A HALF-CENTERED STAR-OPERATION ON AN INTEGRAL DOMAIN

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Abstract. In this paper, we study the natural star-operation defined by the set of associated primes of principal ideals of an integral domain, which is called the g-operation. We are mainly concerned with the ideal-theoretic properties of this star-operation. In particular, we investigate DG-domains (i.e., integral domains in which each ideal is a g-ideal), which form a proper subclass of the DW-domains. In order to provide some original examples, we examine the transfer of the DG-property to pullbacks. As an application of the g-operation, it is shown that w-divisorial Mori domains can be seen as a Gorenstein analogue of Krull domains.

Introduction

All rings considered in this paper are integral domains. Let R be an integral domain with quotient field K. Denote by $F(R)$ (resp., $f(R)$) the set of nonzero fractional (resp., nonzero finitely generated fractional) ideals of R. Recall that a *star-operation* on R is a mapping $\star : F(R) \to F(R)$ such that for all $A, B \in F(R)$, and for all $a \in K^* = K \backslash \{0\}$:

(1) $(aR)_\star = aR$ and $(aA)_\star = aA_\star$,
(2) $A \subseteq A_\star$, and $A \subseteq B$ implies $A_\star \subseteq B_\star$, and
(3) $(A_\star)_\star = A_\star$.

For a brief introduction to star-operations, the reader may consult [17, Sections 32 and 34].

Examples of star-operations include the d-operation ($A_d = A$), the v-operation ($A_v = (A^{-1})^{-1} = \cap\{aR \mid A \subseteq aR, a \in K^*\}$), and the t-operation ($A_t = \cup\{B_v \mid B \subseteq A, B \in f(R)\}$). If $\star$ is a star-operation on R, then we can define a new star-operation $\star_s$ by $A \mapsto A_{\star_s} = \cup\{B_\star \mid B \subseteq A \text{ and } B \in f(R)\}$. A star-operation $\star$ on R is said to be of *finite character* if $\star = \star_s$. Note that $B_\star = B_{\star_s}$ for all $B \in f(R)$. Hence, it is easy to see that $\star_s$ is of finite character. In particular, we have that $t = v_s$ is of finite character. A fractional

ideal $A \in F(R)$ is called a $\star$-*ideal* if $A = A_\star$. When $\star$ has finite character, every proper integral $\star$-ideal is contained in a maximal proper integral $\star$-ideal and maximal proper integral $\star$-ideals are prime. We will denote the set of maximal $\star$-ideals by $\star$-Max(R).

For a given star-operation $\star$ on R, Anderson and Cook [2] construct two new star-operations $\bar{\star}$ and $\star_w$ by setting, for $A \in F(R)$,

$$A_{\bar{\star}} = \{x \in K \mid xJ \subseteq A \text{ for some } J \in F(R) \text{ with } J_\star = R\}$$

and

$$A_{\star_w} = \{x \in K \mid xJ \subseteq A \text{ for some } J \in f(R) \text{ with } J_\star = R\},$$

respectively. The well-known w-operation is given by $w = v_w$.

Another example of a star-operation is the so-called "half-centered star-operation" (see [15]). Let $\mathscr{P}$ be a set of prime ideals of R such that $R = \cap_{\mathfrak{p} \in \mathscr{P}} R_\mathfrak{p}$. Then the mapping $A \mapsto A_{\star_{\mathscr{P}}} = \cap_{\mathfrak{p} \in \mathscr{P}} AR_\mathfrak{p}$ is known to be a star-operation satisfying the following two conditions (see [1, Theorem 1]):

  (1) each proper integral $\star_{\mathscr{P}}$-ideal is contained in a prime $\star_{\mathscr{P}}$-ideal, and
  (2) $(A \cap B)_{\star_{\mathscr{P}}} = A_{\star_{\mathscr{P}}} \cap B_{\star_{\mathscr{P}}}$ for all $A, B \in F(R)$.

In [1, Theorem 4], Anderson showed that any star-operation satisfying these two conditions must have, for some choice of $\mathscr{P}$, the above form. A few years later, García et al. in [15] described the main points of the Anderson's result by using the language of hereditary torsion theories. They also called these operations half-centered star-operations, emphasizing the relation with the half-centered hereditary torsion theories (in the sense of [6]). Thus, star-operations of this type have some homological properties. Roughly speaking, the homological aspect of a half-centered star-operation can be seen as the corresponding half-centered hereditary torsion theory. It is well-known that the hereditary torsion theory is equivalent to the localizing system (or the Gabriel filter). For more on hereditary torsion theories, see the books of Stenström [38] and Golan [20].

The most familiar example of a half-centered star-operation may be the w-operation. Recall that an ideal J of R is called a *Glaz-Vasconcelos ideal* (a *GV-ideal* for short) if J is finitely generated and $J^{-1} = R$. We denote by GV(R) the set of all GV-ideals of R. For any R-module M, set

$$\text{tor}_{\text{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \text{GV}(R)\}.$$

Thus $\text{tor}_{\text{GV}}(M)$ is a submodule of M. Now M is said to be *GV-torsion* (resp., *GV-torsionfree*) if $\text{tor}_{\text{GV}}(M) = M$ (resp., $\text{tor}_{\text{GV}}(M) = 0$). It is easily checked that the pair $\text{tor}_{\text{GV}} = (\{\text{GV-torsion modules}\}, \{\text{GV-torsionfree modules}\})$ is a half-centered hereditary torsion theory on the category of all R-modules. In [45], the second named author and McCasland defined a torsionfree R-module M to be a *w-module* if $Jx \subseteq M$ for $J \in \text{GV}(R)$ and $x \in M \bigotimes_R K$ implies that $x \in M$; w-modules are called *semi-divisorial* in [19] and (in the ideal case) F_∞-ideals in [21]. Then they also defined, for a torsionfree R-module M, the

w-envelope of M as

$$M_w = \{x \in M \bigotimes_R K \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}.$$

The canonical map $A \mapsto A_w$ on $F(R)$ is the w-operation. From the torsion-theoretic point of view, the notion of w-modules (resp., w-envelopes) coincides with that of torsionfree and $\mathrm{tor_{GV}}$-injective modules (resp., $\mathrm{tor_{GV}}$-injective envelopes). So the w-operation is essentially a star-operation defined via $\mathrm{tor_{GV}}$.

It is well-known that for any domain R we have $R = \cap R_\mathfrak{p}$, where the intersection is taken over all associated primes of principal ideals (that is, primes minimal over an ideal of the form $(aR : bR)$) [39, Theorem E(i)]. Thus, there is a natural half-centered star-operation defined by the set of associated primes of principal ideals of a domain, referred to as a g-operation. For this operation, associated primes of principal ideals may play a role similar to that played by maximal w-ideals working with the w-operation. In a recent paper [37], the authors study the homological aspect of the g-operation and use it to exhibit a characterization of coherent domains of weak Gorenstein global dimension at most two, a class of domains arisen in Gorenstein homological algebra. On the other hand, some classes of domains (e.g. the class of *P-domains* [32], that is, the class of domains whose localizations at associated primes of principal ideals are valuation domains) have the natural connection with the g-operation. The purpose of this paper is to investigate the ideal-theoretic properties of g-operations.

In Section 1 we introduce the notion of the g-operation by a certain half-centered torsion theory and study the connection of the g-operation with the w-operation. In Section 2 we investigate DG-domains, that is, domains in which each ideal is a g-ideal. It is shown that a domain R is a DG-domain if and only if $R_\mathfrak{m}$ is a DG-domain for each maximal ideal $\mathfrak{m}$ of R (see Corollary 2.7). We also determine when the t-Nagata ring (in the sense of [27]) is DG (see Theorem 2.16). Section 3 is devoted to a brief discussion of connections with H-domains. Some new properties of H-domains are also given in Theorem 3.4. In Section 4, we study what happens when "DVR" in the definition of Krull domains is replaced by "one-dimensional Gorenstein ring". It turns out that these are precisely Mori w-divisorial domains (see Theorem 4.4). The last section examines the transfer of the DG-property to pullbacks in order to provide some original examples.

Any unexplained terminology is standard as in [17, 28].

1. The g-operation

In this section, we review the g-operation in a more general context than the one induced by overrings and study its connection with the w-operation. We should begin by recalling the notion of g-torsion theories. Let $\mathscr{P}(R)$ denote the set of associated primes of principal ideals of R, that is

$$\mathscr{P}(R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \text{ is minimal over } (aR : bR) \text{ for some } a, b \in R\}.$$

Notice that $\mathscr{P}(R)$ is exactly the associated primes $\text{Ass}_R(K/R)$ of the R-module K/R, and that over a Krull domain R, $\mathscr{P}(R)$ is exactly the set $X^1(R)$ (or just X^1) of prime ideals of height one in R. For each ideal I of R, put $V(I) = \{\mathfrak{p} \in \text{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. To $\mathscr{P}(R)$ we can associate a Gabriel filter

$$\mathscr{L}_g(R) = \{I \mid V(I) \cap \mathscr{P}(R) = \emptyset\}.$$

It follows from [39, Theorem E(ii)] that for a finitely generated ideal J of R, $J \in \mathscr{L}_g(R)$ if and only if $J^{-1} = R$. So $\text{GV}(R) \subseteq \mathscr{L}_g(R)$. We now call the hereditary torsion theory corresponding to $\mathscr{L}_g(R)$ the g-torsion theory and denote it by g. It is known that the g-torsion class consists of all R-modules M with $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathscr{P}(R)$. Thus, the g-torsion theory is half-centered. Moreover, every g-torsion module is a torsion module, or equivalently, every torsionfree module is a g-torsionfree module. Recall that an R-module M is said to be g-injective if it is injective with respect to every R-monomorphism having a g-torsion cokernel, and it is said to be g-closed if it is g-injective and g-torsionfree. Now we call an R-module M a g-module if M is torsionfree and g-injective. Obviously, all g-modules are g-closed. Let N be a submodule of an R-module M. Denote by N_M^c the g-closure of N in M, i.e.,

$$N_M^c = \{x \in M \mid (N : x) \in \mathscr{L}_g(R)\}.$$

In particular, we call the g-closure of M in its injective envelope $E(M)$ the g-injective envelope of M and denote it by $E_g(M)$. Note that N_M^c/N is always a g-torsion R-module. For any torsionfree R-module M, we will denote simply by M_g the g-injective envelope of M. Then

$$M_g = \{x \in E(M) \mid (M : x) \in \mathscr{L}_g(R)\}$$

must be a g-module. Clearly, a torsionfree R-module M is a g-module if and only if $M = M_g$. It is also easy to see that $M_w \subseteq M_g$ for all torsionfree R-modules M. Hence, every g-module is a w-module.

