obtain a proof. □

5. Classification of equivariant vector bundles over a circle

Let a compact Lie group G act continuously on a circle S^1. Denote by Ker the kernel of the G-action on S^1. In [3], they reduced the classification of $\mathrm{Vect}_G(S^1)$ to the classification of $\mathrm{Vect}_{G_\chi}(S^1, \chi)$ for a character χ in each G-orbit of Irr(Ker), where G_χ is the stabilizer subgroup of G at χ and the set $\mathrm{Vect}_{G_\chi}(S^1, \chi)$ is defined as follows:

(5.1)
$$\left\{ E \in \mathrm{Vect}_{G_\chi}(S^1) \,\middle|\, \text{the character of } \mathrm{res}^{G_x}_{\mathrm{Ker}} E_x \text{ for any } x \in S^1 \text{ is a multiple of } \chi \right\}.$$

We note that 'for any x' in (5.1) can be replaced with 'for some x' by Lemma 1.8(1).

From Example 1.5, we recall graphical $\mathbb{Z}_n$- or D_n-actions (for $n \geq 3$) on a simple graph Γ and its geometric realization $|\Gamma|$ which is homeomorphic to a circle. If $G_\chi/\mathrm{Ker} \cong \mathbb{Z}_n$ or D_n for some natural number $n \geq 3$, those graphical $\mathbb{Z}_n$- or D_n-actions on $|\Gamma|$ can be considered as graphical G_χ-actions via the projection map $G_\chi \to G_\chi/\mathrm{Ker}$, respectively. In this section, we classify $\mathrm{Vect}_{G_\chi}(S^1, \chi)$ in the case when the G_χ-space S^1 is equivariantly homeomorphic to a so defined graphical G_χ-action on $|\Gamma|$. This shows how to apply our main result to the classification problem of equivariant vector bundles.

We have two cases according to G_χ/Ker. Pick a vertex v_0 and an edge e_0 of Γ. Let U be the Ker-representation whose character is χ.

5.1. The case when G_χ/Ker is isomorphic to a cyclic group $\mathbb{Z}_n$ for $n \geq 3$

By Theorem 1.13, the map
$$\mathrm{ir}_{\mathcal{Z}} : \mathrm{Vect}_{G_\chi}(|\Gamma|) \longrightarrow \mathrm{IR}_{G_\chi}(|\Gamma|)$$
is bijective. Since the stabilizer subgroup of G_χ at each point $x \in |\Gamma|$ is equal to Ker by Example 1.5, we can check
$$\mathrm{IR}_{G_\chi}(|\Gamma|) = \left\{ (W)_{x \in \mathcal{Z}} \,\middle|\, W \in \mathrm{Rep}(\mathrm{Ker}) \text{ and } {}^gW \cong W \text{ for each } g \in G_\chi \right\}$$
by definition, where $(W)_{x \in \mathcal{Z}}$ is the element $(W_x)_{x \in \mathcal{Z}}$ satisfying $W_x = W$ for each $x \in \mathcal{Z}$. So, we can also check that the injective image of $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$ under $\mathrm{ir}_{\mathcal{Z}}$ is
$$\left\{ (W)_{x \in \mathcal{Z}} \,\middle|\, W \in \mathrm{Rep}(\mathrm{Ker}, \chi) \right\}$$

by definition of $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ because χ is fixed by G_χ and hence ${}^gU \cong U$ for any $g \in G_\chi$, where $\mathrm{Rep}(K,\chi)$ for a closed subgroup $\mathrm{Ker} \subset K \subset G$ is the following:

$$\Big\{V \in \mathrm{Rep}(K) \,\Big|\, \text{the character of } \mathrm{res}^K_{\mathrm{Ker}} V \text{ is a multiple of } \chi\Big\},$$

especially $\mathrm{Rep}(\mathrm{Ker},\chi)$ is the set of multiples of U. Moreover, since each image $(W_x)_{x\in\mathcal{Z}}$ of $\mathrm{ir}_{\mathcal{Z}}$ is determined by W_{v_0}, we have the following classification.

