J. Korean Math. Soc. **54** (2017), No. 1, pp. 177–191
https://doi.org/10.4134/JKMS.j150666
pISSN: 0304-9914 / eISSN: 2234-3008

STRUCTURES CONCERNING GROUP OF UNITS

Young Woo Chung and Yang Lee

Abstract. In this note we consider the right unit-duo ring property on the powers of elements, and introduce the concept of *weakly right unit-duo* ring. We investigate first the properties of weakly right unit-duo rings which are useful to the study of related studies. We observe next various kinds of relations and examples of weakly right unit-duo rings which do roles in ring theory.

1. Basic structure of weakly right unit-duo rings

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring and $a \in R$. We use $U(R)$ to denote the group of all units in R. Let $[a]_\ell = \{ua \mid u \in U(R)\}$ and $[a]_r = \{au \mid u \in U(R)\}$, i.e., $[a]_\ell = U(R)a$ and $[a]_r = aU(R)$. Let $J(R)$, $X(R)$, $I(R)$, and $N(R)$ denote the Jacobson radical, the set of all nonzero nonunits, the set of all idempotents, and the set of all nilpotent elements in R, respectively. $|S|$ denotes the cardinality of a subset S of R. The polynomial (power series) ring, with an indeterminate x over R, is written by $R[x]$ ($R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n), and $\mathbb{Q}$ denotes the field of rational numbers. Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $U_n(R)$), and use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere zeros. Following the literature, we write $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $R^* = R\backslash\{0\}$. We use $\oplus$ to denote the direct sum, and Q_8 denotes the quaternion group.

Due to Feller [11], a ring is called *right* (resp. *left*) duo if every right (resp. left) ideal is an ideal; a ring is called *duo* if it is both right and left duo. We see various kinds of useful results for duo rings in [6, 21, 26]. A ring R (possibly without identity) is usually called *reduced* if $N(R) = 0$. A ring (possibly without identity) is usually called *Abelian* if every idempotent is central. The class of Abelian rings is easily shown to contain reduced rings and right (left) duo rings.

Following Han et al. [14], a ring is called *right unit-duo* if $[a]_\ell \subseteq [a]_r$ for every $a \in R$. Left unit-duo rings are defined similarly. A ring is *unit-duo* if it is both

Received November 5, 2015; Revised May 4, 2016.
2010 *Mathematics Subject Classification.* 16U60, 16U80.
Key words and phrases. weakly right unit-duo ring, right unit-duo ring, group of units, Jacobson radical, Abelian ring, nilpotent element, π-regular ring, right duo ring, weakly right duo ring, group ring.

left and right unit-duo. Note that a ring R is unit-duo if and only if $[a]_\ell = [a]_r$ for each $a \in R$. Commutative rings are clearly unit-duo. Right or left unit-duo rings are Abelian by [14, Theorem 1.2(1)].

We consider next the right unit-duo ring property on the powers of elements.

Definition 1.1. A ring R is said to be *weakly right unit-duo* if for every $a \in R$ there exists $k \geq 1$, depending on a, such that $[a^k]_\ell \subseteq [a^k]_r$, i.e., $U(R)a^k \subseteq a^kU(R)$. Weakly left unit-duo rings are defined similarly. A ring is *weakly unit-duo* if it is both left and right weakly unit-duo.

A ring R is clearly (weakly) unit-duo when $U(R)$ is contained in the center of R. It is also obvious that a ring R is a division ring if and only if $X(R)$ is empty. So we have that a ring R, with $X(R)$ nonempty, is weakly right unit-duo if and only if for every $a \in X(R)$ there exists $k \geq 1$ such that $U(R)a^k \subseteq a^kU(R)$. We will use this fact freely.

We consider first a property of weakly right unit-duo rings which is related to the conjugation by units.

Lemma 1.2. (1) *Let R be a right unit-duo ring. Then $N(R) \subseteq u\{1 - v \mid v \in U(R)\}u^{-1}$ for all $u \in U(R)$.*

(2) *Let R be a weakly right unit-duo ring. Then, for every $u \in U(R)$, $N(R) \subseteq u^k\{1 - v \mid v \in U(R)\}u^{-k}$ for some $k \geq 1$.*

Proof. (1) Let $a \in N(R)$ and $u \in U(R)$. Since R is right unit-duo, there exists $v \in U(R)$ such that $(1 - a)u = uv$, noting $1 - a \in U(R)$. So $a = u(1 - v)u^{-1}$. This yields

$$N(R) \subseteq u\{1 - v \mid v \in U(R)\}u^{-1}.$$

(2) Let $a \in N(R)$ and $u \in U(R)$. Since R is weakly right unit-duo, there exists $v \in U(R)$ and $k \geq 1$ such that $(1 - a)u^k = u^kv$. So $a = u^k(1 - v)u^{-k}$, and this implies

$$N(R) \subseteq u^k\{1 - v \mid v \in U(R)\}u^{-k}.$$ □

In the proof of Lemma 1.2(2), if $u^k(1 - v)u^{-k} \in N(R)$, then $1 - v \in N(R)$ because $0 = [u^k(1 - v)u^{-k}]^n = u^k(1 - v)^nu^{-k}$ for some $n \geq 1$ (hence $(1 - v)^n = 0$). Let $R = D_n(K)$ over a division ring K, where $n \geq 2$. Then R is weakly right (left) unit-duo by Lemma 1.3(2) below. Let $E_{ij} \in R$ with $i < j$ (then $E_{ij} \in N(R)$), and $u \in U(R)$, i.e., $u \in \{(a_{ij}) \in R \mid a_{ii} \neq 0\}$. Then $E_{ij} = u^k(1 - v)u^{-k}$ for some $v \in U(R)$ and $k \geq 1$, by Lemma 1.2(2). Since $E_{ij} \neq 0$, $v \neq 1$ and so v must be of the form $\left(\begin{smallmatrix} 1 & s & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{smallmatrix}\right)$ because $1 - v \in N(R)$.