Proposition 1.1. *Let N be a submodule of an R-module M. Then:*
   (1) *$(N_M^c)_\mathfrak{p} = N_\mathfrak{p}$ for all $\mathfrak{p} \in \mathscr{P}(R)$.*
   (2) *If M is a torsionfree module, then $M_g = \cap_{\mathfrak{p} \in \mathscr{P}(R)} M_\mathfrak{p}$.*

Proof. (1) Since N_M^c/N is g-torsion, $(N_M^c)_\mathfrak{p}/N_\mathfrak{p} \cong (N_M^c/N)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \mathscr{P}(R)$. Hence (1) holds.

(2) If M is a torsionfree module, then so is M_g. Hence, for each $\mathfrak{p} \in \mathscr{P}(R)$, $M_g \subseteq (M_g)_\mathfrak{p} = M_\mathfrak{p}$ by (1), and so $M_g \subseteq \cap_{\mathfrak{p} \in \mathscr{P}(R)} M_\mathfrak{p}$. On the other hand, let $x \in \cap_{\mathfrak{p} \in \mathscr{P}(R)} M_\mathfrak{p}$. Since M is torsionfree, $x \in M_S = E(M)$ where $S = R \backslash \{0\}$. For each $\mathfrak{p} \in \mathscr{P}(R)$, there exists an $s \in R \backslash \mathfrak{p}$ such that $sx \in M$. Thus, $(M : x) \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathscr{P}(R)$, i.e., $(M : x) \in \mathscr{L}_g(R)$, and so $x \in M_g$. It follows that $M_g = \cap_{\mathfrak{p} \in \mathscr{P}(R)} M_\mathfrak{p}$. $\square$

Note that R is a g-module over itself. More generally, it was shown in [37, Proposition 2.4] that all reflexive modules are g-modules.

Corollary 1.2. *Let J be an ideal of R. Then $J \in \mathscr{L}_g(R)$ if and only if $J_g = R$.*

Proof. If $J_g = R$, then $R/J = J_g/J$ is a g-torsion R-module, and so $J \in \mathscr{L}_g(R)$. Conversely, let $J \in \mathscr{L}_g(R)$. Then by Proposition 1.1(2), $J_g = \cap_{\mathfrak{p} \in \mathscr{P}(R)} J_{\mathfrak{p}} = \cap_{\mathfrak{p} \in \mathscr{P}(R)} R_{\mathfrak{p}} = R_g = R$. $\qquad\square$

Corollary 1.3. *For any $A \in F(R)$, we have $A_g = A_K^c$.*

Proof. Since K is g-injective (in fact, injective) over R, $A_g \subseteq K_g = K$, and so $A_g \subseteq A_K^c$. On the other hand, because A_K^c is a torsionfree R-module, it follows from Proposition 1.1(1) that for each $\mathfrak{p} \in \mathscr{P}(R)$, $A_K^c \subseteq (A_K^c)_{\mathfrak{p}} = A_{\mathfrak{p}} \subseteq A_S = E(A)$, where $S = R \setminus \{0\}$. Thus, $A_K^c \subseteq A_g$. $\qquad\square$

Now, by Corollary 1.3 and [15, Lemma 1], we have that the canonical map $A \mapsto A_g$ on $F(R)$ is a half-centered star-operation, called the *g-operation*.

For two star-operations $\star_1$ and $\star_2$ on R, we write $\star_1 \leq \star_2$ if $A_{\star_1} \subseteq A_{\star_2}$ for all $A \in F(R)$ (and $\star_1 < \star_2$ if $\star_1 \leq \star_2$ but $\star_1 \neq \star_2$). It is known that for any star-operation $\star$ on R, $d \leq \star_w \leq \bar{\star} \leq \star \leq v$, and $\star_w \leq \star_s \leq t$ (cf. [2, Theorem 2.4]).

Proposition 1.4.
   (1) $\bar{g} = g$.
   (2) $g_w = w$.

Proof. (1) This follows directly from [2, Theorem 2.6], as the g-operation is half-centered.

(2) This is clear from Corollary 1.2 and the fact that for a finitely generated ideal J of R, $J \in \mathrm{GV}(R)$ if and only if $J \in \mathscr{L}_g(R)$. $\qquad\square$

Observe that the g-operation is similar to the w-operation. Clearly, we have $w \leq g$. When do we have $w = g$? This question leads us to introduce the notion of *GW-domains*, i.e., domains in which the w-operation coincides with the g-operation. In fact, this type of domains has been characterized in [37, Theorem 3.2] by using the language of hereditary torsion theories. Next, we rewrite it from the ideal-theoretic point of view.

Theorem 1.5. *The following statements are equivalent for a domain R.*
   (1) *R is a GW-domain.*
   (2) *Every w-module over R is a g-module.*
   (3) *Every prime w-ideal of R is a g-ideal.*
   (4) *Every maximal w-ideal of R is a g-ideal.*
   (5) *Every prime w-ideal of R is contained in some $\mathfrak{p} \in \mathscr{P}$.*
   (6) *$w\text{-}\mathrm{Max}(R) \subseteq \mathscr{P}(R)$.*
   (7) *$\mathscr{L}_g(R)$ has a cofinite subset of finitely generated ideals.*
   (8) *$\mathscr{P}(R)$ is a compact subspace of $\mathrm{Spec}(R)$ in the Zariski topology.*
   (9) *g has finite character.*

Examples of GW-domains include H-domains (cf. [19]) and domains with Noetherian Spectrum. But a valuation ring is not necessarily a GW-domain. Actually, in [33, p. 223], Papick says that if V is a valuation ring with maximal ideal M such that $M = \cup_{P \in \mathrm{Spec}(R), P \subsetneq M} P$, then $\mathscr{P}(V)$ is not compact. Namely, it is possible to have $w < g$. In fact, in [18, p. 145] Gilmer and Heinzer presents a concrete example of such a valuation ring. Before reviewing their example, let us first recall the notion of unbranched prime ideals (in the sense of [16]). A prime ideal $\mathfrak{p}$ of R is called *unbranched* if $\mathfrak{p}$ is the only $\mathfrak{p}$-primary ideal. From [16, Lemma 4.3], we have that a prime ideal $\mathfrak{p}$ of a valuation ring is unbranched if and only if $\mathfrak{p}$ is the union of the chain of all prime ideals properly contained in $\mathfrak{p}$.

Example 1.6. Let $\{X_i\}_{i=1}^{\infty}$ be a countable collection of indeterminates over a field k. Define a valuation ν on $k(\{X_i\})$ by defining it on $k[\{X_i\}]$, then taking the canonical extension to $k(\{X_i\})$. ν is defined on $k[\{X_i\}]$ and has value group G, the countable direct product of the additive group of integers, ordered lexicographically. ν is defined on a nonzero monomial $aX_1^{e_1} \cdots X_n^{e_n}$ to be $(e_1, \ldots, e_n, 0 \ldots)$, and $\nu(f)$, for a nonzero element $f \in k[\{X_i\}]$ as the minimum of the ν-values of the nonzero monomials in f. Let V be the valuation ring associated with ν and let M be the maximal ideal of V. It is observed that $V = k + M$, and that M is an unbranched ideal of V. In fact, one can check that $M = (X_1, \ldots, X_n, \ldots)$, and that $\{X_iV\}_{i=1}^{\infty} \cup \{0\}$ is the set of all nonmaximal prime ideals of V. Hence, by [26, Proposition 2.6], we have $M \notin \mathscr{P}(V)$. Thus, the valuation ring V does not have $\mathscr{P}(V)$ compact.

The following is another example of a domain which is not GW.

Example 1.7. Assume that R is a P-domain (that is, $R_{\mathfrak{p}}$ is a valuation domain for all $\mathfrak{p} \in \mathscr{P}(R)$) but is not a PVMD (see [32, Example 2.1]). It is well known that a domain D is a PVMD if and only if $D_{\mathfrak{m}}$ is a valuation domain for all $\mathfrak{m} \in w\text{-Max}(D)$. Thus, $w\text{-Max}(R) \nsubseteq \mathscr{P}(R)$, and so R is not a GW-domain by Theorem 1.5.

Remark 1.8.

(1) In view of Theorem 1.5, Examples 1.6 and 1.7 show that the g-operation does not necessarily have finite character.

(2) It is known that every flat module is a w-module (cf. [19, Corollary 2.3]). In [37, Proposition 2.2] the authors strengthened this result, proving that every flat module is actually a g-module. However, this is incorrect. Let us reconsider Example 1.6. Clearly, M is a flat ideal of V. But M is not a g-ideal as $M \notin \mathscr{P}(V)$. By the way, the proof of [37, Proposition 2.2] depends heavily on a result of Vasconcelos [40, Proposition 1.7]. It was stated (without proof in his paper) that, in our terminology, a torsionfree R-module M is a g-module if and only if every R-sequence of at most two elements is an M-sequence. In the proof of [37, Proposition 2.2], we only showed that the latter condition

holds for all flat modules. Thus, Example 1.6 also shows that this result of Vasconcelos does not hold in general.

We end this section with an example of a g-module.

