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RELATING GALOIS POINTS TO WEAK GALOIS WEIERSTRASS POINTS THROUGH DOUBLE COVERINGS OF CURVES

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Abstract. The point $P \in \mathbb{P}^2$ is referred to as a Galois point for a nonsingular plane algebraic curve C if the projection $\pi_P : C \to \mathbb{P}^1$ from P is a Galois covering. In contrast, the point $P' \in C'$ is referred to as a weak Galois Weierstrass point of a nonsingular algebraic curve C' if P' is a Weierstrass point of C' and a total ramification point of some Galois covering $f : C' \to \mathbb{P}^1$. In this paper, we discuss the following phenomena. For a nonsingular plane curve C with a Galois point P and a double covering $\varphi : C \to C'$, if there exists a common ramification point of π_P and φ, then there exists a weak Galois Weierstrass point $P' \in C'$ with its Weierstrass semigroup such that $H(P') = \langle r, 2r-1\rangle$ or $\langle r, 2r+1\rangle$, which is a semigroup generated by two positive integers r and $2r+1$ or $2r-1$, such that P' is a branch point of φ. Conversely, for a weak Galois Weierstrass point $P' \in C'$ with $H(P') = \langle r, 2r-1\rangle$ or $\langle r, 2r+1\rangle$, there exists a nonsingular plane curve C with a Galois point P and a double covering $\varphi : C \to C'$ such that P' is a branch point of φ.

1. Introduction

We work over an algebraically closed field k of characteristic 0. A curve refers to a complete nonsingular irreducible algebraic curve over k, and a plane curve refers to a curve in $\mathbb{P}^2$.

Morrison and Pinkham have introduced the notion of a Galois Weierstrass point as follows.

Definition 1.1 ([12])**.** Let C be a curve of genus $g \geq 2$. A point $P \in C$ is termed a *Galois Weierstrass point* (GW point), if $\Phi_{|aP|} : C \to \mathbb{P}^1$ is a Galois

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covering, where a is the smallest positive integer of the Weierstrass semigroup

$$H(P) := \{n \in \mathbb{Z}_{\geq 0} \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\}.$$

Remark 1.1. A GW point may be a non-Weierstrass point. However, in this paper, we only treat GW points that are Weierstrass points.

Yoshihara introduced the notion of a Galois point for a plane curve as follows.

Definition 1.2 ([10, 14])**.** Let C be a plane curve of degree $d \geq 3$. For a point $P \in \mathbb{P}^2$, the projection $\pi_P : C \to \mathbb{P}^1$ from P induces an extension of function fields $\pi_P^* : k(\mathbb{P}^1) \hookrightarrow k(C)$. P is referred to as a *Galois point* for C if the extension is of the Galois type. Moreover, when $P \in C$ or when $P \notin C$, the point is considered to be an inner or outer Galois point, respectively.

GW points have been studied previously ([7, 8, 12]) and studies on Galois points have also been reported (e.g., [4, 10, 14]). In addition, some studies on double coverings of curves have been conducted (e.g., [9, 13]). The purpose of this work is to study new phenomena pertaining to relationships between certain kinds of GW points and Galois points by examining the branch points of double coverings of curves.

In this regard, we define two new concepts, namely a weak GW point and a pseudo-GW point, as follows.

Definition 1.3. Let C be a curve of genus $g \geq 2$. We refer to $P \in C$ as a *weak GW point* if

(1) P is a total ramification point of some Galois covering $f : C \to \mathbb{P}^1$, and
(2) P is a Weierstrass point of C.

Moreover, if P is not a GW point, we consider P a *pseudo-GW point.* For a weak GW point P, we denote

$$\begin{aligned}&\mathrm{degGW}(P)\\ &:= \{\deg f \mid \text{Galois covering } f : C \to \mathbb{P}^1 \text{ which is totally ramified at } P\}\end{aligned}$$

and we refer to it as *the set of degrees of the weak GW point.*

Remark 1.2. On Definition 1.3, the Galois group of the Galois extension $k(C)/f^*(k(\mathbb{P}^1))$ is isomorphic to the group $\mathrm{Gal}(f) := \{\sigma \in \mathrm{Aut}(C) \mid f \circ \sigma = f\}$, and the group is a cyclic group.

We denote $\langle a, b\rangle$ as the semigroup generated by elements $a, b \in \mathbb{N}$ and $\mathrm{Ramif}(f)$ as the set of ramification points of morphism f. Our main theorem is the following.

Theorem 1.1. *Let $C \subset \mathbb{P}^2$ be a plane curve of degree $d \geq 5$. Assume that there exists a Galois point P for C and a double covering $\varphi : C \to C'$, where C' is a curve, such that* $\mathrm{Ramif}(\pi_P) \cap \mathrm{Ramif}(\varphi) \neq \emptyset$. *We choose $P' \in C'$ as follows*:

(1) *if P is an inner Galois point, then $P' := \varphi(P)$.*

(2) *if P is an outer Galois point, then let $P' = \varphi(Q)$ for some $Q \in \mathrm{Ramif}(\pi_P) \cap \mathrm{Ramif}(\varphi)$.*

Then, P' is a weak GW point. Moreover, on $H(P')$ and $\mathrm{degGW}(P')$ of the weak GW point P' with respect to d and P, we have Table 1.

TABLE 1. The Weierstrass semigroup and the set of degrees of the weak GW point

	P is an inner Galois point	*P is an outer Galois point*
d is odd	$H(P') = \langle (d-1)/2, d\rangle$, $(d-1)/2 \in \mathrm{degGW}(P')$	$H(P') = \langle (d-1)/2, d\rangle$, $d \in \mathrm{degGW}(P')$
d is even	$H(P') = \langle d/2, d-1\rangle$, $d-1 \in \mathrm{degGW}(P')$	$H(P') = \langle d/2, d-1\rangle$, $d/2 \in \mathrm{degGW}(P')$

Conversely, let C' be a curve and P' be a weak GW point with $H(P') = \langle (d-1)/2, d\rangle$ or $\langle d/2, d-1\rangle$ for $d \geq 5$. Then, there exists a plane curve C of degree d, a Galois point P for C, and a double covering $\varphi : C \to C'$ such that P' is obtained as above.

Remark 1.3. Let C be a plane curve of degree $d \geq 5$. Assume that there exists a Galois point P for C. If d is odd and P is an inner point, or d is even and P is an outer point, then there exists a double covering $\varphi : C \to C'$, where C' is a curve, such that $\mathrm{Ramif}(\pi_P) \cap \mathrm{Ramif}(\varphi) \neq \emptyset$. For details, see Lemmas 4.1 and 4.3. When d is even and P is an inner point, or d is odd and P is an outer point, if there exists a double covering $\varphi : C \to C'$, where C' is a curve, then $\mathrm{Ramif}(\pi_P) \cap \mathrm{Ramif}(\varphi) \neq \emptyset$. For details, see Lemmas 4.5 and 4.7.

In Section 2, we describe some preliminary or fundamental results on Galois points and weak GW points. In Section 3, we show some important examples of Galois points and weak GW points. In Section 4, we prove Theorem 1.1.

2. Preliminary

In this section, we describe selected preliminary or fundamental results on Galois points and weak GW points.

First, the results on Galois points are summarized.