In the following we see basic properties of weakly right unit-duo rings.

Lemma 1.3. (1) *A weakly right (left) unit-duo ring is Abelian.*

(2) *A local ring with nil Jacobson radical is weakly unit-duo.*

(3) *The weakly unit-duo property is not left-right symmetric.*

Proof. (1) Let R be a weakly right unit-duo ring. For $r, e^2 = e \in R$, let $x = er(1-e)$ and $y = (1-e)re$. Then $x^2 = 0 = y^2$, and so $1-x, 1-y \in U(R)$. Since R is weakly right unit-duo, we have

$$(1-e) - er(1-e) = (1-x)(1-e) = (1-e)u, \text{ and } e - (1-e)re = (1-y)e = ev$$

for some $u, v \in U(R)$. Multiplying these equalities by $1-e$ and e on the left respectively, we get

$$1 - e = (1-e)u \text{ and } e = ev.$$

This yields $er(1-e) = 0$ and $(1-e)re = 0$, entailing $er = re$. Thus R is Abelian. The proof of the left case is similar.

(2) Let R be a local ring with $J(R)$ nil. Then $X(R) = J(R)\backslash\{0\}$. Since $J(R)$ is nil, R is obviously weakly unit-duo.

(3) There is a weakly left unit-duo ring that is not weakly right unit-duo. We use the ring and apply the argument in [18, Example 1]. Let S be the quotient field of the polynomial ring $F[t]$ with an indeterminate t over a field F, and consider the field monomorphism $\sigma : S \to S$ by $\sigma(\frac{f(t)}{g(t)}) = \frac{f(t^2)}{g(t^2)}$. Next consider the skew power series ring $R = S[[x;\sigma]]$ by σ with an indeterminate x over S, in which every element is of the form $\sum_{i=0}^{\infty} a_i x^i$, only subject to $xa = \sigma(a)x$ for $a \in S$.

Note that $U(R) = \{\sum_{i=0}^{\infty} a_i x^i \in R \mid a_0 \neq 0\}$. Let $f(x) = a_0x^k + a_1x^{k+1} + \cdots \in R$ with $a_0 \neq 0$ and $k \geq 0$, and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in U(R)$ (i.e., $b_0 \neq 0$). Letting $f(x) = h(x)x^k$, we have $h(x) = a_0 + a_1x + \cdots \in U(R)$ because $a_0 \neq 0$.

Let $h(x)^{-1} = a_0^{-1} + c_1x + \cdots$. Then

$$\begin{aligned} g(x)f(x) &= g(x)h(x)x^k = h(x)[h(x)^{-1}g(x)h(x)]x^k \\ &= h(x)[(a_0^{-1}b_0 + (a_0^{-1}b_1 + c_1\sigma(b_0))x + \cdots)h(x)]x^k \\ &= h(x)[a_0^{-1}b_0a_0 + (a_0^{-1}b_0a_1 + a_0^{-1}b_1\sigma(a_0) + c_1\sigma(b_0)\sigma(a_0))x + \cdots]x^k \\ &= h(x)[b_0 + (a_0^{-1}b_0a_1 + a_0^{-1}b_1\sigma(a_0) + c_1\sigma(b_0)\sigma(a_0))x + \cdots]x^k \\ &= h(x)x^kk(x) = f(x)k(x) \end{aligned}$$

for some $k(x) = d_0 + d_1x + \cdots \in U(R)$ (i.e., $d_0 \neq 0$) such that

$$\begin{aligned} &[b_0 + (a_0^{-1}b_0a_1 + a_0^{-1}b_1\sigma(a_0) + c_1\sigma(b_0)\sigma(a_0))x + \cdots]x^k \\ &= x^kk(x) = (\sigma^k(d_0) + \sigma^k(d_1)x + \cdots)x^k, \end{aligned}$$

noting that R is a domain. This implies that

$$b_0 = \sigma^k(d_0) \text{ and } a_0^{-1}b_0a_1 + a_0^{-1}b_1\sigma(a_0) + c_1\sigma(b_0)\sigma(a_0) = \sigma^k(d_1).$$

So it is impossible that b_0 is of the form $\frac{u(t)}{v(t)}$ such that a term of $u(t)$ or $v(t)$ is of odd degree, because $\sigma^k(d_0)$ is of the form $\frac{f(t^{2k})}{g(t^{2k})}$.

Thus, if we let $g(x) = t^{2l+1} + b_1x + \cdots$ (i.e., $b_0 = t^{2l+1}$) with $l \geq 0$, then there cannot exist such $k(x)$. This induces a contradiction, and so R is not weakly right unit-duo.