Proposition 1.9. *Let X be an indeterminate over a domain R. Then $R[X]$ is a g-module over R.*

Proof. Let $y \in R[X]_g$. Then there exists $J \in \mathscr{L}_g(R)$ such that $Jy \subseteq R[X]$, and so $JR[X]y \subseteq R[X]$. We claim that $JR[X] \in \mathscr{L}_g(R[X])$. Indeed, if not, then $JR[X]$ must be contained in some $P \in \mathscr{P}(R[X])$. Hence, it follows from [5, Corollary 8] that $J \subseteq JR[X] \cap R \subseteq P \cap R \in \mathscr{P}(R)$, which is impossible. Therefore, $JR[X] \in \mathscr{L}_g(R[X])$. Since $R[X]$ is a g-module over itself, $y \in R[X]$. Thus, as an R-module, $R[X]$ is a g-module. $\square$

2. DG-domains

Recently, the class of *DW-domains* (that is, domains in which the d-operation and w-operations coincide) has received a good deal of attention in the papers [31, 34, 35]. In fact, it is known that these domains were already known many years before (Dobbs et al. [8, 9, 10]) under the name of t-linkative domains. In this section, we study a subclass of the DW-domains: the *DG-domains*, that is, domains in which the two star-operations d and g coincide; equivalently, domains in which each ideal is a g-ideal. Notice that the valuation ring as in Example 1.6 is a DW-domain, but not a DG-domains.

Proposition 2.1. *The following statements are equivalent for a domain R.*
  (1) *R is a DG-domain.*
  (2) *Every prime ideal of R is a g-ideal.*
  (3) *Every prime ideal of R is contained in an element of $\mathscr{P}(R)$.*
  (4) $\mathrm{Max}(R) \subseteq \mathscr{P}(R)$.
  (5) *Every maximal ideal of R is a g-ideal.*
  (6) $\mathscr{L}_g(R) = \{R\}$.
  (7) *Every torsionfree R-module is a g-module.*

Proof. Straightforward. $\square$

We know that all one-dimensional domains are DW-domains (cf. [35, Proposition 2.9]). Since every prime ideal of height one is an associate prime ideal of a principal ideal, it follows from Proposition 2.1 that these are, in fact, DG-domains. Moreover, every divisorial domain is clearly a DG-domain. However, the converse is not true in general. For an example, one can take a one-dimensional Noetherian local domain which is not Gorenstein.

Recall that a domain R is called a *treed domain* if $\mathrm{Spec}(R)$ being a poset under inclusion is a tree, that is, no prime ideal of R can contain prime ideals of R that are incomparable. It is known that every treed domain is a DW-domain (cf. [9, Corollary 2.7]). In [35, p. 1965], it was shown that the class

of treed domains of finite dimension is a class of DW-domains whose integral closure is DW. Next, we will see that this class of domains is, in fact, a class of DG-domains whose integral closure is also DG.

Corollary 2.2 (cf. [35, Proposition 3.2]). *Let R be a domain such that every maximal ideal $\mathfrak{m}$ is not the union of prime ideals properly contained in $\mathfrak{m}$. Then R and its integral closure R^c are DG-domains.*

Proof. Since $\operatorname{Max}(R) \subseteq \mathscr{P}(R)$, Proposition 2.1 implies that R is a DG-domain. $\square$

Corollary 2.3 (cf. [35, Corollary 3.3]). *Let R be a domain with finite prime spectrum. Then R and R^c are DG-domains.*

Now, let us consider a treed domain R of finite dimension. Then it is easy to check that each maximal of R is minimal over a principal ideal. Hence, we have that the class of treed domains of finite dimension is a class of DG-domains whose integral closure is also DG.

In [31, Theorem 2.9] Mimouni proved that if R is a domain such that $R_{\mathfrak{m}}$ is a DW-domain for all $\mathfrak{m} \in \operatorname{Max}(R)$, then so is R, and the equivalence holds when R is v-coherent. Later, the localization of DW-domains was also studied in [35]. We next consider the localization in connection with the DG-property.

Lemma 2.4. *Let $\mathfrak{p}$ be a prime ideal of R. Assume that S is a multiplicatively closed subset of R and that $\mathfrak{p} \cap S = \emptyset$. Then, $\mathfrak{p} \in \mathscr{P}(R)$ if and only if $\mathfrak{p}R_S \in \mathscr{P}(R_S)$.*

Proof. The proof is essentially the same as that given for [26, Lemma 2.8] $\square$

Lemma 2.5. *Let S be a multiplicatively closed subset of R. If $J \in \mathscr{L}_g(R)$, then $JR_S \in \mathscr{L}_g(R_S)$.*

Proof. Let $J \in \mathscr{L}_g(R)$. If $JR_S \notin \mathscr{L}_g(R_S)$, then $JR_S \subseteq P$ for some $P \in \mathscr{P}(R_S)$. Write $P = \mathfrak{p}R_S$, where $\mathfrak{p}$ is a prime ideal of R with $\mathfrak{p} \cap S = \emptyset$. Since $\mathfrak{p}R_S \in \mathscr{P}(R_S)$, $\mathfrak{p} \in \mathscr{P}(R)$ by Lemma 2.4. However, note that $J \subseteq \mathfrak{p}$, a contradiction. Thus, $JR_S \in \mathscr{L}_g(R_S)$. $\square$

Proposition 2.6. *A domain R is a GW-domain if and only if $R_{\mathfrak{m}}$ is a DG-domain for all $\mathfrak{m} \in w\text{-}\operatorname{Max}(R)$.*

Proof. Assume that R is a GW-domain. Let $\mathfrak{m} \in w\text{-}\operatorname{Max}(R)$. Then by Theorem 1.5, $\mathfrak{m} \in \mathscr{P}(R)$. Hence, Lemma 2.4 says that $\mathfrak{m}R_{\mathfrak{m}} \in \mathscr{P}(R_{\mathfrak{m}})$, and so $R_{\mathfrak{m}}$ is a DG-domain by Proposition 2.1.

Conversely, assume that $R_{\mathfrak{m}}$ is a DG-domain for all $\mathfrak{m} \in w\text{-}\operatorname{Max}(R)$. Let I be an ideal of R and let $x \in I_g$. Then there is a $J \in \mathscr{L}_g(R)$ such that $Jx \subseteq I$. Hence, for each $\mathfrak{m} \in w\text{-}\operatorname{Max}(R)$, we have $xJR_{\mathfrak{m}} \subseteq IR_{\mathfrak{m}}$. Since $R_{\mathfrak{m}}$ is a DG-domain, $IR_{\mathfrak{m}}$ is a g-ideal of $R_{\mathfrak{m}}$. Also, by Lemma 2.5, $JR_{\mathfrak{m}} \in \mathscr{L}_g(R_{\mathfrak{m}})$. Thus, $x \in IR_{\mathfrak{m}}$, and so $x \in \cap_{\mathfrak{m} \in w\text{-}\operatorname{Max}(R)} IR_{\mathfrak{m}} = I_w$ by [2, Corollary 2.13]. Therefore, $I_g = I_w$. It follows that R is a GW-domain. $\square$

Corollary 2.7. *A domain R is a DG-domain if and only if $R_{\mathfrak{m}}$ is a DG-domain for all $\mathfrak{m} \in \operatorname{Max}(R)$.*

Proof. This follows easily from Proposition 2.6 and [31, Theorem 2.9]. $\qquad \square$

The following example shows that a localization of a DG-domain with respect to a nonmaximal prime ideal is not necessarily a DG-domain.

Example 2.8. In [31, Example 2.10] Mimouni presents an example of a local Noetherian domain R which is a DW-domain, but is localized at a nonmaximal prime ideal $\mathfrak{p}$ so that $R_{\mathfrak{p}}$ is not a DW-domain. Since every Noetherian domain is a GW-domain, we see that R is a DG-domain, but $R_{\mathfrak{p}}$ is not a DG-domain.

In [33] Papick provided some interesting characterizations of compactness of $\mathscr{P}(R)$, and showed that if R is a treed domain, then $\mathscr{P}(R)$ is compact if and only if $\operatorname{Max}(R) \subseteq \mathscr{P}(R)$ (see [33, Proposition 3.3]). Since every treed domain is a DW-domain, Theorem 1.5 generalizes this result of Papick. Moreover, it was also proved in [33, Corollary 3.4] that for a treed domain R, $\mathscr{P}(R)$ is compact if and only if $\mathscr{P}(R_{\mathfrak{m}})$ is compact for all $\mathfrak{m} \in \operatorname{Max}(R)$. Now, we notice that the following result can be easily obtained as a corollary of the previous Proposition 2.6 and Corollary 2.7, using the fact that any localization of a treed domain is a treed domain.

Corollary 2.9. *Let R be a treed domain. Then the following statements are equivalent.*

(1) *$\mathscr{P}(R)$ is compact.*
(2) *$R_{\mathfrak{m}}$ is a DG-domain for all $\mathfrak{m} \in \operatorname{Max}(R)$.*
(3) *$\mathscr{P}(R_{\mathfrak{m}})$ is compact for all $\mathfrak{m} \in \operatorname{Max}(R)$.*
(4) *R is a DG-domain.*

Recall from [9] that an overring T of a domain R is said to be *t-linked* over R if $J^{-1} = R$ implies $(T : JT) = T$ for all $J \in f(R)$. Recall also from [43] that an overring T of R is said to be a *w-overring* of R if T is a w-module as an R-module. But these two notions of an overring coincide (cf. [43, Proposition 1.2]). It was shown in [9, Theorem 2.6] that each overring of R is t-linked over R if and only if each maximal ideal of R is a t-ideal. This is exactly a characterization of DW-domains given in [31, Proposition 2.2]. Now we call an overring T of R a *g-overring* of R if T is a g-module as an R-module. Clearly, if R is a DG-domain, then each overring of R is a g-overring. It is natural to ask if the converse is still true. Before answering the question, we give some characterizations of g-overrings in terms of the $\bar{v}$-operation. Set

$$Z_v(R) = \{J \in F(R) \mid J_v = R\} = \{J \in F(R) \mid J^{-1} = R\}.$$

It is easy to see that for each $J \in Z_v(R)$, $J_{\bar{v}} = R$. Moreover, we have $g \leq \bar{v} \leq v$.

Example 2.10. Let (V, M) be a non-discrete rank one valuation domain. Then it is easy to check that $M \in Z_v(V)$. Hence, $M_{\bar{v}} = V$, and so M is not a $\bar{v}$-ideal. But M is a g-ideal as V is DG. Thus, $g < \bar{v}$.