Theorem 2.1 ([5, 10, 14])**.** *Let C be a plane curve of degree d ($d \geq 4$). Then:*

(1) *If $P \in \mathbb{P}^2$ is a Galois point for C, then the Galois group of $k(C)/\pi_P^*(k(\mathbb{P}^1))$ is cyclic.*

(2) *The point P is either an inner or outer Galois point if and only if, by changing the coordinates in a suitable way, we may assume that*

$$P = (0:0:1) \text{ and } C:\ XZ^{d-1} + f_d(X,Y) = 0 \text{ or } Z^d + f_d(X,Y) = 0, \tag{1}$$

where $f_d(X,Y)$ is a homogeneous polynomial of degree d.

(3) *If $d \geq 5$, then the number of inner Galois points for C equals* 0 *or* 1. *If $d = 4$, then the number of inner Galois points for C equals* 0, 1, *or* 4. *Moreover, the number equals* 4 *if and only if C is projectively equivalent to the curve $XZ^3 + X^4 + Y^4 = 0$. The four Galois points for the curve are on $Y = 0$.*
(4) *The number of outer Galois points for C equals* 0, 1, *or* 3. *Moreover, the number equals* 3 *if and only if, C is projectively equivalent to the Fermat curve $X^d + Y^d + Z^d = 0$. The three Galois points for the curve are* $(1 : 0 : 0)$, $(0 : 1 : 0)$, *and* $(0 : 0 : 1)$.
(5) *There exist both an inner and an outer Galois point for C if and only if C is projectively equivalent to the curve $XZ^{d-1} + X^d + Y^d = 0$. If $d \geq 5$ or $d = 4$, then the Galois points for the curve are either* $(0 : 0 : 1)$ *and* $(0 : 1 : 0)$ *or four points of $C \cap (Y = 0)$ and* $(0 : 1 : 0)$, *respectively.*

Remark 2.1. According to [2] or [1, Appendix A, 17 and 18], if C is a plane curve of degree $d \geq 4$, then every element of $\mathrm{Aut}(C)$ is a restriction of some projective transformation of $\mathbb{P}^2$. In this paper, we often express an element of $\mathrm{Aut}(C)$ as an element of $PGL(3, k)$. We note that an element $A \in PGL(3, k)$ induces a projective transformation $(X' : Y' : Z') \mapsto (X : Y : Z)$ such that $(X\,Y\,Z)^{\mathrm{tr}} = A\,(X'\,Y'\,Z')^{\mathrm{tr}}$, where $(X\,Y\,Z)^{\mathrm{tr}}$ and $(X'\,Y'\,Z')^{\mathrm{tr}}$ indicate transposed matrices of $(X\,Y\,Z)$ and $(X'\,Y'\,Z')$. Assume that P is a Galois point for a plane curve C. Then, a generator σ of its Galois group can be expressed as a projective transformation $T_\sigma \in \mathrm{Aut}(\mathbb{P}^2)$. We denote $\mathrm{Fix}(\sigma) := \{Q \in \mathbb{P}^2 \mid T_\sigma(Q) = Q\}$. Based on [14], if P and C are given by (1), then T_σ is expressed as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix}, \tag{2}$$

where ζ is a primitive $(d-1)$th or dth root of unity, respectively. We have $\mathrm{Fix}(\sigma) = \{P\} \cup \ell_Z$, where ℓ_Z is the line defined by $Z = 0$.

Remark 2.2. Let C be a plane curve of degree $d \geq 5$. Assume that C is not projectively equivalent to a Fermat curve, and there exists a Galois point P for C. Then, by Theorem 2.1, there exists at most one inner or at most one outer Galois point. Hence, $\tau(P) = P$ for every automorphism τ.

For a specific point in a plane curve, we can determine its Weierstrass semigroup as follows. For a point P on a plane curve C, we denote $I_P(C, T_P C)$ as the local intersection multiplicity at P of C and the tangent line $T_P C$ to C at P.

Lemma 2.1 ([3]). *Let C be a plane curve of degree d and P be a point on C. Then*:

(1) *if $I_P(C, T_P C) = d$, then $H(P) = \langle d-1, d\rangle$;*
(2) *if $I_P(C, T_P C) = d-1$, then $H(P) = \langle \{i(d-2)+1 \mid i = 1, \ldots, d-1\}\rangle$.*

For a branch point of a double covering, we can determine its Weierstrass semigroup as follows.

Lemma 2.2 ([6, page 392])**.** *Let $\pi : C \to C'$ be a double covering of curves, $P \in C$ be a ramification point of π, and $P' := \pi(P)$. Then $H(P') = \{n \in \mathbb{Z}_{\geq 0} \mid 2n \in H(P)\}$.*

For the case in which the Weierstrass semigroup is generated by two integers, we have the following fundamental result on $\deg\mathrm{GW}(P)$.

Proposition 2.1. *If P is a weak GW point of a curve C with $H(P) = \langle a, b\rangle$ ($a < b$), then $a \in \deg\mathrm{GW}(P)$ or $b \in \deg\mathrm{GW}(P)$.*

Proof. Let $g : C \to \mathbb{P}^1$ be a Galois covering such that P is a total ramification point, and $(g)_\infty = nP$. Note that $n \in H(P)$, that is, $\mathrm{Bs}\,|nP| = \emptyset$. Let σ be a generator of the cyclic group $\mathrm{Gal}(g) := \{\sigma \in \mathrm{Aut}(C) \mid g \circ \sigma = g\}$. Let x and y be functions such that $(x)_\infty = aP$ and $(y)_\infty = bP$.

Claim 2.1. We may assume $\sigma^* x = \alpha x$, where $\alpha \in k \setminus \{0\}$.

We prove Claim 2.1. Because $(\sigma^* x)_\infty = aP$ and $H(P) = \langle a, b\rangle$, let $\sigma^* x = \alpha_1 x + \alpha_2$, where $\alpha_1, \alpha_2 \in k$. Assume that $\alpha_2 \neq 0$. Because the order of σ is finite, we have $\alpha_1 \neq 1$. Then, $X := (\alpha_1 - 1)x + \alpha_2$ holds that $(X)_\infty = aP$ and $\sigma^*(X) = \alpha_1 X$. We conclude Claim 2.1.

Claim 2.2. We may assume that $\sigma^* y = \beta y$, where $\beta \in k \setminus \{0\}$.

We prove Claim 2.2. Let $l := [b/a] = \max\{n \in \mathbb{N} \mid an \leq b\}$. Then, $H^0(C, \mathcal{O}_C(bP)) = \{f \in k(C) \mid (f) + bP \geq 0\} = \langle 1, x, x^2, \ldots, x^l, y\rangle$. Let $\sigma^* y = \beta y + \beta_l x^l + \cdots + \beta_1 x + \beta_0$, where $\beta, \beta_i \in k$. Assume that $\beta_i \neq 0$ for some i. Let $B_i := \beta_i/(\beta - \alpha^i)$ if $\beta \neq \alpha^i$, and $B_i := 0$ (actually B_i may be any number) if otherwise ($i = 0, \ldots, l$). Note that if $\beta = \alpha^i$, then $\beta_i = 0$, because the order of σ is finite. We have that $Y := y + B_l x^l + \cdots + B_0$ holds $(Y)_\infty = bP$ and $\sigma^*(Y) = \beta Y$. We conclude Claim 2.2.