We claim next that R is (weakly) left unit-duo. For the power series $f(x)$, $g(x)$ above, we have

$$f(x)g(x) = h(x)x^k g(x) = h(x)g_1(x)x^k = h(x)g_1(x)h(x)^{-1}h(x)x^k$$
$$= h(x)g_1(x)h(x)^{-1}f(x) \in U(R)f(x),$$

where

$$x^k g(x) = g_1(x)x^k \text{ and } g_1(x) = \sum_{j=0}^{\infty} \sigma^k(b_j)x^j.$$

But $h(x)g_1(x)h(x)^{-1} = a_0\sigma^k(b_0)a_0^{-1} + \cdots \in U(R)$ because $a_0\sigma^k(b_0)a_0^{-1} \neq 0$, so $f(x)U(R) \subseteq U(R)f(x)$. Therefore R is (weakly) left unit-duo.

Next, in the preceding construction, let $R = S[[x;\sigma]]$ be the skew power series ring, with the coefficients written on the right, only subject to $ax = x\sigma(a)$ for $a \in S$. Then R can be shown to be a weakly right unit-duo ring that is not weakly left unit-duo, through a similar computation. □

Let K be a division ring. Then $D_n(K)$ is clearly a local ring with nilpotent Jacobson radical, and so it is weakly unit-duo by Lemma 1.3(2).

The monomorphism σ in the proof of Lemma 1.3(3) is not surjective. If σ is bijective, then the ring R is unit-duo as we see in the following.

Theorem 1.4. *Let K be a field and σ be a monomorphism of K. If σ is bijective, then the skew power series ring $R = K[[x;\sigma]]$ by σ, with an indeterminate x over K, is a unit-duo ring, where every element is of R is of the form $\sum_{i=0}^{\infty} a_i x^i$, only subject to $xa = \sigma(a)x$ for all $a \in K$.*

Proof. Note first that R is left unit-duo by the computation in the proof of Lemma 1.3(3). Suppose that σ is bijective. Let $f(x) = \sum_{i=m}^{\infty} a_i x^i \in R$ with $a_m \neq 0$, and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in U(R)$ (i.e., $b_0 \neq 0$), where $m \geq 0$. Let

$$f_1(x) = \sum_{k=0}^{\infty} c_k x^k \text{ with } c_k = a_{m+k} \text{ for all } k.$$

Then $f(x) = f_1(x)x^m$ and $f_1(x) \in U(R)$. We will show that $g(x)f(x) \in f(x)U(R)$. Write

$$f_1(x)^{-1}g(x)f_1(x) = \sum_{t=0}^{\infty} d_t x^t,$$

noting that $d_0 \neq 0$ because $c_0, b_0 \neq 0$. Then

$$g(x)f(x) = g(x)f_1(x)x^m = f_1(x)f_1(x)^{-1}g(x)f_1(x)x^m$$
$$= f_1(x)(\sum_{t=0}^{\infty} d_t x^t)x^m = f_1(x)x^m(\sum_{t=0}^{\infty} \sigma^{-m}(d_t)x^t)$$
$$= f(x)(\sum_{t=0}^{\infty} \sigma^{-m}(d_t)x^t).$$

Note here that $\sum_{t=0}^{\infty} \sigma^{-m}(d_t)x^t \in U(R)$ because $\sigma^{-m}(d_0) \neq 0$. This implies $g(x)f(x) \in f(x)U(R)$, entailing $U(R)f(x) \subseteq f(x)U(R)$. Therefore R is right unit-duo. □

Right unit-duo rings are clearly weakly right unit-duo, but the converse need not hold by the following.

Example 1.5. (1) Let K be a division ring and $R = D_3(K)$. Let $A = (a_{ij})$. It is easily checked that A is a unit if and only if $a_{11} = \cdots = a_{nn} \neq 0$, entailing $U(R) = \{(a_{ij}) \in R \mid a_{ii} \neq 0\}$. Note also that A is a nonunit if and only if $a_{11} = \cdots = a_{nn} = 0$ if and only if A is nilpotent. It then follows that R is weakly unit-duo by Lemma 1.3(2), noting that R is a local ring with $J(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\}$ and $\frac{R}{J(R)} \cong K$.

However R is neither left nor right unit-duo. Indeed,

$$U(R)E_{12} = RE_{12} = KE_{12} \subsetneqq KE_{12} + KE_{13} = E_{12}R = E_{12}U(R),$$

implying that R is not left unit-duo; and

$$U(R)E_{23} = RE_{23} = KE_{13} + KE_{23} \supsetneqq KE_{23} = E_{23}R = E_{23}U(R),$$

implying that R is not right unit-duo.

(2) We apply the construction of ring in [25, Definition 1.3] and argument in [20, pp. 1693–1694]. Let R be a commutative ring with an endomorphism σ and M be an R-module. To give $R \oplus M$ a ring structure, define the addition and multiplication are given by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2) \text{ and}$$
$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + m_1 r_2).$$

Then this construction forms a ring and usually called the *skew-trivial extension* of R by M, denoted by $R \propto M$. Let K be a field with a monomorphism σ and M be a K-module. If σ is not surjective, then $K \propto M$ is a right unit-duo ring which is not left unit-duo by [14, Theorem 1.1(1)].

In the preceding construction of R, define the multiplication by

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 \sigma(r_2))$$

as in [14, Example 1.2(2)]. Then R is not right but left unit-duo by the argument in [14, Example 1.2(2)].

But $J(R) = 0 \propto M$ and $\frac{R}{J(R)} \cong K$, so R is weakly unit-duo by Lemma 1.3(2).