Note that over a Noetherian domain, $w = g = \bar{v}$. So for a one-dimensional Noetherian local domain which is not Gorenstein, we have $\bar{v} < v$.

Proposition 2.11. *Let T be an overring of a domain R. Then the following statements are equivalent.*

(1) $I_g \subseteq (IT)_{\bar{v}_T}$ *for each ideal I of R.*
(2) $(I_g T)_{\bar{v}_T} = (IT)_{\bar{v}_T}$ *for each ideal I of R.*
(3) $A \cap R$ *is a g-ideal of R for each $\bar{v}$-ideal A of T.*
(4) $(IT)_{\bar{v}_T} \cap R$ *is a g-ideal of R for each ideal I of R.*
(5) *If $J \in \mathscr{L}_g(R)$, then $JT \in Z_v(T)$.*
(6) T *is a g-overring of R.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal of R. Then by (1), we have $I_g T \subseteq (IT)_{\bar{v}_T}$, and so $(I_g T)_{\bar{v}_T} \subseteq ((IT)_{\bar{v}_T})_{\bar{v}_T} = (IT)_{\bar{v}_T}$. The reverse containment is obvious.

(2) $\Rightarrow$ (3) Let A be a $\bar{v}$-ideal of T and write $I = A \cap R$. Then $I_g \subseteq (I_g T)_{\bar{v}_T} \cap R = (IT)_{\bar{v}_T} \cap R \subseteq A_{\bar{v}_T} \cap R = I$ by (2). Thus, $I = I_g$.

(3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (5) If $J \in \mathscr{L}_g(R)$, then by (4), $R = J_g \subseteq (JT)_{\bar{v}_T} \cap R \subseteq (JT)_{\bar{v}_T}$. Hence, $(JT)_{\bar{v}_T} = T$, and so $JT \in Z_v(T)$.

(5) $\Rightarrow$ (1) Let I be an ideal of R and let $x \in I_g$. Then there is a $J \in \mathscr{L}_g(R)$ such that $Jx \subseteq I$, and so $JTx \subseteq IT$. Since $JT \in Z_v(T)$ by (5), we have $x \in (IT)_{\bar{v}_T}$. Thus, $I_g \subseteq (IT)_{\bar{v}}$.

(5) $\Rightarrow$ (6) Let $x \in T_g$. Then there exists a $J \in \mathscr{L}_g(R)$ such that $Jx \subseteq T$, and so $JTx \subseteq T$. Therefore, it follows from (5) that $x \in (T : JT) = T$. So $T = T_g$.

(6) $\Rightarrow$ (5) Let $J \in \mathscr{L}_g(R)$ and $x \in (T : JT)$. Then $xJ \subseteq JTx \subseteq T$, and so $x \in T_g = T$ by (6). Thus, $(T : JT) = T$, i.e., $JT \in Z_v(T)$. $\square$

Corollary 2.12. *Let S be a multiplicatively closed subset of a domain R. Then R_S is a g-overring of R.*

Proof. It follows from Lemma 2.5 and Proposition 2.11. $\square$

Corollary 2.13. *Every overring of a valuation ring is a g-overring.*

Proof. This follows since every overring of a valuation ring R has the form $R_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of R. $\square$

In the light of Corollary 2.13, Example 1.6 shows that the answer to the question mentioned earlier is negative.

In [31, Proposition 2.12], it was shown that for a domain R, $R[X]$ is a DW-domain if and only if R is a field. So $R[X]$ is a DG-domain if and only if R is a field. In [27] Kang extended the construction of the Nagata ring referring to an arbitrary given star-operation $\star$ on R as follows. Set

$$N_\star = \{h \in R[X] \mid c(h)_\star = R\},$$

where $c(h)$ is the content of the polynomial h. Then the $\star$-Nagata ring is $R[X]_{N_\star}$. When $\star = d$, we obtain the classical Nagata ring, usually denoted by

$R(X)$. In [35], the authors remarked that $R[X]_{N_v}$ is always DW, and they also showed that a domain R is DW if and only if $R[X]_{N_v} = R(X)$, if and only if $R(X)$ is DW. From this it is natural to ask the following: When is $R[X]_{N_g}$ a DG-domain? Note that $N_g = N_v$, and that the maximal ideals of $R[X]_{N_v}$ are of the type $\mathfrak{m}R[X]_{N_v}$, where $\mathfrak{m}$ ranges among the maximal t-ideals (or, equivalently, the maximal w-ideals) of R (cf. [27, Proposition 2.1]).

Lemma 2.14. *Let M be a maximal w-ideal of $R[X]$ with $M \cap R \neq 0$. Then $M = (M \cap R)R[X]$ and $M \cap R \in w\text{-Max}(R)$.*

Proof. See [24, Proposition 1.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.15. *Let $\mathfrak{m}$ be a prime ideal of a domain R. Then $\mathfrak{m} \in w\text{-Max}(R)$ if and only if $\mathfrak{m}R[X] \in w\text{-Max}(R[X])$.*

Proof. Let $\mathfrak{m} \in w\text{-Max}(R)$. Then by [21, Proposition 4.6], $\mathfrak{m}R[X]$ is a prime w-ideal of $R[X]$, and so it must be contained in some $M \in w\text{-Max}(R[X])$. Note that if $0 \in w\text{-Max}(R)$, then $R = K$ is a field. Thus, by Lemma 2.14, we have that $M = (M \cap R)R[X]$ and $\mathfrak{m} \subseteq M \cap R \in w\text{-Max}(R)$. Hence, it follows that $M = \mathfrak{m}R[X] \in w\text{-Max}(R[X])$. The converse follows from Lemma 2.14. $\qquad\square$

Theorem 2.16. *The following statements are equivalent for a domain R.*
 (1) $R[X]_{N_v}$ *is a DG-domain.*
 (2) $R[X]_{N_v}$ *is a GW-domain.*
 (3) $R[X]$ *is a GW-domain.*
 (4) R *is a GW-domain.*

Proof. The equivalence of (1) and (2) follows from the fact that $R[X]_{N_v}$ is a DW-domain. From Theorem 1.5 and [33, Proposition 3.2], one can deduce the equivalence of (3) and (4).

 (1) $\Rightarrow$ (4) Assume that $R[X]_{N_v}$ is a DG-domain and let $\mathfrak{m} \in w\text{-Max}(R)$. Then $\mathfrak{m}R[X]_{N_v}$ is the maximal ideal of $R[X]_{N_v}$, and so $\mathfrak{m}R[X]_{N_v} \in \mathscr{P}(R[X]_{N_v})$ by Proposition 2.1. Since $\mathfrak{m}R[X] \cap N_v = \emptyset$ [27, Proposition 2.1(1)], Lemma 2.4 says that $\mathfrak{m}R[X] \in \mathscr{P}(R[X])$. Therefore, by [5, Corollary 8], we have $\mathfrak{m} = \mathfrak{m}R[X] \cap R \in \mathscr{P}(R)$. Thus, it follows from Theorem 1.5 that R is a GW-domain.

 (3) $\Rightarrow$ (1) Assume that (3) holds and let M be a maximal ideal of $R[X]_{N_v}$. Then $M = \mathfrak{m}R[X]_{N_v}$ for some $\mathfrak{m} \in w\text{-Max}(R)$. Since $R[X]$ is GW and $\mathfrak{m}R[X]$ is a maximal w-ideal of $R[X]$ by Corollary 2.15, we see from Theorem 1.5 that $\mathfrak{m}R[X] \in \mathscr{P}(R[X])$. Hence, by Lemma 2.4, $M = \mathfrak{m}R[X]_{N_v} \in \mathscr{P}(R[X]_{N_v})$. Therefore, (1) follows from Proposition 2.1. $\qquad\square$

3. H-domains

In [19], Glaz and Vasconcelos introduced the concept of an *H-domain*: a domain R in which every ideal I with $I^{-1} = R$ is quasi-finite (i.e., $I^{-1} = J^{-1}$ for some finitely generated subideal J of I). Later, in [23, Proposition 2.4], it

was shown that a domain R is an H-domain if and only if each maximal t-ideal of R is divisorial. In this section, we will give a brief discussion of H-domains.

As mentioned in Section 1, the class of H-domains is an important class of GW-domains. In fact, it is remarked in [19, (3.2b)] that a domain R is an H-domain if and only if it satisfies the following conditions:

(1) $\mathscr{P}(R)$ is compact as a subset of $\mathrm{Spec}(R)$.
(2) $\mathfrak{p}^{-1} \neq R$ for each $\mathfrak{p} \in \mathscr{P}(R)$.

Now we know that the compactness of $\mathscr{P}(R)$ is equivalent to the GW-property of R. Our net result gives a characterization of the above condition (2) in terms of the g-operation.

Proposition 3.1. *Let R be a domain. Then $\mathfrak{p}^{-1} \neq R$ for each $\mathfrak{p} \in \mathscr{P}(R)$ if and only if $g = \bar{v}$.*

Proof. Suppose that $g = \bar{v}$. If $\mathfrak{p}^{-1} = R$ for some $\mathfrak{p} \in \mathscr{P}(R)$, then $\mathfrak{p} = \mathfrak{p}_g = \mathfrak{p}_{\bar{v}} = R$, a contradiction.

Conversely, assume that $\mathfrak{p}^{-1} \neq R$ for each $\mathfrak{p} \in \mathscr{P}(R)$. If $J \in Z_v(R)$, then J is not contained in any element of $\mathscr{P}(R)$, i.e., $J \in \mathscr{L}_g(R)$; if not, then there is a $\mathfrak{p} \in \mathscr{P}(R)$ such that $J \subseteq \mathfrak{p}$. Thus, $R \subseteq \mathfrak{p}^{-1} \subseteq J^{-1} = R$, a contradiction. Hence, $Z_v(R) = \mathscr{L}_g(R)$, and so $g = \bar{v}$. $\qquad\qquad\square$

As a consequence of Proposition 3.1, we state the following:

Corollary 3.2. *A domain R is an H-domain if and only if $w = \bar{v}$.*

The next result characterizes domains in which every ideal is a $\bar{v}$-ideal, whose proof is easy.