We show that the morphism $C \to \mathbb{P}^1$ defined by x or y is a Galois covering. As $H(P) = \langle a, b\rangle$, we have that $H^0(C, \mathcal{O}_C(nP)) = \{f \in k(C) \mid (f) + nP \geq 0\} = \langle 1, x, \ldots, y, \ldots, x^s y^t\rangle$ and $(x^s y^t)_\infty = nP$. The Galois covering g is given by $z := A_{s,t} x^s y^t + \cdots + A_{0,0}$, where $A_{i,j} \in k$. Because $\sigma^* z = z$, $A_{s,t} \neq 0$ and $1, x, \ldots, y, \ldots, x^s y^t$ are linearly independent over k, we have that $\alpha^s \beta^t = 1$. Hence, $\sigma^*(x^s y^t) = x^s y^t$. Thus, the morphism defined by $x^s y^t$, which we denote as $\varphi : C \to \mathbb{P}^1$, is a Galois covering of degree n, and P is its total ramification point.

Claim 2.3. $st = 0$.

We prove Claim 2.3. For a sufficiently large $N \in \mathbb{N}$, NP is very ample and the embedding $\Phi_{|NP|}$ is defined by functions $x^i y^j$. Because the Galois covering φ is defined by the function $x^s y^t$ and the Galois group is generated by σ, which is given by $\sigma^* x = \alpha x$ and $\sigma^* y = \beta y$, if $st \neq 0$, then we see that the branch points of φ are only $(0:1)$ and $(1:0)$. According to the Riemann–Hurwitz formula, the genus of C must be equal to zero, which is a contradiction. We conclude Claim 2.3.

Therefore, the morphism defined by the function x^s or y^t is of the Galois type. Hence, we have that the morphism defined by the function x or y is of the Galois type, of which the degree is either a or b. □

In the case for which a curve has a weak GW point and its Weierstrass semigroup is generated by two integers, the structure of the function field of the curve is as follows.

Theorem 2.2. *Let P be either a GW point or pseudo-GW point, respectively, on a curve C with $H(P) = \langle a, b\rangle$. Then, there exist $x, y \in k(C)$ with $(x)_\infty = aP$ and $(y)_\infty = bP$ such that $k(C) = k(x, y)$ and $y^a = \prod_{i=1}^{b}(x - c_i)$ or $x^b = \prod_{i=1}^{a}(y - c_i)$, respectively, where each c_i is a distinct element of k.*

We conclude Theorem 2.2 by Lemmas 2.3 and 2.4 below.

Lemma 2.3. *Let P be a GW point of a curve C with $H(P) = \langle a, b\rangle$. Then, for any $x \in k(C)$ with $(x)_\infty = aP$, there exists a rational function $y \in k(C)$ with $(y)_\infty = bP$ such that $k(C) = k(x, y)$ and $y^a = \prod_{i=1}^{b}(x - c_i)$, where each c_i is a distinct element of k.*

Proof. First, we show that we have $k(C) = k(x, y)$ for any $x, y \in k(C)$ with $(x)_\infty = aP$ and $(y)_\infty = bP$. We have that $H^0(C, \mathcal{O}_C(aP)) = \langle 1, x\rangle$. We may assume that $\Phi_{|aP|}$ is a morphism sending Q to $(1 : x(Q))$, which implies that $k(C) \supset k(x)$. Then, $y^a \in H^0(C, \mathcal{O}_C(abP))$. We have that

$$H^0(C, \mathcal{O}_C(abP)) = \langle \{x^i y^j \mid 0 \le i, 0 \le j, ai + bj < ab\} \cup \{x^b\}\rangle.$$

Hence, we obtain

$$y^a = \sum_{j=0}^{a-1} \sum_{ai+bj<ab} c_{ij} x^i y^j + c_{b0} x^b.$$

Thus, we have that $k(x) \subset k(x, y) \subset k(C)$ and the degree of the extension $k(x, y)/k(x)$ equals a, which implies that $k(x, y) = k(C)$.

Because $\Phi_{|aP|} : C \to \mathbb{P}^1$ is cyclic, there exists an automorphism σ of C of which the order equals a, which induces an automorphism σ^* of $H^0(C, \mathcal{O}_C(bP)) = \langle 1, x, x^2, \ldots, x^t, y\rangle$, where $t = [b/a] = \max\{n \in \mathbb{Z} \mid na \le b\}$, satisfying $\sigma^* 1 = 1$, $\sigma^* x = x$, $\sigma^* x^2 = x^2$,…, $\sigma^* x^t = x^t$ and $\sigma^* y \ne y$. The eigenvalues of σ^* are $1, 1, \ldots. 1, \zeta$, where ζ is a primitive ath root of unity. Take y as an eigenvector of ζ. Then we obtain $(y)_\infty = bP'$ and $\sigma^* y = \zeta y$. We have

$$y^a + f_{a-1}(x) y^{a-1} + \cdots + f_1(x) y + f_0(x) = 0.$$

Hence, we obtain

$$\begin{aligned} 0 &= \sigma^*(y)^a + f_{r-1}(x)\sigma^*(y)^{a-1} + \cdots + f_1(x)\sigma^*(y) + f_0(x) \\ &= \zeta^a y^a + f_{a-1}(x)\zeta^{a-1} y^{a-1} + \cdots + f_1(x)\zeta y + f_0(x) \end{aligned}$$

$$= y^a + f_{a-1}(x)\zeta^{a-1}y^{a-1} + \cdots + f_1(x)\zeta y + f_0(x),$$

which implies that

$$(\zeta^{a-1} - 1)f_{a-1}(x)y^{a-1} + \cdots + (\zeta - 1)f_1(x)y = 0.$$

Hence, we obtain $f_{a-1}(x) = 0, \ldots, f_1(x) = 0$, which implies that $y^a = -f_0(x)$, where the degree of $f_0(x)$ equals b. In view of $H(P) = \langle a, b\rangle$, the genus of C equals $(a-1)(b-1)/2$. Following the Riemann–Hurwitz formula, the sum of the ramification indices of $\Phi_{|aP|}$ equals $(a-1)(b+1)$. Therefore, the morphism $\Phi_{|aP|}$ must have at least $b+1$ branch points on $\mathbb{P}^1$, which implies that $f_0(x)$ has no multiple factors. $\square$

Lemma 2.4. *Let P be a weak GW point of a curve C with $b \in \mathrm{degGW}(P)$ and $H(P) = \langle a, b\rangle$. Then, there exist rational functions x and $y \in k(C)$ with $(x)_\infty = aP$ and $(y)_\infty = bP$ such that $k(C) = k(x, y)$ and $x^b = \prod_{i=1}^{a}(y - c_i)$, where each c_i is a distinct element of k.*

Proof. We assume that P is a total ramification point of a Galois covering $\Phi : C \to \mathbb{P}^1$ of degree b, and for some rational function y on C with $(y)_\infty = bP$ the morphism Φ sends a point Q of C to $(1 : y(Q))$. Let x be a rational function on C with $(x)_\infty = aP$. Then we have that $H^0(C, \mathcal{O}_C(bP)) = \langle 1, x, x^2, \ldots, x^t, y\rangle$ where we set $t = [b/a] = \max\{n \in \mathbb{Z} \mid na \le b\}$. Moreover, the inclusion $k(y) \subset k(C)$ is a cyclic extension of degree b. We have that

$$x^b \in H^0(C, \mathcal{O}_C(abP)) = \langle \{x^i y^j \mid 0 \le i, 0 \le j, ai + bj < ab\} \cup \{y^a\}\rangle.$$

Hence, we obtain $k(C) = k(x, y)$.