Following Yao [28], a ring R is said to be *weakly right duo* if for each $a \in R$ there exists a positive integer $n \geq 1$ such that $Ra^n \subseteq a^n R$, i.e., $a^n R = Ra^n R$. The weakly left duo rings can be defined similarly. A ring is a *weakly duo* ring if it is both weakly left and weakly right duo. A right duo ring is obviously weakly right but the converse need not hold as can be seen by $D_3(K)$ over a division ring K. Indeed, every matrix in $D_3(K)$ is either a unit or nilpotent, so $D_3(K)$ is weakly duo; but $D_3(K)E_{12} = KE_{12}$ (resp., $E_{23}D_3(K) = KE_{23}$) does not contain $E_{12}D_3(K) = KE_{12} + KE_{13}$ (resp., $D_3(K)E_{23} = KE_{13} + KE_{23}$), so $D_3(K)$ is neither right nor left duo. A weakly right (left) duo ring is Abelian by [28, Lemma 4]. There exists a unit-duo ring which is neither weakly right nor weakly left duo by Example 2.8 below.

We see in the following a condition under which the weakly unit-duo property is left-right symmetric. Recall that an involution on a ring R is a function $* : R \to R$ which satisfies the properties that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, and $(x^*)^* = x$ for all $x, y \in R$. The following is similar to [14, Theorem 1.3(4)].

Proposition 1.6. *Let R be a ring with an involution $*$.*

(1) *R is weakly left unit-duo if and only if it is weakly right unit-duo.*

(2) *R is weakly left duo if and only if it is weakly right duo.*

Proof. (1) Suppose that R is a weakly left unit-duo ring. Let $a \in R$ and $u \in U(R)$. Then

$$(a^*)^n u^* = v(a^*)^n$$

for some $n \geq 1$ and $v \in U(R)$, noting $u^* \in U(R)$. Observe $(a^n)^* = (a^*)^n$. It then follows that

$$ua^n = ((ua^n)^*)^* = ((a^*)^n u^*)^* = (v(a^*)^n)^* = (v(a^n)^*)^* = ((a^n)^*)^* v^* = a^n v^*.$$

Since $v^* \in U(R)$, we have now $U(R)a^n \subseteq a^n U(R)$. This implies that R is weakly right unit-duo. The proof of the converse is similar.

(2) Suppose that R is a weakly left duo ring, and let $a, b \in R$. Then

$$(a^*)^n b^* = c(a^*)^n$$

for some $n \geq 1$ and $c \in R$, and we have

$$ba^n = ((ba^n)^*)^* = ((a^*)^n b^*)^* = (c(a^*)^n)^* = (c(a^n)^*)^* = ((a^n)^*)^* c^* = a^n c^*.$$

This implies that R is weakly right duo. The proof of the converse is similar. □

Let G be any group G and K be any commutative ring, and consider the group ring KG of G over K. It is well-known that KG is left duo if and only if it is right duo, with the standard involution $*$ on KG defined by $(\sum a_i g_i)^* = \sum a_i g_i^{-1}$ for all $a_i \in K$ and $g_i \in G$. We have also that KG is left unit-duo if and only if it is right unit-duo by [14, Theorem 1.3(4)]. Recall that Q_8 denotes the quaternion group.

Proposition 1.7. *Let K be a field of characteristic zero and R be the group ring KQ_8. Then the following conditions are equivalent:*

(1) *R is weakly right (left) unit-duo;*

(2) *R is right (left) unit-duo;*

(3) *R is right (left) duo;*

(4) *R is weakly right (left) duo;*

(5) *R is Abelian.*

Proof. Note first that R is isomorphic to $K\oplus K\oplus K\oplus K\oplus H(K)$ such that $H(K)$ is either a division ring, D say, or $\mathrm{Mat}_2(K)$, by [24, Theorem 7.4.6 and Lemma 7.4.9]. Here if R is Abelian, then R is isomorphic to $K \oplus K \oplus K \oplus K \oplus D$. So the proof is done by help of Proposition 1.6, [14, Proposition 1.8], [28, Lemma 4], Lemma 1.3(1), and the argument prior to this proposition. □

In the preceding proof, $K\oplus K\oplus K\oplus K\oplus H(K)$, where $H(K)$ is the quaternion algebra over K, is called the Wedderburn decomposition in [24].

Due to Bell [3], a ring R is called to satisfy the *insertion-of-factors-property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. IFP rings are easily shown to be Abelian. The ring R in Lemma 1.3(3) is a domain (hence IFP), but R is not weakly right unit-duo. Consider next $D_n(K)$ over a division ring K for $n \geq 4$. Then $D_n(K)$ is weakly unit-duo as above, but it is not IFP by [19, Example 1.3]. So the concepts of weakly unit-duo and IFP are independent of each other. We will see conditions under which these two concepts are equivalent.

In the following, the class of weakly unit-duo rings of minimal order is completely characterized by help of Lemma 1.3(2).

Lemma 1.8. (1) [23, Corollary 6] *A ring R is a noncommutative local ring of minimal order if and only if R is a noncommutative IFP ring of minimal order.*

(2) [10, Theorem] *Let R be a finite ring of order m with identity. If m has a cube free factorization, then R is a commutative ring.*

(3) [10, Proposition] *If R is a noncommutative of order p^3, p a prime, then R is isomorphic to $U_2(GF(p))$.*

Xu and Xue [27, Theorem 8] proved that a noncommutative IFP ring of minimal order is a local ring of order 16, and if R is such a ring, then $R \cong R_i$ for some $i \in \{1, 2, 3, 4, 5\}$, where R_i's are the rings in Example 1.9 to follow. So a noncommutative local ring of minimal order is also isomorphic to one of the R_i's by Lemma 1.8(1). $GF(p^n)$ denotes the Galois field of order p^n.