Proposition 3.3. *The following statements are equivalent for a domain R.*

(1) $d = \bar{v}$.
(2) *Each prime ideal of R is a $\bar{v}$-ideal.*
(3) *Each maximal ideal of R is a $\bar{v}$-ideal.*
(4) $Z_v(R) = \{R\}$.
(5) $I^{-1} \neq R$ *for each proper ideal I of R.*
(6) $\mathfrak{p}^{-1} \neq R$ *for each prime ideal $\mathfrak{p}$ of R.*
(7) $\mathfrak{m}^{-1} \neq R$ *for each maximal ideal $\mathfrak{m}$ of R.*
(8) *Each maximal ideal of R is divisorial.*

Clearly, every divisorial domain has $d = \bar{v}$. The converse is false. In fact, a one-dimensional Noetherian local domain which is not Gorenstein has $d = \bar{v}$, but is not divisorial. Note that generalized Dedekind domains (see, for instance, [12]) are examples of domains with $d = \bar{v}$ (since their prime ideals are divisorial). From Corollary 3.2, it is easy to see that Proposition 3.3 generalizes [31, Corollary 2.8]. It is known that a completely integrally closed divisorial domain is a Dedekind domain [22, Proposition 5.5]. More generally, it is easily seen that a completely integrally closed domain with $d = \bar{v}$ is a Dedekind domain.

It was proved in [19, (3.2c)] that if R is an H-domain, then so is $R[X]$. We next examine the question of when $R[X]_{N_v}$ is an H-domain.

Theorem 3.4. *The following statements are equivalent for a domain R.*

(1) $R[X]_{N_v}$ *is an H-domain.*
(2) $d = \bar{v}$ *over $R[X]_{N_v}$.*
(3) R *is an H-domain.*
(4) $R[X]$ *is an H-domain.*

Proof. The equivalence of (1) and (2) follows from Corollary 3.2 and the fact that $R[X]_{N_v}$ is a DW-domain.

(2) $\Rightarrow$ (3) Assume that (2) holds and let $\mathfrak{m} \in w\text{-Max}(R)$. Then $\mathfrak{m}R[X]_{N_v}$ is a maximal ideal of $R[X]_{N_v}$, and so it is divisorial over $R[X]_{N_v}$ by Proposition 3.3. Therefore, it follows from [27, Corollary 2.3] that $\mathfrak{m}_v R[X]_{N_v} = \mathfrak{m}R[X]_{N_v}$. Since both $\mathfrak{m}$ and $\mathfrak{m}_v$ are w-ideals, [44, Proposition 3.9(2)] says that $\mathfrak{m} = \mathfrak{m}_v$, i.e., $\mathfrak{m}$ is divisorial. Thus, R is an H-domain.

(3) $\Rightarrow$ (2) Assume that R is an H-domain and let M be a maximal ideal of $R[X]_{N_v}$. Then $M = \mathfrak{m}R[X]_{N_v}$ for some $\mathfrak{m} \in w\text{-Max}(R)$, and hence $\mathfrak{m}$ is divisorial. By [27, Corollary 2.3], $M_{v_{R[X]_{N_v}}} = \mathfrak{m}_v R[X]_{N_v} = M$, i.e., M is divisorial over $R[X]_{N_v}$. So (2) holds by Proposition 3.3.

(3) $\Rightarrow$ (4) See [19, (3.2c)].

(4) $\Rightarrow$ (3) Assume that $R[X]$ is an H-domain and let $\mathfrak{m} \in w\text{-Max}(R)$. Then by Corollary 2.15, $\mathfrak{m}R[X] \in w\text{-Max}(R[X])$, and so $\mathfrak{m}R[X]$ is divisorial over $R[X]$. Therefore, by [21, Proposition 4.3], $\mathfrak{m}_v R[X] = \mathfrak{m}R[X]$, and so $\mathfrak{m} = \mathfrak{m}_v$. Thus, R is also an H-domain. $\square$

4. Gorenstein Krull domains

It is well known that from the homological algebra point of view, Dedekind domains (resp., Prüfer domains) are exactly the domains of the global dimension (resp., weak global dimension) at most one. Recently, several classical results and notions on global homological dimensions have been extended to Gorenstein global homological dimensions. In the paper [29], the authors introduce the domains of Gorenstein homological dimensions at most one, which they call, by analogy to the classical ones, Gorenstein Dedekind and Prüfer domains, respectively. Although these domains come from the homological theory, they can also be characterized in terms of the ideal-theoretic concept. Indeed, a domain R is a Gorenstein Dedekind domain (for short, G-Dedekind domain) if and only if R is a 1-Gorenstein domain (i.e., R is a Noetherian domain with self-injective dimension ≤ 1), if and only if R is a Noetherian divisorial domain (see [4, Proposition 1.5 and Theorem 3.4]); and R is a Gorenstein Prüfer domain if and only if R is a coherent FGV-domain (see [36, Theorem 4.2]). Recall from [48] that a domain R is called an *FGV-domain* if every finitely generated ideal of R is divisorial, i.e., $d = t$.

The following inclusions are well known:

Dedekind domains $\subset$ Noetherian integrally closed domains $\subset$ Krull domains

Krull's normality criterion (cf. [11, Theorem 4.1]) says that a Noetherian domain R is integrally closed (i.e., normal) if and only if R satisfies the conditions:

(1) $\mathrm{depth}(R_{\mathfrak{p}}) \geq \inf\{2, \dim(R_{\mathfrak{p}})\}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
(2) If $\mathfrak{q} \in X^1$, then $R_{\mathfrak{q}}$ is a regular local ring (i.e., a DVR).

In [40], Vasconcelos extended the notion of Noetherian normal domains to that of quasi-normal domains by substituting "DVR" into "one-dimensional Gorenstein ring" in the Krull's normality criterion. In fact, the quasi-normality is introduced for Noetherian rings which are not necessarily domains and used for the purpose of studying reflexive modules (for ideal-theoretic properties of quasi-normal rings, see [41]). Examples of quasi-normal rings include Gorenstein rings and G-Dedekind domains. Following the lead of Vasconcelos, we introduce the notion of Gorenstein Krull domains as follows:

Definition. A domain R is called a *Gorenstein Krull domain* (for short, G-Krull domain) if R satisfies the following three conditions:

(1) For each $\mathfrak{p} \in X^1$, $R_{\mathfrak{p}}$ is a Gorenstein ring.
(2) $R = \cap_{\mathfrak{p} \in X^1} R_{\mathfrak{p}}$.
(3) Each nonzero element of R is contained in only finitely many elements of X^1.

Remark 4.1.

(1) By [28, Theorem 222], we have that condition (1) of the above definition is equivalent to (1)′ for each $\mathfrak{p} \in X^1$, $R_{\mathfrak{p}}$ is a G-Dedekind domain.
(2) Obviously, every Krull domain is a G-Krull domains. Notice that for a quasi-normal domain R, $X^1(R) = \mathscr{P}(R)$. It follows that every quasi-normal domain is a G-Krull domain. The converse is not true in general. In fact, each non-Noetherian Krull domain is a G-Krull domain, but not quasi-normal. Moreover, not all G-Krull domains are Krull. In [25, Example 1], Hu and the second named author present an example of a G-Dedekind domain R which is not integrally closed. Then R is a G-Krull domain, but not a Krull domain.

Recall that a domain R is a Krull domain if and only if for $\mathfrak{p} \in \mathscr{P}(R)$, $R_{\mathfrak{p}}$ is a DVR, and each nonzero element of R is contained in only finitely many elements of $\mathscr{P}(R)$ (see the proof of [19, (3.2d)]). We next provide a Gorenstein analogue of this result.

Proposition 4.2. *If R is a G-Krull domain, then $X^1(R) = \mathscr{P}(R)$.*

Proof. Let R be a G-Krull domain. Then by applying [1, Theorem 1] to X^1, we see that the mapping $\star : A \mapsto A_\star = \cap_{\mathfrak{q} \in X^1} A_{\mathfrak{q}}$ is a finite character star-operation on R such that each proper $\star$-ideal is contained in an element of X^1. Let $\mathfrak{p} \in \mathscr{P}(R)$. Then $\mathfrak{p}$ is minimal over $I = (aR : bR)$, where $a, b \in R$ and

$b \notin aR$. Note that I is a v-ideal, and hence it is a $\star$-ideal. Therefore, by [21, Proposition 1.1(5)], $\mathfrak{p}$ is a $\star$-ideal, and so it is contained in some $\mathfrak{q} \in X^1$. But $\mathrm{ht}(\mathfrak{q}) = 1$, so $\mathfrak{p} = \mathfrak{q} \in X^1$. Thus, $\mathscr{P}(R) = X^1$. $\square$

Corollary 4.3. *A domain R is a G-Krull domain if and only if R satisfies the following conditions*:

(1) *For each $\mathfrak{p} \in \mathscr{P}(R)$, $R_\mathfrak{p}$ is a G-Dedekind domain.*
(2) *Each nonzero element of R is contained in only finitely many elements of $\mathscr{P}(R)$.*