Because Φ is cyclic, there exists an automorphism σ of C with $\mathrm{ord}(\sigma) = b$ such that σ induces an automorphism σ^* of $H^0(C, \mathcal{O}_C(bP)) = \langle 1, x, x^2, \ldots, x^t, y\rangle$ satisfying $\sigma^*1 = 1$, $\sigma^*y = y$ and $\sigma^*x \ne x$. Because $\sigma(P) = P$, we have that $\sigma^*x \in H^0(C, \mathcal{O}_C(aP)) = \langle 1, x\rangle$. Let $\sigma^*x = d_1x + d_0$, where $d_1, d_0 \in k$. As the order of σ equals $b \ne \infty$, we have $d_1 \ne 1$. Then, $X := (d_1 - 1)x + d_0$ holds that $(X)_\infty = aP$ and $\sigma^*(X) = d_1X$. By taking X instead of x, we may assume that $\sigma^*x = \zeta x$ where ζ is a primitive bth root of unity. We have that

$$x^b + f_{b-1}(y)x^{b-1} + \cdots + f_1(y)x + f_0(y) = 0,$$

where the degree of $f_0(y)$ is a. By applying σ^* to the equation we obtain

$$x^b + f_{b-1}(y)\zeta^{b-1}x^{b-1} + \cdots + f_1(y)\zeta x + f_0(y) = 0.$$

Hence, we obtain $x^b = -f_0(y)$. As the genus of C equals $(a-1)(b-1)/2$ and the degree of $f_0(y)$ equals a, we have that $f_0(y)$ cannot have a multiple factor. $\square$

It is possible to find weak GW points directly from Galois points for a plane curve, as follows.

Theorem 2.3. *If a point $P \in \mathbb{P}^2$ is a Galois point for a plane curve C, then some of the ramification points of π_P are weak GW points with $H(Q) = \langle d-1, d\rangle$. More precisely,*

(1) *if P is an inner Galois point, then every ramification point Q of π_P is a GW point, and*
$$H(Q) = \begin{cases} \langle d-1, d\rangle \text{ if } Q = P \\ \langle \{i(d-2)+1 \mid i = 1, \dots, d-1\}\rangle \text{ if } Q \neq P \end{cases};$$

(2) *if P is an outer Galois point, then every ramification point Q of π_P is a pseudo-GW point with $H(Q) = \langle d-1, d\rangle$, except for the case where C is projectively equivalent to $XZ^{d-1} + X^d + Y^d = 0$. (For the case where C is defined by $XZ^{d-1} + X^d + Y^d = 0$, see Example 3.1.)*

Conversely, if Q is a weak GW point of a curve C with $H(Q) = \langle d-1, d\rangle$, then C is isomorphic to a plane curve and Q is a ramification point of the projection from a Galois point.

Proof. Let P be an inner Galois point for a plane curve C and Q be another ramification point of π_P. Based on Theorem 2.1, we may assume that C and P are expressed as (1). Because $I_P(C, T_PC) = d$ and $I_Q(C, T_QC) = d-1$, Lemma 2.1 indicates that $H(P) = \langle d-1, d\rangle$ and $\langle \{i(d-2)+1 \mid i = 1, \dots, d-1\}\rangle$. Further, P and Q are total ramification points of π_P and $\deg(\pi_P) = d-1$; therefore, P and Q are GW points.

Let P be a GW point of a curve C with $H(P) = \langle d-1, d\rangle$ ($d \geq 4$). According to Theorem 2.2, there exist $x, y \in k(C)$ with $(x)_\infty = (d-1)P$ and $(y)_\infty = dP$ such that $k(C) = k(x, y)$ and $y^{d-1} = \prod_{i=1}^{d}(x - c_i)$, where each c_i is a distinct element of k. Then, we have that $H^0(C, \mathcal{O}_C(dP)) = \langle 1, x, y\rangle$, $\Phi_{|dP|} : C \to \mathbb{P}^2$ is an embedding and its image is the plane curve $XZ^{d-1} - \prod_{i=1}^{d}(Y - c_iX) = 0$. Moreover, $\Phi_{|dP|}(P) = (0:0:1)$ is an inner Galois point and the point $(0:0:1)$ is a total ramification point of $\pi_{(0:0:1)}$.

Let P be an outer Galois point for a plane curve C and Q be a ramification point of π_P. According to Theorem 2.1, we may assume that C and P are expressed as (1). Because $I_Q(C, T_QC) = d$, by Lemma 2.1, we see that $H(Q) = \langle d-1, d\rangle$. Further, Q is a total ramification point of π_P and $\deg(\pi_P) = d$; thus, Q is a weak GW point. Assume that C is not projectively equivalent to the curve $XZ^{d-1} + X^d + Y^d = 0$. Then, Q is not an inner Galois point; hence, Q is a pseudo-GW point.

Let Q be a pseudo-GW point of a curve C with $H(Q) = \langle d-1, d\rangle$ ($d \geq 4$). According to Theorem 2.2, there exist $x, y \in k(C)$ with $(x)_\infty = (d-1)P$ and $(y)_\infty = dP$ such that $k(C) = k(x, y)$ and $x^d = \prod_{i=1}^{d-1}(y - c_i)$, where each c_i is a distinct element of k. Then, we have that $H^0(C, \mathcal{O}_C(dP)) = \langle 1, x, y\rangle$, $\Phi_{|dP|} : C \to \mathbb{P}^2$ is an embedding, its image is the plane curve $Y^d - X\prod_{i=1}^{d-1}(Z - c_iX) = 0$. The point $P := (0:1:0)$ is an outer Galois point and $\Phi_{|dP|}(Q) = (0:0:1)$ is a total ramification point of π_P. $\square$

3. Examples

In this section, we state some important examples of Theorems 2.3 and 1.1.

Example 3.1. Let $C \subset \mathbb{P}^2$ be a plane curve defined by $XZ^{d-1}+X^d+Y^d=0$. Note that if $d \geq 5$ or $d=4$, then there exist either two or five Galois points, respectively, for C, which either consist of one inner Galois point $P := (0:0:1)$ or four inner Galois points $P_j := ((-1)^{j/3} : 0 : 1)$, respectively, where $j = 0, 1, 2$, and $P_3 := (0:0:1)$, and one outer Galois point $P_{\text{out}} = (0:1:0)$.

(1) Assume that $d \geq 5$. We study C as an example of Theorem 2.3.