Example 1.9. In [27, Example 7], there exist five kinds of noncommutative finite local rings of order 16. Let $A\langle x, y\rangle$ be the free algebra generated by noncommuting indeterminates x, y over given a commutative ring A, and (x, y) denote the ideal of $A\langle x, y\rangle$ generated by x, y.

(1) Let $R_1 = \mathbb{Z}_2\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$.

(2) Let $R_2 = \mathbb{Z}_4\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$.

(3) Let $R_3 = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \in U_2(GF(2^2)) \mid a, b \in GF(2^2)\right\}$.

(4) Let $R_4 = \mathbb{Z}_2\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_2\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy$. Identify x, y with their images in R_4 for simplicity. It is simply checked that R_4 is isomorphic to $D_3(\mathbb{Z}_2)$ through the corresponding $x \mapsto E_{12} + E_{23}$ and $y \mapsto E_{23}$.

(5) Let $R_5 = \mathbb{Z}_4\langle x, y\rangle/I$, where I is the ideal of $\mathbb{Z}_4\langle x, y\rangle$ generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$.

We now have a complete structure of noncommutative unit-duo rings of minimal order by help of Example 1.2.

Theorem 1.10. *If R is a noncommutative weakly right (left) unit-duo ring of minimal order, then $|R| = 16$ and R is isomorphic to one of R_i's in Example 1.9.*

Proof. Let R be a noncommutative weakly right (left) unit-duo ring of minimal order. Then R is Abelian by Lemma 1.3(1). Moreover R is a local ring of order 16 by help of the proof of [14, Theorem 2.3]. Finite local rings are weakly unit-duo by Lemma 1.3(2). These results lead us to conclude that R is isomorphic to one of R_i's in Example 1.9. □

The following is an immediate consequence of Theorem 1.10 and [14, Theorem 2.3 and Corollary 2.4].

Corollary 1.11. (1) *A ring is a noncommutative weakly right (left) unit-duo ring of minimal order if and only if it is a right (left) unit-duo ring of minimal order.*

(2) *A ring is a noncommutative weakly right unit-duo ring of minimal order if and only if it is an IFP ring of minimal order.*

(3) *A ring is a noncommutative weakly right unit-duo ring of minimal order if and only if it is a noncommutative weakly unit-duo ring of minimal order if and only if it is a noncommutative weakly left unit-duo ring of minimal order.*

2. Relations and examples

In this section we study first useful relations between weakly right duo rings and nearby ring concepts. We investigate next various kinds of examples of weakly right unit-duo rings which do roles in ring theory.

Following Cohn [7], a ring R is *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Commutative rings are clearly reversible, and reduced rings are easily shown to be reversible. There exist many reversible rings which are neither commutative nor reduced as can seen by $D_2(K)$, over a noncommutative division ring K. The class of IFP rings contains both reversible rings and right (left) duo rings. Each converse does not hold in general as can be seen by $D_3(A)$ over a reduced ring A (see [19, Proposition 1.2 and Example 1.5]).

The concepts of reversible and right (unit-)duo are independent of each other as the following example shows. Recall $Q_8 = \langle i, j\rangle$ with $ji = i^3j$.

Example 2.1. (1) There exist reversible rings which are not left or right duo. We use the ring constructed by Marks in [21, Example 5]. Let K be a field and $F = K\langle x, y, z\rangle$ be the free algebra with noncommuting indeterminates x, y, z over K. Next consider the ideal $I = (FxF)^2 + (FyF)^2 + (FzF)^2 + FxyzF +$

$FyzxF + FzxyF$, and put $R = F/I$. Then, by the argument in [21, Example 5], R is a reversible ring that is not left or right duo.

(2) There exist reversible rings which are neither right nor left unit-duo. We use the ring constructed by Bell and Li in [4, Example 1.2]. Let R be the integral group ring $\mathbb{Z}Q_8$ of Q_8 over $\mathbb{Z}$. Then R is a reversible ring by help of [13, Theorem 3.1]. Gutan and Kisielewicz showed that $(i+2j)i \notin R(i+2j)$ in [13, Theorem 3.1], noting $i \in U(R)$. This implies that R is not left (unit-)duo. A similar argument can show that R is not right (unit-)duo.

(3) There exist one-sided duo rings which are not reversible. We use the ring constructed by Courter in [8, Example 3]. Let K be a field and R_0 be the row finite infinite matrix ring over K. Let P be the K-subspace of R_0 generated by the matrices $\{a_0, a_1, \ldots, a_j, \ldots\}$, where

$$a_0 = E_{12},\ a_1 = E_{13},\ a_2 = E_{14} + E_{32},\ \ldots,\ a_j = E_{1,j+2} + E_{j+1,2}$$

for $j \geq 2$. Next let I be the identity matrix in R_0, and $R = KI \oplus P$, the K-subspace of R_0 with a basic $\{I, a_0, a_1, \ldots, a_j, \ldots\}$. Then R is a right duo ring by the computation in [8, Example 3]. However $a_2a_1 = 0$ but $a_1a_2 = E_{12}$, entailing that R is not reversible.