Proof. The necessity is obtained by Proposition 4.2, and the sufficiency follows from the fact that every G-Dedekind domain is one-dimensional (cf. [22, Corollary 4.3]). $\square$
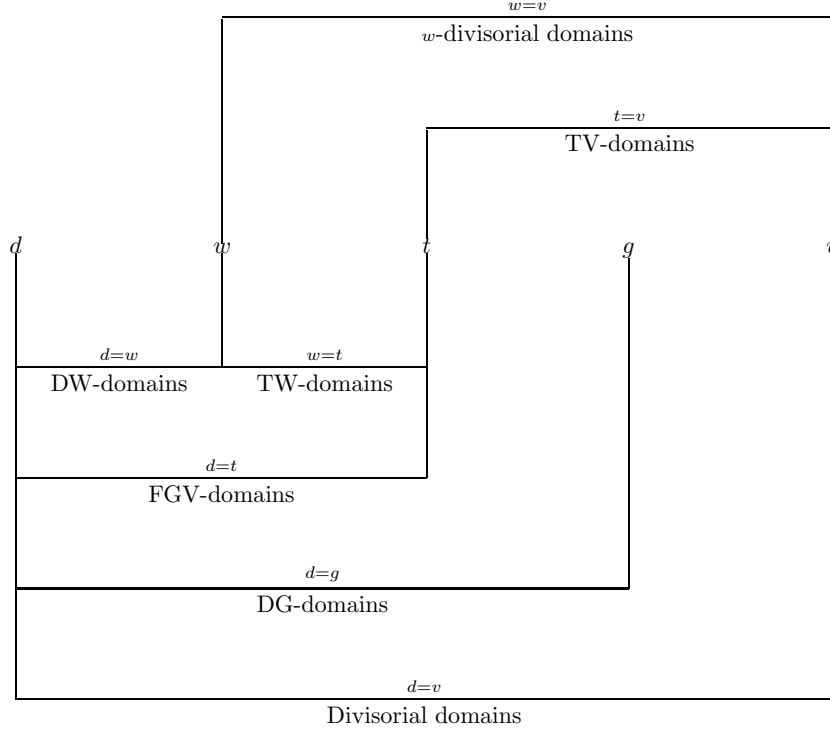
Before giving more characterizations of G-Krull domains, we recall some basic definitions and results. Let $\star$ be a star-operation on a domain R. An ideal I of R is said to be $\star$-*finitely generated* if there exists a finitely generated ideal $J \subseteq I$ such that $J_\star = I_\star$. Recall from [49] that R is called $\star$-*Noetherian* if R satisfies ACC (ascending chain condition) on integral $\star$-ideals or, equivalently, if each ideal of R is $\star$-finitely generated. Note that the d-Noetherian domains are just the usual Noetherian domains, and that the notions of v-Noetherian (resp., w-Noetherian) domain and Mori (resp., strong Mori) domain coincide. Clearly, every $\star$-Noetherian domain is a Mori domain. It was shown in [37, Corollary 3.4] that a domain is a g-Noetherian domain if and only if it is a strong Mori domain. Now we recall the following classes of domains defined by the equality of two star-operations. A domain R is:

(1) a *TV-domain* if each t-ideal of R is divisorial, i.e., $t = v$ (see [23]);
(2) a *w-divisorial domain* if each w-ideal of R is divisorial, i.e., $w = v$ (see [3]);
(3) a *TW-domain* if each w-ideal of R is a t-ideal, i.e., $w = t$ (see [30]).

There are the following implications:

$$\text{Mori} \Rightarrow \text{TV-domain} \Rightarrow \text{H-domain}$$

We pause here to compare the g-operation to the t-operation. Since the g-operation is not of finite character, $g \neq t$. Note that if R is the valuation domain as in Example 1.6, then $t = w < g$; and that if R is a TV-domain which is not a TW-domain (see for example, [31, Example 2.1(3)]), then $g = w < v = t$. In fact, since $w\text{-Max}(R) = t\text{-Max}(R)$ and by Theorem 1.5, it is easily seen that $g < t$ if and only if $g = w$. Thus, if $g \neq w$, then we have $d < w < t < g < v$. The following diagram summarizes the classes of domains defined by the equality of two star-operations.

$$w=v$$
w-divisorial domains

$$t=v$$
TV-domains

d w t g v

$$d=w$$ $$w=t$$
DW-domains   TW-domains

$$d=t$$
FGV-domains

$$d=g$$
DG-domains

$$d=v$$
Divisorial domains

Now, we characterize G-Krull domains as follows.

Theorem 4.4. *The following statements are equivalent for a domain R.*

 (1) *R is a G-Krull domain.*
 (2) *R is a g-Noetherian domain with $g = v$.*
 (3) *R is a w-divisorial strong Mori domain.*
 (4) *R is a w-divisorial Mori domain.*
 (5) *R is a strong Mori domain which is a TW-domain.*
 (6) *R is a Mori domain which is a TW-domain.*
 (7) *R a strong Mori domain and $R_{\mathfrak{m}}$ is divisorial for all $\mathfrak{m} \in w\text{-Max}(R)$.*
 (8) *R is a strong Mori domain and $R_{\mathfrak{m}}$ is an FGV-domain for all $\mathfrak{m} \in w\text{-Max}(R)$.*
 (9) *R is a Mori domain and $R_{\mathfrak{m}}$ is a G-Dedekind domain for all $\mathfrak{m} \in w\text{-Max}(R)$.*

Proof. $(1) \Rightarrow (2)$ Assume that R is a G-Krull domain and let I be a nonzero proper ideal of R. Set $0 \neq x \in I$. Then by Corollary 4.3, x is contained in only finitely many elements of $\mathscr{P}(R)$, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Since $R_{\mathfrak{p}_i}$ is G-Dedekind for each $\mathfrak{p}_i$, $IR_{\mathfrak{p}_i}$ is a finitely generated ideal of $R_{\mathfrak{p}_i}$. Write $IR_{\mathfrak{p}_i} = (a_{i1}, \ldots, a_{in_i})R_{\mathfrak{p}_i}$, where $a_{ij} \in I$, $i = 1, \ldots, n$, and $j = 1, \ldots, n_i$. Let A be the ideal of R generated by x and all a_{ij}. Then $A \subseteq I$. Let $\mathfrak{p} \in \mathscr{P}$. If $\mathfrak{p} = \mathfrak{p}_i$

for some i, then $AR_{\mathfrak{p}} = (a_{i1}, \ldots, a_{in_i})R_{\mathfrak{p}_i} = IR_{\mathfrak{p}}$; otherwise, $\mathfrak{p} \neq \mathfrak{p}_i$ for any i, and we derive that $AR_{\mathfrak{p}} = R_{\mathfrak{p}} = IR_{\mathfrak{p}}$ as $x \notin \mathfrak{p}$. Hence, $A_g = \cap_{\mathfrak{p} \in \mathscr{P}(R)} AR_{\mathfrak{p}} = \cap_{\mathfrak{p} \in \mathscr{P}(R)} IR_{\mathfrak{p}} = I_g$, and so I is g-finitely generated. Thus, it follows that R is g-Noetherian.

To see $g = v$, let J be a g-ideal of R and let $\mathfrak{p} \in \mathscr{P}(R)$. Since R is Mori and $R_{\mathfrak{p}}$ is G-Dedekind, it is easy to see that $J_v R_{\mathfrak{p}} = (JR_{\mathfrak{p}})_{v_{R_{\mathfrak{p}}}} = JR_{\mathfrak{p}}$. But both J and J_v are g-ideals, so $J_v = J$.

$(2) \Rightarrow (3)$ This follows from the fact that the notions of g-Noetherian domain and strong Mori domain coincide, and that every g-Noetherian domain is a GW-domain.

$(3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$ and $(7) \Rightarrow (8)$ are obvious.

$(6) \Rightarrow (7)$ It follows immediately from [30, Corollary 2.5 and Theorem 2.4].

$(8) \Rightarrow (9)$ By [45, Proposition 4.6].

$(9) \Rightarrow (1)$ Assume that (9) holds. Then R is a GW-domain, and so w-Max$(R) \subseteq \mathscr{P}(R)$. To see the reverse inclusion, let $\mathfrak{p} \in \mathscr{P}(R)$. Then $\mathfrak{p}$ is a w-ideal, and hence it is contained in some $\mathfrak{m}_0 \in w$-Max(R). But note that ht$(\mathfrak{m}) = 1$ for all $\mathfrak{m} \in w$-Max(R). Therefore, $\mathfrak{p} = \mathfrak{m}_0$. Thus, w-Max$(R) = \mathscr{P}(R)$. Since R is a TV-domain, it follows from [23, Theorem 1.3] that each nonzero element of R is contained in only finitely many elements of $\mathscr{P}(R)$. Thus, by Corollary 4.3, R is a G-Krull domain. $\square$

It is worth noting that the class of w-divisorial Mori domains has already been studied in [3, 14].

Recall from [42, 43] that the w-*dimension* of a domain R, abbreviated w-dim(R), is by definition equal to the length of the longest chain of prime w-ideals of R. Since every prime subideal of a prime w-ideal of R is a w-ideal, it is easy to see that

$$w\text{-dim}(R) = \sup\{\text{ht}(\mathfrak{m}) \mid \mathfrak{m} \in w\text{-Max}(R)\}.$$

As a consequence of Theorem 4.4, one has the following:

Corollary 4.5. *Every G-Krull domain has w-dimension one.*

It is well known that a domain R is Dedekind if and only if R is one-dimensional integrally closed Noetherian, if and only if R is a one-dimensional Krull domain. The following is a Gorenstein analogue of the result.

Corollary 4.6. *The following statements are equivalent for a domain R.*

(1) *R is a G-Dedekind domain.*
(2) *R is a one-dimensional quasi-normal domain.*
(3) *R is a one-dimensional G-Krull domain.*

Proof. Use Theorem 4.4 and [3, Proposition 4.1]. $\square$

Proposition 4.7. *Let R be a G-Krull domain and let S be a multiplicatively closed subset of R. Then R_S is a G-Krull domain.*

Proof. It follows from Theorem 4.4, [30, Theorem 2.2], and [45, Proposition 4.7]. $\qquad\square$

Proposition 4.8. *If R is a G-Krull domain, then so is $R[X]$.*

Proof. It follows from Theorem 4.4, [46, Theorem 1.13], and [14, Proposition 3.6]. $\qquad\square$

Proposition 4.9. *A domain R is a Krull domain if and only if R is an integrally closed G-Krull domain.*

Proof. Note that an integrally closed TW-domain is a PVMD (cf. [27, Theorem 3.5]), and that a Mori domain which is a PVMD is a Krull domain. Thus, the proof follows by Theorem 4.4. $\qquad\square$

Proposition 4.10. *The following statements are equivalent for a domain R.*

(1) *R is a G-Krull domain.*
(2) *$R[X]_{N_v}$ is a G-Dedekind domain.*
(3) *$R[X]_{N_v}$ is a G-Krull domain.*
(4) *$R[X]$ is a G-Krull domain.*

Proof. Note first that a domain is a G-Krull domain if and only if it is a w-divisorial strong Mori domain. Therefore, the equivalence of (2) and (3) follows from the fact that $R[X]_{N_v}$ is a DW-domain. Moreover, the proof of $(1) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$ follows easily from [7, Theorem 2.2] and [14, Propositions 3.2 and 3.6]. $\qquad\square$

5. Pullbacks

In [30, 31] Mimouni investigated the w-operation on a pullback of domains. To provide some original examples, we examine, in this section, the transfer of the DG-property in pullback diagrams.