Let us consider on the Galois covering π_P. The ramification points of the projection π_P are P and $Q_j := ((-1)^{j/d} : 1 : 0)$, where $j = 0, 1, \ldots, d-1$. Because the tangent line at P to C is $T_PC : X = 0$, we have that $I_P(C, T_PC) = d$. Hence, according to Lemma 2.1, we have that $H(P) = \langle d-1, d\rangle$ and P is a Weierstrass point. The tangent line at Q_j to C is $T_{Q_j}C : X - (-1)^{j/d}Y = 0$; thus, we have that $I_{Q_j}(C, T_{Q_j}C) = d-1$. Hence, by Lemma 2.1, we have that $H(Q_j) = \langle \{i(d-2)+1 \mid i = 1, \ldots, d-1\}\rangle$ and Q_j is a Weierstrass point. Because P and Q_j are total ramification points of the Galois covering π_P of degree $d-1$, these are GW points.

Let us consider the Galois covering $\pi_{P_{\text{out}}}$. The ramification points of the projection $\pi_{P_{\text{out}}}$ are P and $Q'_j := ((-1)^{j/(d-1)} : 0 : 1)$, where $j = 0, 1, \ldots, d-2$. The tangent line at Q'_j to C is $T_{Q'_j} : X - (-1)^{j/(d-1)}Z = 0$; therefore, we have that $I_{Q'_j}(C, T_{Q'_j}C) = d$. Hence, according to Lemma 2.1, we have that $H(Q'_j) = \langle d-1, d\rangle$ and Q'_j is a Weierstrass point. Because P and Q'_j are total ramification points of the Galois covering $\pi_{P_{\text{out}}}$ of degree d, these are weak GW points. Further, as P is a GW point, P is not a pseudo-GW point. As Q'_j is not a Galois point, we infer that Q'_j is not a GW point; thus, Q'_j is a pseudo-GW point.

(2) Assume that $d = 4$. We study C as an example of Theorem 2.3.

By an argument similar to that used for $d \geq 5$, we have the following. The points P_j are GW points with $H(P_j) = \langle 3, 4\rangle$. The ramification points of the projection π_{P_j} are P_j and three other points Q_{jk}, where $k = 1, 2, 3$. The points Q_{jk} are total ramification points of π_{P_j} and GW points with $H(Q_{jk}) = \langle 3, 5, 7\rangle$. The ramification points of $\pi_{P_{\text{out}}}$ are four inner Galois points P_j, and these are GW points.

(3) Assume that $d \geq 5$ and d is odd. We study C as an example of Theorem 1.1. Let σ_P be an automorphism belonging to the inner Galois point P, and $\iota := \sigma_P^{(d-1)/2}$. According to Lemma 4.1, a non-trivial involution of C is only ι. We have that σ_P is expressed as (2), and

$$\iota_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Considering that d is odd, let C' be the curve given by the equation $\bar{X}\bar{Z}^{(d-1)/2}+\bar{X}^d+\bar{Y}^d=0$ in the weighted projective space $\mathbb{P}(1,1,2)$, where $(\bar{X},\bar{Y},\bar{Z})$ are weighted homogeneous coordinates. Then, C' is nonsingular. (Note that the point $(0,0,1)$ is a singular point of $\mathbb{P}(1,1,2)$ and C is nonsingular at $(0,0,1)$.) Let $\varphi : C \to C'$ be a double covering defined by $(X:Y:Z) \mapsto (X,Y,Z^2)$. Then, $\varphi\circ\iota=\varphi$. Let $P' := \varphi(P)$. Because P is a ramification point of φ, according to Lemma 2.2 we have that $H(P')=\langle (d-1)/2, d\rangle$ and P' is a Weierstrass point. Let $f : C' \to \mathbb{P}^1$ be a morphism defined by $(\bar{X},\bar{Y},\bar{Z}) \mapsto (\bar{X}:\bar{Y})$. Then, f is a Galois covering of degree $(d-1)/2$ and P' is a total ramification point of f. Hence, P' is a GW point with $H(P')=\langle (d-1)/2, d\rangle$.

(4) Assume that $d \geq 5$ and that d is even. We study C as an example of Theorem 1.1. Let ι be an involution of C expressed as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Note that $\iota=\sigma_{P_{\mathrm{out}}}^{d/2}$, where $\sigma_{P_{\mathrm{out}}}$ is an automorphism belonging to the outer Galois point P_{out}.) Considering that d is even, let C' be the curve given by the equation $\bar{X}\bar{Z}^{d-1}+\bar{X}^d+\bar{Y}^{d/2}=0$ in the weighted projective space $\mathbb{P}(1,2,1)$, where $(\bar{X},\bar{Y},\bar{Z})$ are weighted homogeneous coordinates. Then, C' is nonsingular. Let $\varphi : C \to C'$ be a double covering defined by $(X:Y:Z)\mapsto(X,Y^2,Z)$. Then, $\varphi\circ\iota=\varphi$. Let $P' := \varphi(P)$. Because P is a ramification point of φ, according to Lemma 2.2, we have that $H(P')=\langle d/2, d-1\rangle$ and P' is a Weierstrass point. Let $f : C' \to \mathbb{P}^1$ be a morphism defined by $(\bar{X},\bar{Y},\bar{Z})\mapsto(\bar{X}^2:\bar{Y})$. Then, f is a Galois covering of degree $d-1$, and P' is a total ramification point of f. Hence, P' is a weak GW point with $d-1\in \mathrm{degGW}(P')$ and $H(P')=\langle d/2, d-1\rangle$. On the other hand, as the morphism defined by $(\bar{X},\bar{Y},\bar{Z})\mapsto(\bar{X}:\bar{Z})$, which is $\Phi_{|(d/2)P'|}$, is a Galois morphism of degree $d/2$, we have that P' is a GW point with $H(P')=\langle d/2, d-1\rangle$.

Example 3.2. Let $C\subset\mathbb{P}^2$ be a Fermat curve defined by $X^d+Y^d+Z^d=0$ ($d\geq 4$). Then, there exist three outer Galois points for C, which are $P_1 := (0:0:1)$, $P_2 := (0:1:0)$, $P_3 := (1:0:0)$.

(1) We study C as an example of Theorem 2.3. The ramification points of π_{P_1}, π_{P_2}, and π_{P_3} are $Q_{1j} := ((-1)^{j/d}:1:0)$, $Q_{2j} := (1:0:(-1)^{j/d})$, and $Q_{3j} := (0:(-1)^{j/d}:1)$, respectively, where $j=0,1,\ldots,d-1$. By an argument similar to that in Example 3.1(1), we see that every Q_{ij} is a pseudo-GW point with $H(Q_{ij})=\langle d-1, d\rangle$.

(2) Assume that $d\geq 5$ and that d is even. We study C as an example of Theorem 1.1. Let σ_{P_i} be an automorphism of C belonging to P_i and let $\iota_i := \sigma_{P_i}^{d/2}$. According to Lemma 4.3, a non-trivial involution such that

$\mathrm{Fix}(\sigma_{P_i}) \cap \mathrm{Fix}(\iota_i) \cap C \neq \emptyset$ is only ι_i. Let $\varphi_i : C \to C'_i$ be the double covering obtained by ι_i. Note that $\mathrm{Fix}(\iota_i) = \mathrm{Fix}(\sigma_{P_i}) = \{P_i\} \cup \ell_i$, where $\ell_1 : Z = 0$, $\ell_2 : Y = 0$, $\ell_3 : X = 0$. Let Q_i be a common ramification point of π_{P_i} and φ_i, i.e., $Q_i \in C \cap \ell_i$, and $Q'_i := \varphi_i(Q_i)$. Then, Q'_i is a GW point of C'_i with $H(Q'_i) = \langle d/2, d-1\rangle$.