Proposition 5.1 ([3, Theorem A and Theorem 6.1])**.** *When G_χ/Ker is isomorphic to $\mathbb{Z}_n$ for $n \geq 3$, the map*

$$\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi) \longrightarrow \mathrm{Rep}(\mathrm{Ker},\chi), \quad E \longmapsto E_{v_0}$$

is bijective.

As corollaries, we obtain the following:

Corollary 5.2 ([3, Corollary D])**.** *Each bundle in $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ is equivariantly trivial.*

Proof. Pick an arbitrary bundle E in $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$. We may assume that $E \not\cong |\Gamma|\times(0)$. Note that E_{v_0} is contained in $\mathrm{Rep}(\mathrm{Ker},\chi)$ by Proposition 5.1. So, we may put $E_{v_0} \cong kU$ for some natural number k by definition of $\mathrm{Rep}(\mathrm{Ker},\chi)$ and the assumption $E \not\cong |\Gamma|\times(0)$. Since the character χ is fixed by G_χ and hence there exists a G_χ-extension V of U by Theorem 4.1, kV is a G_χ-extension of E_{v_0}. So, the equivariantly trivial vector bundle $|\Gamma|\times kV$ and E satisfy $(|\Gamma|\times kV)_{v_0} \cong E_{v_0}$. Therefore, we obtain a proof by Proposition 5.1. □

Corollary 5.3 ([3, Theorem B])**.** *The abelian semigroup $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ is generated by one element.*

Proof. By definition, $\mathrm{Rep}(\mathrm{Ker},\chi)$ is the set of multiples of U, so it is generated by U. Since the map of Proposition 5.1 is a semigroup isomorphism, $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ is also generated by one element. □

5.2. The case when G_χ/Ker is isomorphic to a dihedral group D_n for $n \geq 3$

Since the stabilizer subgroup of G_χ at each point $x \in \mathcal{Z} \subset |\Gamma|$ satisfies $(G_\chi)_x/\mathrm{Ker} \cong \mathbb{Z}_2$ and stabilizer subgroups of G_χ at the other points in $|\Gamma|$ are equal to Ker by Example 1.5, the set $\mathrm{IR}_{G_\chi}(|\Gamma|)$ is equal to the following set:

$$\Big\{(W_x)_{x\in\mathcal{Z}} \,\Big|\, W_x \in \mathrm{Rep}\Big((G_\chi)_x\Big) \text{ for each } x \in \mathcal{Z}, \text{ and}$$
$$(1)\ \mathrm{res}^{(G_\chi)_x}_{\mathrm{Ker}} W_x \cong \mathrm{res}^{(G_\chi)_{x'}}_{\mathrm{Ker}} W_{x'} \text{ for any } x, x' \in \mathcal{Z},$$
$$(2)\ W_{gx} \cong {}^gW_x \text{ for any } g \in G_\chi \text{ and } x \in \mathcal{Z}\Big\}$$

by definition because $G_{[x,x']} = \mathrm{Ker}$ for any bisegment $[x, x']$ of $|\Gamma|$. So, we can check that the injective image of $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$ under $\mathrm{ir}_{\mathcal{Z}}$ is

$$\text{(5.2)} \qquad \Big\{ (W_x)_{x\in\mathcal{Z}} \,\Big|\, W_x \in \mathrm{Rep}\Big((G_\chi)_x, \chi\Big) \text{ for each } x \in \mathcal{Z}, \text{ and} \\ (1)\ \mathrm{res}^{(G_\chi)_x}_{\mathrm{Ker}} W_x \cong \mathrm{res}^{(G_\chi)_{x'}}_{\mathrm{Ker}} W_{x'} \text{ for any } x, x' \in \mathcal{Z}, \\ (2)\ W_{gx} \cong {}^g W_x \text{ for any } g \in G_\chi \text{ and } x \in \mathcal{Z} \Big\}$$