We claim moreover that R is right unit-duo. Let $a \in R$ and $u \in U(R)$. We will show that $ua \in aU(R)$, and it suffices to consider the case of $a \in X(R)$. Then u and a can be expressed by

$$u = kI + \sum_{i=0}^{m} c_i a_i \text{ and } a = \sum_{i=0}^{m} d_i a_i,$$

where $m \geq 1$ and $k \neq 0, c_0, \ldots, c_m, d_0, \ldots, d_m \in K$. We apply the argument in [8, Example 3]. Note $ua = \sum_{i=0}^{m}(kd_i)a_i + (c_1d_2 + c_2d_3 + \cdots + c_{m-1}d_m + kd_0)a_0$. Then, letting $v = kI + c_1a_3 + c_2a_4 + \cdots + c_{m-1}a_{m+1} \in U(R)$, we have $ua = av$. Thus R is a right unit-duo ring.

Marks proved in [21, Proposition 6] that the group ring kG is reversible if and only if kG is IFP, where k is a commutative ring and G is a finite group. Recall that right or left duo ring is IFP. So if kG is right or left duo, then it is reversible by applying [21, Proposition 6].

We see a condition under which the ring properties mentioned above are equivalent. Let K be a field and G be a non-Abelian torsion group. Here if KG is a duo ring, then the characteristic of K is zero or 2 by [4, Theorem 4.2]. In this note we study the following relations in the case of characteristic zero, and will concern one of characteristic 2 in the future study.

Proposition 2.2. *Let K be a field of characteristic zero and R be the group ring KQ_8, where Q_8 is the quaternion group. Then the following conditions are equivalent*:

(1) *R is weakly right* (*left*) *unit-duo*;
(2) *R is right* (*left*) *unit-duo*;
(3) *R is right* (*left*) *duo*;

(4) *R is weakly right (left) duo*;
(5) *R is Abelian*;
(6) *The equation* $1 + x^2 + y^2 = 0$ *has no solutions in* K;
(7) *R is an IFP ring*;
(8) *R is a reversible ring.*

Proof. The conditions (3) and (8) are equivalent by Bell and Li [4, Theorem 2.1]. The conditions (7) and (8) are equivalent by Marks [21, Proposition 6]. The equivalence of the conditions (6) and (8) is proved by Gutan and Kisielewicz [13, Theorem 3.1]. The remainder is done by Proposition 1.7. □

$\mathbb{Q}Q_8$ is a right (unit-)duo ring by Proposition 2.2 when it is an Abelian ring. In this situation, $\mathbb{Z}Q_8$ is not a right (unit-)duo ring by Example 2.1. So we can say that the class of right (unit-)duo rings is not closed under subrings.

We consider next a sort of a domain over which the polynomial is weakly right unit-duo. One may compare the following with [14, Proposition 3.4].

Proposition 2.3. (1) *Let* R *be a ring and suppose that* $R[x]$ *is weakly right (left) unit-duo. Then, for* $a \in R$ *and* $u \in G(R)$, *there exists* $k \geq 1$ *such that* $a^k u = ua^k$ *and* $k(au) = k(ua)$.

(2) *Let* R *be a domain of characteristic zero. If* $R[x]$ *is weakly right (left) unit-duo, then* $au = ua$ *for all* $a \in R$ *and* $u \in G(R)$ *(hence* R *is unit-duo).*

(3) *Let* R *be a division ring of characteristic zero. Then the following are equivalent*:

(a) $R[x]$ *is weakly right (left) unit-duo*;
(b) R *is right (left) unit-duo*;
(c) R *is a field*;
(d) $R[x]$ *is right (left) unit-duo.*

Proof. (1) Let $a \in R$ and $u \in U(R)$. Since $R[x]$ is weakly right unit-duo, there exists $k \geq 1$ such that $u(a+x)^k = (a+x)^k u'$ for some $u' \in U(R[x])$. Comparing the degrees of this equality, we get $u' \in R$ and $u = u'$. Then we have $ua^k = a^k u$ and $k(au) = k(ua)$ from the equality $u(a^k + kax + \cdots + x^k) = (a^k + kax + \cdots + x^k)u$. The proof of the left case is similar.

(2) Suppose that $R[x]$ is weakly right unit-duo, and let $a \in R$ and $u \in U(R)$. Then $k(au) = k(ua)$ for some $k \geq 1$ by (1). Since R is a domain of characteristic zero, we have $au = ua$ because $k \neq 0$. Thus R is also weakly right unit-duo. The proof of the left case is similar.

(3) By (2), (a) implies (b). So it suffices to show (b) implies (c). Suppose that R is right (left) unit-duo, and let $0 \neq a, b \in R$. Then, since $a, b \in U(R)$, we have $ab = ba$ by (2). So R is a filed. □

Let R be the Hamilton quaternions over the real numbers. Then $R[x]$ cannot be (weakly) right unit-duo by Proposition 2.3.

Following [12], a ring R is called *(von Neumann) regular* (resp., *unit-regular*) if for every $x \in R$ there exists $y \in R$ (resp., $u \in U(R)$) such that $xyx = x$

(resp., $xux = x$). In this case, every element of a regular (resp., unit-regular) ring is also said to be *regular* (resp., *unit-regular*. Abelian regulars are unit-regular by [12, Corollary 4.2]. Let R be a unit-regular ring and $a \in R$. Then $a = aua$ for some $u \in U(R)$, so this implies that $a = (au)u^{-1} = u^{-1}(ua)$ with $au, ua \in I(R)$.

Proposition 2.4. *Let R be a regular ring. Then the following conditions are equivalent*:

(1) *R is weakly right* (*left*) *unit-duo*;
(2) *R is right* (*left*) *unit-duo*;
(3) *R is right* (*left*) *duo*;
(4) *R is IFP*;
(5) *R is Abelian.*

Proof. The proof is done by Lemma 1.3(1) and [14, Corollary 1.4]. □

We also have that a regular ring is weakly right unit-duo if and only if it is reduced by help of Proposition 2.4 and [12, Theorem 3.2].