For example, let us consider the outer Galois point P_1. We may assume that σ_{P_1} is expressed as (2). Thus,

$$\iota_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $Q_1 := ((-1)^{1/d} : 1 : 0) \in C \cap \ell_1$. Considering that d is even, let C'_1 be the curve given by the equation $\bar{X}^d + \bar{Y}^d + \bar{Z}^{d/2} = 0$ in the weighted projective space $\mathbb{P}(1, 1, 2)$, where $(\bar{X}, \bar{Y}, \bar{Z})$ are weighted homogeneous coordinates. Then, C'_1 is nonsingular. Let $\varphi_1 : C \to C'_1$ be the double covering obtained by $(X : Y : Z) \mapsto (X, Y, Z^2)$. Then, $\varphi_1 \circ \iota_1 = \varphi_1$. Let $Q'_1 := \varphi_1(Q_1) = ((-1)^{1/d}, 1, 0)$. As Q_1 is a ramification point of φ, according to Lemma 2.2, we have that $H(Q'_1) = \langle d/2, d-1\rangle$ and that Q'_1 is a Weierstrass point. Let $f : C' \to \mathbb{P}^1$ be the morphism defined by $(\bar{X}, \bar{Y}, \bar{Z}) \mapsto (\bar{X} : \bar{Y})$. Then, f is a Galois covering of degree $d/2$, and Q'_1 is a total ramification point of f. Hence, Q'_1 is a GW point with $H(Q'_1) = \langle d/2, d-1\rangle$.

(3) Assume that $d \geq 5$ and that d is odd. We study C as an example of Theorem 1.1. For an involution $\iota \in \mathrm{Aut}(C)$, because $\mathrm{ord}(\iota) = 2$ and the number of outer Galois points for C is three, we have that one of the Galois points satisfies $\iota(P_i) = P_i$. Note that $\mathrm{Fix}(\sigma_{P_i}) = \{P_i\} \cup \ell_i$, where $\ell_1 : Z = 0$, $\ell_2 : Y = 0$, $\ell_3 : X = 0$. As $\#(C \cap \ell_i) = d$ is odd, there exists a point Q in $\mathrm{Fix}(\sigma_{P_i}) \cap \mathrm{Fix}(\iota) \cap C$. Let $\varphi : C \to C'$ be the double covering obtained by ι and $Q' := \varphi(Q)$. Then, the point Q' is either a GW point or a pseudo-GW point, respectively, of C' with $H(Q') := \langle (d-1)/2, d\rangle$ if $d = 5$ or $d \geq 7$, respectively.

For example, let

$$\iota := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, ι is an involution of C, and $\iota(P_1) = P_1$. Because σ_{P_1} is expressed as (2), we have that $\mathrm{Fix}(\sigma_{P_i}) \cap \mathrm{Fix}(\iota) \cap C = \{Q = (-1 : 1 : 0)\}$. By the projective transformation

$$T = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can express that $C : (X-Y)^d + (X+Y)^d + Z^d = 0$, $P_1 = (0:0:1)$, $Q = (0:1:0)$,

$$\sigma_{P_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta \end{pmatrix} \text{ and } \iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let C' be the curve given by the equation $(\bar{X} - \bar{Y}^{1/2})^d + (\bar{X} + \bar{Y}^{1/2})^d + \bar{Z}^d = 0$ in the weighted projective space $\mathbb{P}(1,2,1)$, where $(\bar{X}, \bar{Y}, \bar{Z})$ are the weighted homogeneous coordinates. Note that the left side of the equation will be a polynomial when expanded, because d is odd. (Note that the point $(0,1,0)$ is a singular point of $\mathbb{P}(1,2,1)$, C is nonsingular at $(0,1,0)$.) Let $\varphi : C \to C'$ be the double covering defined by $(X : Y : Z) \mapsto (X, Y^2, Z)$. Then, $\varphi \circ \iota = \varphi$. Let $Q' := \varphi(Q)$. Because Q is a ramification point of φ, according to Lemma 2.2, we have that $H(Q') = \langle (d-1)/2, d \rangle$ and that Q' is a Weierstrass point. Let $f : C' \to \mathbb{P}^1$ be the morphism obtained by $(\bar{X}, \bar{Y}, \bar{Z}) \mapsto (\bar{X}^2 : \bar{Y})$. Then, f is a Galois covering of degree d and Q' is a total ramification point of f. Hence, Q' is a weak GW point with $H(Q') = \langle (d-1)/2, d \rangle$.

If $d \geq 7$, then it is clear that Q' is not a GW point, i.e., $\Phi_{|((d-1)/2)Q'|} : C' \to \mathbb{P}^1$ is not of the Galois type, as follows. Because every inflection point of C is a total inflection point, the ramification index of every ramification point of π_Q is equal to 2 except for Q. Thus, as $\Phi_{|((d-1)/2)Q'|} \circ \varphi = \pi_Q$, the ramification index of every ramification point is equal to 2 except for Q'. According to the Riemann–Hurwitz formula, the number of ramification points of $\Phi_{|((d-1)/2)Q'|}$ with ramification index 2 is equal to $d(d-3)/2$. If $\Phi_{|((d-1)/2)Q'|}$ is of the Galois type, then $(d-1)/4$ is an integer and $(d-1)/4$ divides $d(d-3)/2$, which is a contradiction except for the case $d = 5$. Hence, if $d \geq 7$, then Q' is a pseudo-GW point. If $d = 5$, then Q' is a GW point because $\deg \Phi_{|((d-1)/2)Q'|} = (d-1)/2 = 2$.

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. First, we separately discuss four cases in which, for the first half of the theorem, d is odd or even and a Galois point P is inner or outer. Lastly, we prove the latter half of the theorem.

Lemma 4.1. *Let C be a plane curve of odd degree $d \geq 5$ and P an inner Galois point for C. Let σ be a generator of its Galois group, which is a cyclic group. Then, $\sigma^{(d-1)/2}$ is the unique involution of C.*

Proof. We may assume that C, P, and σ are expressed as (1) and (2). Let ι be an involution of C, i.e., $\iota \in \mathrm{Aut}(C)$, $\iota \neq \mathrm{id}_C$, and $\iota^2 = \mathrm{id}_C$. According to Theorem 2.1(3), we have that $\iota(P) = P$. The line $\ell_X : X = 0$ is the tangent line at P for C; thus, $\iota(\ell_X) = \ell_X$. Because $\mathrm{Fix}(\sigma) = \{P\} \cup \ell_Z$, where $\ell_Z : Z = 0$ and $\iota(P) = P$, we have that $\iota(\ell_Z) = \ell_Z$; hence, $\iota((0:1:0)) = (0:1:0)$.