by definition of $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$. Here, we note that $\mathrm{res}^{G_x}_{\mathrm{Ker}} W_x$ and $\mathrm{res}^{G_{x'}}_{\mathrm{Ker}} W_{x'}$ for any $x, x' \in \mathcal{Z}$ are determined by their complex dimensions because they are multiples of U. So, the condition (1) of (5.2) can be replaced with the following:

$$\dim_{\mathbb{C}} W_x \cong \dim_{\mathbb{C}} W_{x'} \qquad \text{for any } x, x' \in \mathcal{Z}.$$

Also, each image $(W_x)_{x\in\mathcal{Z}}$ of $\mathrm{ir}_{\mathcal{Z}}$ is determined by W_{v_0} and $W_{\mu(e_0)}$ by (2) of (5.2) because the G_χ-action on $|\Gamma|$ is transitive on vertices and also on midpoints of edges. So, we have the following classification.

Proposition 5.4 ([3, Theorem A and Theorem 6.1])**.**

(a) *For each bundle E in* $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$,

$$E_{v_0} \in \mathrm{Rep}\Big((G_\chi)_{v_0}, \chi\Big) \quad \textit{and} \quad E_{\mu(e_0)} \in \mathrm{Rep}\Big((G_\chi)_{\mu(e_0)}, \chi\Big).$$

(b) *For any $W_1 \in \mathrm{Rep}\Big((G_\chi)_{v_0}, \chi\Big)$ and $W_2 \in \mathrm{Rep}\Big((G_\chi)_{\mu(e_0)}, \chi\Big)$ with the same dimension, there exists a unique bundle E in* $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$ *such that*

$$E_{v_0} \cong W_1 \quad \textit{and} \quad E_{\mu(e_0)} \cong W_2. \qquad \square$$

Now, we give four generators of the abelian semigroup $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$. First, we define the following four representation extensions of U: let V_1 and V_2 be two $(G_\chi)_{v_0}$-extensions of U, and let V_+ and V_- be two $(G_\chi)_{\mu(e_0)}$-extensions of U. Their existence is guaranteed by Theorem 4.1 because χ is fixed by G_χ (so, fixed by $(G_\chi)_{v_0}$ and $(G_\chi)_{\mu(e_0)}$). By Proposition 5.4, there exist four bundles $L_{1,+}, L_{1,-}, L_{2,+}, L_{2,-}$ in $\mathrm{Vect}_{G_\chi}(|\Gamma|, \chi)$ whose isotropy representations at v_0 and $\mu(e_0)$ satisfy the following:

$$\begin{aligned}
(L_{1,+})_{v_0} &\cong V_1 \quad \text{and} \quad (L_{1,+})_{\mu(e_0)} \cong V_+,\\
(L_{1,-})_{v_0} &\cong V_1 \quad \text{and} \quad (L_{1,-})_{\mu(e_0)} \cong V_-,\\
(L_{2,+})_{v_0} &\cong V_2 \quad \text{and} \quad (L_{2,+})_{\mu(e_0)} \cong V_+,\\
(L_{2,-})_{v_0} &\cong V_2 \quad \text{and} \quad (L_{2,-})_{\mu(e_0)} \cong V_-.
\end{aligned}$$

And, $L_{1,+} \oplus L_{2,-} \cong L_{1,-} \oplus L_{2,+}$ holds by Proposition 5.4 because

$$\begin{aligned}
(L_{1,+} \oplus L_{2,-})_{v_0} \;&\cong\; V_1 \oplus V_2 \;\cong\; (L_{1,-} \oplus L_{2,+})_{v_0} \qquad \text{and}\\
(L_{1,+} \oplus L_{2,-})_{\mu(e_0)} \;&\cong\; V_+ \oplus V_- \;\cong\; (L_{1,-} \oplus L_{2,+})_{\mu(e_0)}.
\end{aligned}$$

Then, we obtain the following:

Proposition 5.5 ([3, Theorem B]). *The abelian semigroup* $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ *is generated by*

$$L_{1,+}, \quad L_{1,-}, \quad L_{2,+}, \quad L_{2,-}$$

which satisfy $L_{1,+} \oplus L_{2,-} \cong L_{1,-} \oplus L_{2,+}$.