Following the literature, a ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are obviously π-regular, letting $n(a) = 1$ for all a. But there exist π-regular rings which are not regular as can be seen by $D_n(A)$ over a division ring A. Indeed, $D_n(A)$ is π-regular by [5, Corollary 6], but it is clearly not regular when $n \geq 2$. There exist many non-Abelian π-regular rings by [5, Corollary 6], as can be seen by $U_n(K)$ over a division ring K for $n \geq 2$.

Proposition 2.5. *Let R be a π-regular ring. Then the following conditions are equivalent*:

(1) *R is weakly right* (*left*) *unit-duo*;
(2) *R is Abelian.*

Proof. It suffices to prove (2) implies (1) by help of Lemma 1.3(1). Let R be an Abelian ring. Consider $a \in R$ and $u \in U(R)$. Since R is π-regular, a^n is regular for some $n \geq 1$. Then, by [1, Theorem 2], a^n is unit-regular because R is Abelian. Say that $a^n = a^n v a^n$ for some $v \in U(R)$. Note $a^n v, v a^n \in I(R)$. Since R is Abelian, we have

$$\begin{aligned} ua^n &= u(a^n v a^n) = u(a^n v)a^n = (a^n v)ua^n = (a^n v)u(a^n v)v^{-1} \\ &= (a^n v)^2 uv^{-1} = a^n vuv^{-1}, \end{aligned}$$

implying $U(R)a^n \subseteq a^n U(R)$. Thus R is weakly right unit-duo. The proof for the left case is similar. □

Considering Propositions 2.4 and 2.5, one may ask whether a π-regular ring is weakly right unit-duo if and only if it is right unit-duo. However the answer is negative as follows. Let $R = D_n(A)$ over a division ring A for $n \geq 2$. Then R is weakly right unit-duo and π-regular by Lemma 1.3(2) and [5, Corollary

6]. However R is neither right nor left unit-duo by the argument in Example 1.5(1).

Badawi show in [2, Lemma 5] that $J(R) = N(R)$ for an Abelian π-regular ring R, entailing that $R/J(R)$ is a reduced π-regular ring. Then $R/J(R)$ is moreover a regular ring by [2, Theorem 3] or [15, Lemma 4]. It then follows that R/P is a division ring for every prime ideal P of R with $J(R) \subseteq P$, by help of [12, Theorem 3.2]. In this situation, we need the condition "$J(R) \subseteq P$" by the following.

Example 2.6. We use the construction in [17, Theorem 2.2(2)]. Let K be a division ring, $n \geq 1$, and define a map $\sigma : D_{2^n}(K) \to D_{2^{n+1}}(K)$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then $D_{2^n}(K)$ can be considered as a subring of $D_{2^{n+1}}(K)$ via σ (i.e., $A = \sigma(A)$ for $A \in D_{2^n}(K)$). Let R be the direct limit of the direct system $(D_{2^n}(K), \sigma_{ij})$, where $\sigma_{ij} = \sigma^{j-i}$. Then $J(R) = N(R) = \{(a_{st}) \in R \mid$ the diagonal entries of (a_{st}) are zero$\}$ and $R/J(R) \cong K$. This implies that R is a local ring, so R is Abelian. Moreover, R is π-regular by [2, Theorem 3]. Note that R is weakly unit-duo by Lemma 1.3(2).

We can show next that R is a prime ring by applying the proof of [16, Proposition 1.3]. This implies that this prime ring R is not a division ring, noting that $J(R) \neq 0$.

One can compare the following with [14, Proposition 1.6] which states a similar argument for right (left) unit-duo rings.

Proposition 2.7. *Let R be a weakly right* (*resp. left*) *unit-duo ring. If R is a local ring, then R is weakly right* (*resp. left*) *duo.*

Proof. Let R be a weakly right unit-duo ring, and $a, b \in R$. Then $U(R)a^n \subseteq a^nU(R)$ for some $n \geq 1$.

Assume that R is local. Then $b \in U(R)$ or $b \in J(R)$ (hence $1 - b \in U(R)$). Here if $a \in N(R)$, then we are done, so we suppose $a \notin N(R)$.

If $b \in U(R)$, then $ba^n = a^nu$ and $u \in U(R)$ because R is weakly right unit-duo.

If $1 - b \in U(R)$, then $(1 - b)a^n = a^nv$ and $v \in U(R)$ because R is weakly right unit-duo. This yields $ba^n = a^n(1 - v)$.

We have then $Ra^n \subseteq a^nR$, and therefore R is weakly right duo. The proof of the left weakly duo is similar. □

The condition "local" in Proposition 2.7 is not superfluous by the following.

Example 2.8. We refer to the arguments in [14, Example 1.7] and [22, Theorem 1.3.5, Corollary 2.1.14, and Theorem 2.1.15]. Let $\mathbb{Q}\langle x, y\rangle$ be the free algebra with noncommuting indetermiantes x, y over $\mathbb{Q}$. The first Weyl algebra $A_1(\mathbb{Q}) \cong \frac{\mathbb{Q}\langle x,y\rangle}{(yx-xy-1)}$, R say, is a domain whose invertible elements are nonzero rational numbers (i.e., $U(R) = \mathbb{Q}^*$), where $(yx - xy - 1)$ is the ideal of $\mathbb{Q}\langle x, y\rangle$ generated by $yx - xy - 1$. We identify x and y with their images in R for simplicity. Then R is not local since it is a simple ring.