As $\#(C \cap \ell_Z) = d$ is odd and the order of ι equals two, there exists a point $Q \in C \cap \ell_Z$ such that $\iota(Q) = Q$. We may assume that $Q = (1 : 0 : 0)$ by a suitable transformation retaining the expressions (1) and (2). Then,

$$\iota = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If ι is either the first or second matrix, then from $\iota^* C = C$ we infer that X divides F_d; thus C is reducible, which is a contradiction. Hence, ι must be the third matrix, and $\iota = \sigma^{(d-1)/2}$, which is the unique non-trivial involution of C. □

Lemma 4.2. *With the same notations and assumptions as in Lemma 4.1, let $\iota := \sigma^{(d-1)/2}$, φ be a double covering $C \to C' := C/\langle \iota \rangle$ and $P' := \varphi(P)$. Then:*

(1) $H(P') = \langle (d-1)/2, d \rangle$;
(2) *P' is a GW point;*
(3) $\Phi_{|((d-1)/2)P'|} : C' \to \mathbb{P}^1$ *is a Galois covering and* $\pi_P = \Phi_{|((d-1)/2)P'|} \circ \varphi$.

Proof. According to Theorem 2.3(1) and Lemma 2.2, assertion (1) holds true.

The projection $\pi_P = \Phi_{|(d-1)P|}$ is a Galois covering, P is a total ramification point of π_P, and $\mathrm{Gal}(C/\mathbb{P}^1) = \langle \sigma \rangle$. The subgroup $\langle \iota \rangle$ is a normal subgroup of $\mathrm{Gal}(C/\mathbb{P}^1)$. Hence, we conclude (2) and (3). □

Lemma 4.3. *Let C be a plane curve of even degree $d \geq 5$. Let P be an outer Galois point for C and let σ be a generator of its Galois group, which is a cyclic group. Then, $\iota := \sigma^{d/2}$ is the unique involution of C such that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$.*

Proof. First, we assume that C is not a projective equivalent to a Fermat curve. We may assume that C, P, and σ are expressed as (1) and (2). Let ι be an involution of C such that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$. According to Theorem 2.1(3), we have that $\iota(P) = P$. Because $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$, $\mathrm{Fix}(\sigma) = \{P\} \cup \ell_Z$, where $\ell_Z : Z = 0$, and $\#(C \cap \ell_Z) = d$ is even, there exist two points $Q_1, Q_2 \in \mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C$. We may assume that $Q_1 = (1 : 0 : 0)$ and $Q_2 = (0 : 1 : 0)$ by a suitable transformation, thereby retaining the expressions (1) and (2). Then,

$$\iota = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If ι is either the first or second matrix, then from $\iota^* C = C$ and $Q_1, Q_2 \in C$ we infer that X^2 or Y^2 divides F_d; thus, C is singular, which is a contradiction. Hence, ι must be the third matrix, and $\iota = \sigma^{d/2}$, which is the unique non-trivial involution such that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$.

Assuming that C is a Fermat curve, that is, C is defined by $X^d + Y^d + Z^d = 0$; then, there exist three Galois points $P_1 := (0 : 0 : 1)$, $P_2 := (0 : 1 : 0)$, and

$P_3 := (1:0:0)$. We may assume that $P = P_1$ and that σ is expressed as (2). Let ι be an involution of C such that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$. If $\iota(P) = P$, then by an argument similar to that above, we have that $\iota = \sigma^{d/2}$. If $\iota(P) \neq P$, then, because P_1, P_2 and P_3 are all Galois points, we have that $\iota(P) = P_2$ or P_3. Let us assume that $\iota(P) = P_2$. Then, $\iota(P_2) = P$ and $\iota(P_3) = P_3$. Thus ι must be

$$\iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & c \\ 0 & c^{-1} & 0 \end{pmatrix},$$

where $c \in k \setminus \{0\}$. However, $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C = \emptyset$, which is a contradiction. Similarly, $\iota(P) \neq P_3$. Therefore, in the case that C is a Fermat curve, a non-trivial involution ι such that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$ must be $\iota = \sigma^{d/2}$, which is unique. □

Lemma 4.4. *With the same notations and assumptions as in Lemma 4.3, let $\iota := \sigma^{d/2}$, φ be a double covering $C \to C' := C/\langle\iota\rangle$ and $P' := \varphi(P)$. Then, for a fixed point $Q \in C$ of σ, $Q' := \varphi(Q)$ is a GW point with $H(Q') = \langle d/2, d-1\rangle$ and $\Phi_{|(d/2)\,Q'|} : C' \to \mathbb{P}^1$ is of the Galois type.*

Proof. According to Theorem 2.3(2) and Lemma 2.2, we have $H(Q') = \langle d/2, d-1\rangle$. Because $\pi_P = \Phi_{|dQ|}$ and $\langle\iota\rangle$ is a normal subgroup of $\mathrm{Gal}(C/\mathbb{P}^1) = \langle\sigma\rangle$, we have that $\pi_P = \Phi_{|d/2\,Q'|} \circ \varphi$ and $\Phi_{|d/2\,Q'|}$ is of the Galois type. □

Lemma 4.5. *Let C be a plane curve of even degree $d \geq 5$, and let P be an inner Galois point for C. Assume that there exists an involution ι. Then, $\iota(P) = P$.*

Proof. According to Theorem 2.1, the point P is only an inner Galois point; thus, $\iota(P) = P$. □

Lemma 4.6. *With the same notations and assumptions as in Lemma 4.5, let $\varphi : C \to C' := C/\langle\iota\rangle$ be the double covering obtained by ι, and $P' := \varphi(P)$. Then*:

(1) *P' is a weak GW point, and $H(P') = \langle d/2, d-1\rangle$;*
(2) *there exists a Galois covering $f : C' \to \mathbb{P}^1$ of degree $d-1$ such that P' is a total ramification point of f.*

Proof. We may assume that C, P, and σ are expressed as (1) and (2). Following Theorem 2.1(3), we have that $\iota(P) = P$. Because the line $\ell_X : X = 0$ is the tangent line at P for C, $\iota(\ell_X) = \ell_X$. As $\mathrm{Fix}(\sigma) = \{P\} \cup \ell_Z$, where $\ell_Z : Z = 0$, and $\iota(P) = P$, we have that $\iota(\ell_Z) = \ell_Z$. Hence,

$$\iota = \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a, b, c \in k$. Thus, $\iota\sigma = \sigma\iota$. By taking a suitable projective transformation, we can diagonalize ι such that expressions (1) and (2) are retained. Because $\iota^2 = \mathrm{id}_C$,

$$\iota = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Because $\iota^* C = C$, if ι is the first matrix, then we infer that C is reducible, which is a contradiction. Furthermore, ι cannot be the third matrix, because $\iota^* C = C$. Hence, ι must be the second matrix. Because $\iota^* C = C$ again, $f_d(X, -Y) = f_d(X, Y)$. Namely, there exists a polynomial $g(\bar{X}, \bar{Y})$ such that $g(X, Y^2) = f_d(X, Y)$. Let C' be the curve defined by $\bar{X}\bar{Z}^{d-1} + g(\bar{X}, \bar{Y}) = 0$ in $\mathbb{P}(1, 2, 1)$, where $(\bar{X}, \bar{Y}, \bar{Z})$ are the weighted homogeneous coordinates of $\mathbb{P}(1, 2, 1)$. We have that C' is nonsingular, $C' \cong C/\langle\iota\rangle$, $\varphi : (X : Y : Z) \mapsto (X, Y^2, Z)$, and $P' = \varphi(P) = (0, 0, 1)$. Because P is a ramification point of φ, according to Theorem 2.3(1) and Lemma 2.2, we have that $H(P') = \langle d/2, d-1\rangle$. Let $f : C' \to \mathbb{P}^1$ be the morphism defined by $(\bar{X}, \bar{Y}, \bar{Z}) \mapsto (\bar{X}^2 : \bar{Y})$. Then, f is a Galois covering of degree $d - 1$, and P' is its total ramification point. Therefore, P' is a weak GW point of C' with $H(P') = \langle d/2, d - 1\rangle$. □