Proof. We will show that an arbitrary bundle E in $\mathrm{Vect}_{G_\chi}(|\Gamma|,\chi)$ is generated by those four bundles. Note that isotropy representations E_{v_0} and $E_{\mu(e_0)}$ are contained in $\mathrm{Rep}\Big((G_\chi)_{v_0},\chi\Big)$ and $\mathrm{Rep}\Big((G_\chi)_{\mu(e_0)},\chi\Big)$ by Proposition 5.4, respectively. So, E_{v_0} is generated by V_1 and V_2, and also $E_{\mu(e_0)}$ is generated by V_+ and V_- by Corollary 4.2, i.e., we have the following:

$$E_{v_0} \cong aV_1 \oplus bV_2 \qquad \text{and} \qquad E_{\mu(e_0)} \cong cV_+ \oplus dV_- \tag{5.3}$$

for some nonnegative integers a, b, c, d. Since E_{v_0} and $E_{\mu(e_0)}$ have the same dimension, we have

$$a + b = c + d. \tag{5.4}$$

Without loss of generality, we may assume that $a \geq c$ and $b \leq d$. Then, we may rewrite (5.3) as follows:

$$\begin{aligned} E_{v_0} &\cong (a-c)V_1 \oplus cV_1 \oplus bV_2 && \text{and} \\ E_{\mu(e_0)} &\cong cV_+ \oplus bV_- \oplus (d-b)V_-. \end{aligned} \tag{5.5}$$

If we put $F := cL_{1,+} \oplus bL_{2,-} \oplus (a-c)L_{1,-}$, then F satisfies

$$F_{v_0} \cong aV_1 \oplus bV_2 \qquad \text{and} \qquad F_{\mu(e_0)} \cong cV_+ \oplus dV_-$$

because $a - c = d - b$ by (5.4). So, E and F have the same isotropy representations at v_0 and $\mu(e_0)$ by (5.3). Therefore, we obtain a proof by Proposition 5.4. □

Acknowledgement. The author is supported by 2012-2013 Sabbatical Research Fund of Gyeongin National University of Education.

References

[1] M. F. Atiyah, *K-Theory*, Addison-Wesley, 1989.
[2] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985.
[3] J.-H. Cho, S. S. Kim, M. Masuda, and D. Y. Suh, *Classification of equivariant complex vector bundles over a circle*, J. Math. Kyoto Univ. **41** (2001), no. 3, 517–534.
[4] J.-H. Cho, M. Masuda, and D. Y. Suh, *Extending representations of H to G with discrete G/H*, J. Korean Math. Soc. **43** (2006), no. 1, 29–43.
[5] T. tom Dieck, *Transformation Groups*, De Gruyter Studies in Mathematics, vol. 8, De Gruyter, Berlin-New York, 1987.
[6] K. Kawakubo, *The Theory of Transformation Groups*, Oxford University Press, 1991.
[7] M. K. Kim, *Classification of equivariant vector bundles over two-sphere*, arXiv:1005.0681.
[8] ———, *Equivariant pointwise clutching maps*, arXiv:1504.00159.
[9] F. P. Peterson, *Some remarks on Chern classes*, Ann. of Math. (2) **69** (1959), 414–420.

Min Kyu Kim
Department of Mathematics Education
Gyeongin National University of Education
Incheon, 407-753, Korea
E-mail address: mkkim@kias.re.kr