Consider $x \in R$. Then, for any $n \geq 1$, yx^n (resp., $x^n y$) is not contained in $x^n R$ (resp., Rx^n). Thus R is neither weakly right nor weakly left duo. But R is unit-duo (hence an elementwise-uniform weakly unit-duo ring) by the argument in [14, Example 1.7].

Following the literature, a ring R is *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Abelian rings are easily shown to be directly finite, so both weakly right duo rings and weakly right unit-duo rings are directly finite by the results above. We use this fact freely. One can compare the following with [14, Theorem 1.3(3)].

Proposition 2.9. *Let R be a domain. If R is weakly right duo, then R is weakly right unit-duo.*

Proof. Let R be a weakly right duo ring and suppose that $0 \neq a \in R$ and $u \in U(R)$. Then there exist $k \geq 1$ and $b \in R$ such that $ua^k = a^k b$ and $a^k = u^{-1}a^k b$. Also since R is weakly right duo, there exist $m \geq 1$ and $c \in R$ such that $u^{-1}(a^k b)^m = (a^k b)^m c$. Consequently we have

$$a^k = u^{-1}a^k b = (a^k b)^m c = a^k b \cdots a^k bc \text{ and } a^k(1 - b \cdots a^k bc) = 0.$$

But R is a domain, so $b \cdots a^k bc = 1$. Thus $b \in u(R)$, noting that R is directly finite. Hence R is weakly right unit-duo. □

The converse of Proposition 2.9 need not hold by the domain R in Example 2.8.

Let A be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [9], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(a_1, s_1)(a_2, s_2) = (a_1 r_2 + s_1 a_2 + s_2 a_1, s_1 s_2)$ for $a_i \in A$ and $s_i \in S$.

Proposition 2.10. *Let A be a nil algebra over a commutative ring S. If S is a field, then the Dorroh extension of A by S is weakly unit-duo.*

Proof. Let S be a field and R be the Dorroh extension of A by S. Then $J(R) = N(R) = \{(a, s) \mid s = 0\}$ clearly, entailing $R/J(R) \cong S$. So R is weakly unit-duo by Lemma 1.3(2). □

Acknowledgements. The authors thank the referee by much for very careful reading of the manuscript and valuable suggestions that improved the paper. This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2013R1A1A2A10004687).

References

[1] A. Badawi, *On semicommutative π-regular rings*, Comm. Algebra **22** (1994), no. 1, 151–157.

[2] ———, *On Abelian π-regular rings*, Comm. Algebra **25** (1997), no. 4, 1009–1021.

[3] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.
[4] H. E. Bell and Y. Li, *Duo group rings*, J. Pure Appl. Algebra **209** (2007), no. 3, 833–838.
[5] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 53–58.
[6] H. H. Brungs, *Three questions on duo rings*, Pacific J. Math. **58** (1975), no. 2, 345–349.
[7] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648.
[8] R. C. Courter, *Finite dimensional right duo algebras are duo*, Proc. Amer. Math. Soc. **84** (1982), no. 2, 157–161.
[9] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), no. 2, 85–88.
[10] K. E. Eldridge, *Orders for finite noncommutative rings with unity*, Amer. Math. Monthly **75** (1968), 512–514.
[11] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.
[12] K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.
[13] M. Gutan and A. Kisielewicz, *Reversible group rings*, J. Algebra **279** (2004), no. 1, 280–291.
[14] J. Han, Y. Lee, and S. Park, *Duo ring property restricted to group of units*, J. Korean Math. Soc. **52** (2015), no. 3, 489–501.
[15] C. Huh and Y. Lee, *A note on π-regular rings*, Kyungpook Math. J. **38** (1998), no. 1, 157–161.
[16] S. U. Hwang, Y. C. Jeon, and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), no. 1, 186–199.
[17] Y. C. Jeon, H. K. Kim, Y. Lee, and J. S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), no. 1, 135–146.
[18] H. K. Kim, N. K. Kim, and Y. Lee, *Weakly duo rings with nil Jacobson radical*, J. Korean Math. Soc. **42** (2005), no. 3, 455–468.
[19] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.
[20] Y. Lee, *On generalizations of commutativity*, Comm. Algebra **43** (2015), no. 4, 1687–1697.
[21] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002), no. 3, 311–318.
[22] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, John Wiley & Sons Ltd., Chichester, New York, Brisbane, Toronto, Singapore, 1987.
[23] S. B. Nam, *Finite local rings of order ≤ 16 with nonzero Jacobson radical*, Korean J. Math. **21** (2013), 23–28.
[24] C. Polcino Milies and S. K. Sehgal, *An Introduction to Group Rings*, Kluwer Academic Publishers, Dordrecht, 2002.
[25] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.
[26] G. Thierrin, *On duo rings*, Canad. Math. Bull. **3** (1960), 167–172.
[27] L. Xu and W. Xue, *Structure of minimal non-commutative zero-insertive rings*, Math. J. Okayama Univ. **40** (1998), 69–76.
[28] X. Yao, *Weakly right duo rings*, Pure Appl. Math. Sci. **21** (1985), no. 1-2, 19–24.

Young Woo Chung
School of Mathematics and Applied Statistics
Kyungsung University
Busan 48434, Korea
E-mail address: young38woo@daum.net

Yang Lee
Department of Mathematics Education
Pusan National University
Busan 46241, Korea
E-mail address: ylee@pusan.ac.kr