Lemma 4.7. *Let C be a plane curve of odd degree $d \geq 5$, and let P be an outer Galois point for C. Let σ be a generator of its Galois group. Assume that there exists an involution ι such that $\iota(P) = P$. Then,* $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C$ *consists of one point.*

Proof. We may assume that C, P, and σ are expressed as (1) and (2). Then, $\mathrm{Fix}(\sigma) \cap C = \ell_Z \cap C$, where $\ell_Z : Z = 0$, and these d points are total inflexion points. Because $C \cap l_Z = \{ Q \in C \mid Q \text{ is a total inflexion point and } P \in T_Q C \}$ and $\iota(P) = P$, we have that $\iota(C \cap l_Z) = C \cap l_Z$. As d is odd, $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C \neq \emptyset$. Let $Q \in \mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C$. By changing the coordinates suitably, we may assume that $Q = (0 : 1 : 0)$. Because $\iota^2 = \mathrm{id}_C$, $\iota(P) = P$, $\iota(Q) = Q$ and $\iota(l_Z) = l_Z$, we have that

$$\iota = \begin{pmatrix} 1 & 0 & 0 \\ c & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{3}$$

where $c \in k$. If ι is the third matrix, then $\iota^* C \neq C$, which is a contradiction. Hence, $\iota|_{l_Z} \neq \mathrm{id}$. Because d is odd and $Q = (0 : 1 : 0) \in \mathrm{Fix}(\iota) \cap C$, we have that $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C = \{Q\}$. □

Lemma 4.8. *With the same notations and assumptions as in Lemma 4.7, let $\varphi : C \to C' := C/\langle\iota\rangle$ be the double covering obtained by ι, and let Q be the element of* $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C$. *Then*:

(1) *$Q' := \varphi(Q)$ is a weak GW point, and $H(Q') = \langle (d-1)/2, d\rangle$;*

(2) *there exists a Galois covering $f : C' \to \mathbb{P}^1$ of degree d such that Q' is a total ramification point of f.*

Proof. We may assume that C, P, and σ are expressed as (1) and (2). According to the proof of Lemma 4.7, we may assume that $Q = (0 : 1 : 0)$ in which case ι can either be expressed as the first or second matrix of (3). Because $\iota\sigma = \sigma\iota$, by changing the coordinates in a suitable manner, we can diagonalize ι and we may assume that

$$\iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Because $\mathrm{Fix}(\sigma) \cap \mathrm{Fix}(\iota) \cap C = \{Q\}$, we have that $(1 : 0 : 0) \notin C$. As $\iota^* C = C$, this indicates that ι must be the first matrix of (4), and C is expressed as $Z^d + g(X, Y^2) = 0$, where $g(X, Y^2)$ is a homogeneous polynomial of degree d.

Let C' be the curve in $\mathbb{P}(1, 2, 1)$ defined by $\bar{Z}^d + g(\bar{X}, \bar{Y}) = 0$, where $(\bar{X}, \bar{Y}, \bar{Z})$ are the weighted homogeneous coordinates. Considering that C' is nonsingular, we have that $C' \cong C/\langle\iota\rangle$ and $\varphi : (X : Y : Y) \mapsto (X, Y^2, Z)$. Because Q is a ramification point of φ, according to Lemma 2.2, we have that $H(Q') = \langle (d-1)/2, d\rangle$. Let $f : C' \to \mathbb{P}^1$ be the morphism obtained by $(\bar{X}, \bar{Y}, \bar{Z}) \mapsto (\bar{X}^2 : \bar{Y})$. Then, f is a Galois covering of degree d and Q' is its total ramification point. Therefore, Q' is a weak GW point of C' with $H(P') = \langle (d-1)/2, d\rangle$. □

According to Lemmas 4.1–4.8, we conclude the former part of Theorem 1.1.

Lemma 4.9. *Let C' be a curve and P' a weak GW point.*

(1) *If $H(P') = \langle r, 2r+1\rangle$ $(r \geq 2)$ and P' is a GW point, then there exists a plane curve C of degree $d = 2r+1$, an inner Galois point P, and a double covering $\varphi : C \to C'$ such that $P' = \varphi(P)$.*
(2) *If $H(P') = \langle r, 2r-1\rangle$ $(r \geq 3)$ and P' is a GW point, then there exists a plane curve C of degree $d = 2r$, an outer Galois point P, and a double covering $\varphi : C \to C'$ such that $P' = \varphi(Q)$, where Q is a total ramification point of π_P and φ.*
(3) *If $H(P') = \langle r, 2r-1\rangle$ $(r \geq 3)$ and P' is a weak GW point with $2r-1 \in$* degGW(P')*, then there exists a plane curve C of degree $d = 2r$, an inner Galois point P, and a double covering $\varphi : C \to C'$ such that $P' = \varphi(P)$.*
(4) *If $H(P') = \langle r, 2r+1\rangle$ $(r \geq 2)$ and P' is a weak GW point with $2r+1 \in$* degGW(P')*, then there exists a plane curve C of degree $d = 2r+1$, an outer Galois point P and a double covering $\varphi : C \to C'$ such that $P' = \varphi(Q)$, where Q is a total ramification point of π_P and φ.*

Proof. We prove assertion (1). Let $f : C' \to \mathbb{P}^1$ be a Galois covering of degree r with P' as its total ramification point. According to Theorem 2.2, we may assume that $k(C') = k(u, v)$, $f^*(k(\mathbb{P}^1)) = k(u)$, and the minimal equation of v is $v^r = \prod_{i=1}^{2r+1}(u - d_i)$, where $d_i \in k$. Let C be the plane curve given by the equation $XZ^{d-1} = \prod_{i=1}^{d}(X - d_iY)$, where $d = 2r + 1$. Then, C is nonsingular and $P := (0 : 0 : 1)$ is an inner Galois point. Moreover, we have an involution $\iota := \sigma^{(d-1)/2}$, where σ is an automorphism of C belonging to the Galois point P. We see that the natural morphism $\varphi : C \to C/\langle\iota\rangle$ is a double covering, $C/\langle\iota\rangle$ is isomorphic to C', and $\varphi(P) = P'$. Indeed, $C/\langle\iota\rangle$ is birationally equivalent to the curve

$$\bar{x}\bar{z}^{(d-1)/2} = \prod_{i=1}^{d}(\bar{x} - d_i\bar{y})$$

in the weighted projective space $\mathbb{P}(1, 1, 2)$, and $\Phi_{|(d-1)/2\,\varphi(P)|}$ is expressed as $(\bar{x}, \bar{y}, \bar{z}) \mapsto (\bar{x} : \bar{y})$, which induces the extension of function fields $k(u, v)/k(u)$, where $u = \bar{x}/\bar{y}$ and $v = \bar{z}/\bar{y}^2$.

By an argument similar to that above, we can prove assertions (2)–(4). $\square$

By Lemma 4.9, we conclude the latter part of Theorem 1.1.

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