Let M be a (nonzero) maximal ideal of a domain T, let $\phi : T \to F := T/M$ be the natural projection, and let D be a proper subring of F. Then let $R = \phi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms:

$$(\square) \qquad \begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\;\phi\;} & F = T/M. \end{array}$$

The case where $T = V$ is a valuation ring of the form $k + M$, where k is a field and M is the maximal ideal of V, is of particular interest. This is known as the classical $D + M$ construction.

Proposition 5.1. *For the diagram $(\square)$, T is a g-overring of R.*

Proof. Let $x \in T_g$ and set $A = (T : x)$. Then $A \in \mathscr{L}_g(R)$, and so $A^{-1} = R$. Now, the rest of the proof that $x \in T$ is the same as the proof of [30, Lemma 3.3]. $\qquad\square$

Lemma 5.2. *For the diagram* ($\square$), *let* $\mathfrak{p}$ *be a prime ideal of* R *containing* M. *Then* $\mathfrak{p} \in \mathscr{P}(R)$ *if and only if* $\phi(\mathfrak{p}) \in \mathscr{P}(D)$.

Proof. Let us first consider the special case when R is local with maximal ideal $\mathfrak{p}$. Let $\mathfrak{p}$ be an associated prime of xR for some $x \in R$. Clearly, $x \neq 0$ (otherwise, $\mathfrak{p} = M = 0$, a contradiction). If $\mathfrak{p} = M$, then $\phi(\mathfrak{p}) = 0 \in \mathscr{P}(D)$. If $M \subsetneq \mathfrak{p}$, then we can choose an element $y \in \mathfrak{p} \backslash M$. By [5, Theorem 3], $\mathfrak{p}$ is an associated prime of yR. Then there exists $z \in R$ such that $\mathfrak{p}$ is minimal over $(yR \underset{R}{:} zR)$. To show $\phi(\mathfrak{p}) \in \mathscr{P}(D)$, it suffices to show that $\phi(\mathfrak{p})$ is minimal over $(\phi(y)D \underset{D}{:} \phi(z)D)$. If $\phi(z) \in \phi(y)D$, then there is $u \in R$ such that $\phi(z) = \phi(y)\phi(u)$, i.e., $z - yu \in M$. Since $M \subseteq yR_{\mathfrak{p}}$ ([13, Lemma 2.25]), there exists $s \in R \backslash \mathfrak{p}$ such that $s(z - yu) \in yR$. Note that s is a unit of R. So $z \in yR$, a contradiction. Thus, $(\phi(y)D \underset{D}{:} \phi(z)D) \neq D$, and so $(\phi(y)D \underset{D}{:} \phi(z)D) \subseteq \phi(\mathfrak{p})$. Moreover, if Q is a prime ideal of D satisfying $(\phi(y)D \underset{D}{:} \phi(z)D) \subsetneq Q \subseteq \phi(\mathfrak{p})$, then $(yR \underset{R}{:} zR) \subsetneq \phi^{-1}(Q) \subseteq \mathfrak{p}$. Indeed, it suffices to observe that $(\phi(y)D \underset{D}{:} \phi(z)D) = \phi((yR \underset{R}{:} zR))$ by using the fact that $M \subseteq yR_{\mathfrak{p}}$. Hence, $\mathfrak{p} = \phi^{-1}(Q)$, and so $\phi(\mathfrak{p}) = Q$. It follows that $\phi(\mathfrak{p}) \in \mathscr{P}(D)$. Conversely, assume that $\phi(\mathfrak{p})$ is an associated prime of $\phi(y)D$ for some $y \in R$. If $y \in M$, then $\phi(y) = 0$, and so $\phi(\mathfrak{p}) = 0$. Therefore, $\mathfrak{p} = M$ is v-maximal in R. By the proof of [23, Proposition 1.6], it is easy to see that $\mathfrak{p} \in \mathscr{P}(R)$. Now assume that $y \notin M$. There exists $z \in R$ such that $\phi(\mathfrak{p})$ is minimal over $(\phi(y)D \underset{D}{:} \phi(z)D)$. Then $(yR \underset{R}{:} zR) \neq R$, and so $(yR \underset{R}{:} zR) \subseteq \mathfrak{p}$. To show $\mathfrak{p} \in \mathscr{P}(R)$, we need only show that $\mathfrak{p}$ is minimal over $(yR \underset{R}{:} zR)$. For this, let $\mathfrak{q}$ be a prime ideal of R with $(yR \underset{R}{:} zR) \subsetneq \mathfrak{q} \subseteq \mathfrak{p}$. Note that $M \subseteq (yR \underset{R}{:} zR)$ as $M \subseteq yR_P$. So we have $(\phi(y)D \underset{D}{:} \phi(z)D) = \phi((yR \underset{R}{:} zR)) \subsetneq \phi(\mathfrak{q}) \subseteq \phi(\mathfrak{p})$. Thus, $\phi(\mathfrak{q}) = \phi(\mathfrak{p})$, and so $\mathfrak{q} = \mathfrak{p}$.

Now let R be not local. By [13, Lemma 1.4], the following is also a pullback diagram of type ($\square$):

$$
\begin{array}{ccc}
R_{\mathfrak{p}} & \longrightarrow & D_{\phi(\mathfrak{p})} \\
\downarrow & & \downarrow \\
T_{\mathfrak{p}} & \overset{\phi'}{\longrightarrow} & F
\end{array}
$$

where ϕ' is induced from the map ϕ. Note that $\phi'(\mathfrak{p}R_{\mathfrak{p}}) = \phi(\mathfrak{p})D_{\phi(\mathfrak{p})}$. Hence, by Lemma 2.4, $\mathfrak{p} \in \mathscr{P}(R) \Leftrightarrow \mathfrak{p}R_{\mathfrak{p}} \in \mathscr{P}(R_{\mathfrak{p}}) \Leftrightarrow \phi'(\mathfrak{p}R_{\mathfrak{p}}) \in \mathscr{P}(D_{\phi(\mathfrak{p})}) \Leftrightarrow \phi(\mathfrak{p}) \in \mathscr{P}(D)$, which completes the proof. $\qquad\square$

Proposition 5.3. *For the diagram* $(\square)$, *let I be an ideal of R and let A be an ideal of D. Then:*

(1) *If $I \in \mathscr{L}_g(R)$, then $\phi(I) \in \mathscr{L}_g(D)$.*

(2) *If $A \in \mathscr{L}_g(D)$, $\phi^{-1}(A) \in \mathscr{L}_g(R)$.*

(3) $\phi^{-1}(A_{g_D}) = (\phi^{-1}(A))_g$.

(4) *If $I \nsubseteq M$, then $(\phi(I_g))_{g_D} = (\phi(I))_{g_D}$.*

(5) *If $M \subsetneq I$, then $\phi(I_g) = (\phi(I))_{g_D}$.*

Proof. (1) Let $I \in \mathscr{L}_g(R)$. If $\phi(I) \notin \mathscr{L}_g(D)$, then $\phi(I)$ must be contained in some $P \in \mathscr{P}(D)$. Therefore, $I \subseteq I + M \subseteq \phi^{-1}(P)$. Note that $\phi^{-1}(P)$ is a prime ideal of R containing M, and that $\phi(\phi^{-1}(P)) = P$. Thus, by Lemma 5.2, $\phi^{-1}(P) \in \mathscr{P}(R)$, a contradiction.

(2) Let $A \in \mathscr{L}_g(D)$. If $\phi^{-1}(A) \notin \mathscr{L}_g(R)$, then $M \subseteq \phi^{-1}(A) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathscr{P}(R)$, and so $A \subseteq \phi(\mathfrak{p})$. However, by Lemma 5.2, we have $\phi(\mathfrak{p}) \in \mathscr{P}(D)$, which is impossible.

(3) Let $x \in \phi^{-1}(A_{g_D})$. Then $\phi(x) \in A_{g_D}$, and so there is $B \in \mathscr{L}_g(D)$ such that $B\phi(x) \subseteq A$. Set $J = \phi^{-1}(B)$. Then by (2), $J \in \mathscr{L}_g(R)$. Since $\phi(Jx) = B\phi(x) \subseteq A$, $Jx \subseteq \phi^{-1}(A)$, and so $x \in (\phi^{-1}(A))_g$. Therefore, $\phi^{-1}(A_{g_D}) \subseteq (\phi^{-1}(A))_g$. To see the reverse inclusion, let $x \in (\phi^{-1}(A))_g$. Then $Jx \subseteq \phi^{-1}(A)$ for some $J \in \mathscr{L}_g(R)$. By (1), $\phi(J) \in \mathscr{L}_g(D)$. Since $\phi(J)\phi(x) = \phi(Jx) \subseteq A$, then $\phi(x) \in A_{g_D}$, and so $x \in \phi^{-1}(A_{g_D})$. Hence, $(\phi^{-1}(A))_g \subseteq \phi^{-1}(A_{g_D})$.

(4) Let $I \nsubseteq M$ and write $C = \phi(I)$. Then $C \neq 0$ and $I \subseteq \phi^{-1}(C)$. By (3), we have $I_g \subseteq (\phi^{-1}(C))_g = \phi^{-1}(C_{g_D})$, and so $\phi(I_g) \subseteq C_{g_D} = (\phi(I))_{g_D}$. Hence, $(\phi(I_g))_{g_D} \subseteq (\phi(I))_{g_D}$. The reverse inclusion is obvious.

(5) Let $M \subsetneq I$. Then, by (4), $\phi(I_g) \subseteq (\phi(I_g))_{g_D} = (\phi(I))_{g_D}$. Conversely, let $x \in (\phi(I))_{g_D}$. Then there exists $B \in \mathscr{L}_g(D)$ such that $Bx \subseteq \phi(I)$. Set $J = \phi^{-1}(B)$ and $x = \phi(y)$, where $y \in R$. Then $J \in \mathscr{L}_g(R)$ by (2). Since $\phi(Jy) = \phi(J)\phi(y) = Bx \subseteq \phi(I)$, then $Jy \subseteq I$, and so $y \in I_g$. Thus, $x \in \phi(I_g)$ and therefore $(\phi(I))_{g_D} \subseteq \phi(I_g)$. $\square$

Theorem 5.4. *For the diagram* $(\square)$,

(1) *If R is a DG-domain, then so is D.*

(2) *If D is a DG-domain and $d_T = \bar{v}_T$, then R is a DG-domain.*

(3) *If T is local and D is a DG-domain, then R is a DG-domain.*

Proof. (1) Assume that R is DG, and let A be an ideal of D. Then by Proposition 5.3(3), $\phi^{-1}(A) = (\phi^{-1}(A))_g = \phi^{-1}(A_{g_D})$, whence $A = A_{g_D}$, that is, A is a g-ideal of D.

(2) Assume that D is DG and $d_T = \bar{v}_T$. By Proposition 2.1, it suffices to show that every maximal ideal of R is a g-ideal. Let $\mathfrak{m}$ be a maximal ideal of R. If $\mathfrak{m} = M$, then $\mathfrak{m}$ is a v-ideal (and hence a g-ideal) of R. If $\mathfrak{m} \supsetneq M$, then by Proposition 5.3(5), $\phi(\mathfrak{m}_g) = (\phi(\mathfrak{m}))_{g_D} = \phi(\mathfrak{m})$, and so $\mathfrak{m}_g = \mathfrak{m}$. If $\mathfrak{m} \nsupseteq M$, then it follows from [13, Proposition 1.9] that $\mathfrak{m} = P \cap R$ for some prime ideal P of T. Then P is a $\bar{v}$-ideal of T as $d_T = \bar{v}_T$. Also, note that T is a g-overring of R (Proposition 5.1). Thus, $\mathfrak{m}$ is a g-ideal of R by Proposition 2.11.

(3) Assume that T is local and D is DG. Then by [13, Proposition 1.6], each ideal of R is comparable to M. Let $\mathfrak{m}$ be a maximal ideal of R. Then we have either $\mathfrak{m} = M$ or $\mathfrak{m} \supsetneq M$. Thus, the rest of the proof follows as in the proof of (2). $\square$

Remark 5.5.

(1) In [31, Theorem 3.1], it was shown that for the diagram of ($\square$), if D is a DW-domain and T is either local or a DW-domain, then R is DW. Thus, if we take D to be a DW-domain that is not DG (e.g., the valuation ring of Example 1.6) and T is either local or a DW-domain, then it follows from Theorem 5.4(1) that R is a DW-domain that is not DG.

(2) For the diagram of ($\square$), if R is a DG-domain, then T is not necessary a DG-domain even if T is local (cf. [31, Example 3.4(1)]).

(3) The assumption that D is a DG-domain is not sufficient to get R is DG when T is not local (cf. [31, Example 3.4(2)]).

(4) We do not know whether "$d_T = \bar{v}_T$" (in the assumption of Theorem 5.4(2)) can be weakened to "T is a DG-domain".

Corollary 5.6. *Let R be the classical $D + M$ construction. Then R is a DG-domain if and only if so is D.*

Example 5.7. In [31, Example 3.11(2)] Mimouni gives a classical $D + M$ construction R which is a DW-domain but $d \neq \bar{v}$, where D is a one-dimensional Prüfer domain. Note that D is a DG-domain. Thus, by Corollary 5.6, R is in fact a DG-domain.

References

[1] D. D. Anderson, *Star-operations induced by overrings*, Comm. Algebra **16** (1988), no. 12, 2535–2553.

[2] D. D. Anderson and S. Cook, *Two star-operations and their induced lattices*, Comm. Algebra **28** (2000), no. 5, 2461–2475.

[3] S. El Baghdadi and S. Gabelli, *w-divisorial domains*, J. Algebra **285** (2005), no. 1, 335–355.

[4] D. Bennis, *A note on Gorenstein global dimension of pullback rings*, Int. Electron. J. Algebra **8** (2010), 30–44.

[5] J. W. Brewer and W. J. Heinzer, *Associated primes of principal ideals*, Duke Math. J. **41** (1974), 1–7.

[6] P.-J. Cahen, *Torsion theory and associated primes*, Proc. Amer. Math. Soc. **38** (1973), 471–476.

[7] G. W. Chang, *Strong Mori domains and the ring $D[X]_{N_v}$*, J. Pure Appl. Algebra **197** (2005), no. 1-3, 293–304.

[8] D. E. Dobbs, E. G. Houston, T. G. Lucas, M. Roitman, and M. Zafrullah, *On t-linked overrings*, Comm. Algebra **20** (1992), no. 5, 1463–1488.

[9] D. E. Dobbs, E. G. Houston, T. G. Lucas, and M. Zafrullah, *t-linked overrings and Prüfer v-multiplication domains*, Comm. Algebra **17** (1989), no. 11, 2835–2852.

[10] _____, *t-linked overrings as intersections of localizations*, Proc. Amer. Math. Soc. **109** (1990), no. 3, 637–646.

[11] R. M. Fossum, *The Divisor Class Group of a Krull Domain*, Springer-Verlag, New York-Heidelberg, 1973.

[12] S. Gabelli, *Generalized Dedekind domains*, Multiplicative Ideal Theory in Commutative Algebra, pp. 189–206, Springer, New York, 2006.

[13] S. Gabelli and E. Houston, *Ideal theory in pullbacks*, Non-Noetherian commutative ring theory, 199–227, Math. Appl., vol. 520, Kluwer Acad. Publ., Dordrecht, 2000.

[14] S. Gabelli, E. Houston, and G. Picozza, *w-divisoriality in polynomial rings*, Comm. Algebra **37** (2009), no. 3, 1117–1127.

[15] J. M. García, P. Jara, and E. Santos, *Prüfer ∗-multiplication domains and torsion theories*, Comm. Algebra **27** (1999), no. 3, 1275–1295.

[16] R. Gilmer, *A class of domains in which primary ideals are valuation ideals*, Math. Ann. **161** (1965), 247–254.

[17] _____, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[18] R. Gilmer and W. J. Heinzer, *Intersections of quotient rings of an integral domain*, J. Math. Kyoto Univ. **7** (1967), 133–150.

[19] S. Glaz and W. V. Vasconcelos, *Flat ideals II*, Manuscripta Math. **22** (1977), no. 4, 325–341.

[20] J. S. Golan, *Torsion Theories*, Longman Scientific & Technical, Harlow, 1986.

[21] J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.

[22] W. Heinzer, *Integral domains in which each non-zero ideal is divisorial*, Mathematika **15** (1968), 164–170.

[23] E. Houston and M. Zafrullah, *Integral domains in which each t-ideal is divisorial*, Michigan Math. J. **35** (1988), no. 2, 291–300.

[24] _____, *On t-invertibility. II*, Comm. Algebra **17** (1989), no. 8, 1955–1969.

[25] K. Hu and F. Wang, *Some results on Gorenstein Dedekind domains and their factor rings*, Comm. Algebra **41** (2013), no. 1, 284–293.

[26] J. A. Huckaba and I. J. Papick, *A localization of R[x]*, Canad. J. Math. **33** (1981), no. 1, 103–115.

[27] B. G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), no. 1, 151–170.

[28] I. Kaplansky, *Commutative Rings* (Revised edition), Univ. Chicago Press, Chicago, 1974.

[29] N. Mahdou and M. Tamekkante, *On (strongly) Gorenstein (semi) hereditary rings*, Arab. J. Sci. Eng. **36** (2011), no. 3, 431–440.

[30] A. Mimouni, *TW-domains and strong Mori domains*, J. Pure Appl. Algebra **177** (2003), no. 1, 79–93.

[31] _____, *Integral domains in which each ideal is a W-ideal*, Comm. Algebra **33** (2005), no. 5, 1345–1355.

[32] J. L. Mott and M. Zafrullah, *On Prüfer v-multiplication domains*, Manuscripta Math. **35** (1981), no. 1-2, 1–26.

[33] I. J. Papick, *Super-primitive elements*, Pacific J. Math. **105** (1983), no. 1, 217–226.

[34] M. H. Park and F. Tartarone, *Divisibility properties related to star-operations on integral domains*, Int. Electron. J. Algebra **12** (2012), 53–74.

[35] G. Picozza and F. Tartarone, *When the semistar operation ⋆̃ is the identity*, Comm. Algebra **36** (2008), no. 5, 1954–1975.

[36] L. Qiao and F. Wang, *A Gorenstein analogue of a result of Bertin*, J. Algebra Appl. **14** (2015), no. 2, 1550019, 13 pp.

[37] _____, *A hereditary torsion theory for modules over integral domains and its applications*, Comm. Algebra **44** (2016), no. 4, 1574–1587.

[38] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.

[39] H. Tang, *Gauss' Lemma*, Proc. Amer. Math. Soc. **35** (1972), 372–376.

[40] W. V. Vasconcelos, *Reflexive modules over Gorenstein rings*, Proc. Amer. Math. Soc. **19** (1968), 1349–1355.

[41] _____, *Quasi-normal rings*, Illinois J. Math. **14** (1970), 268–273.

[42] F. Wang, *w-dimension of domains*, Comm. Algebra **27** (1999), no. 5, 2267–2276.

[43] _____, *w-dimension of domains. II*, Comm. Algebra **29** (2001), no. 6, 2419–2428.

[44] F. Wang and H. Kim, *Two generalizations of projective modules and their applications*, J. Pure Appl. Algebra **219** (2015), no. 6, 2099–2123.

[45] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.

[46] _____, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.

[47] F. Wang and L. Qiao, *The w-weak global dimension of commutative rings*, Bull. Korean Math. Soc. **52** (2015), no. 4, 1327–1338.

[48] M. Zafrullah, *The v-operation and intersections of quotient rings of integral domains*, Comm. Algebra **13** (1985), no. 8, 1699–1712.

[49] _____, *Ascending chain conditions and star operations*, Comm. Algebra **17** (1989), no. 6, 1523–1533.

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