

## IMPROVED CONVERGENCE OF STEFFENSEN'S METHOD FOR APPROXIMATING FIXED POINTS OF OPERATORS IN BANACH SPACE

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ABSTRACT. We present a new local as well as a semilocal convergence analysis for Steffensen's method in order to locate fixed points of operators on a Banach space setting. Using more precise majorizing sequences we show under the same or less computational cost that our convergence criteria can be weaker than in earlier studies such as [1–13], [21, 22]. Numerical examples are provided to illustrate the theoretical results.

### 1. Introduction

In this study we are concerned with the problem of locating a locally unique fixed point of equation

$$(1.1) \quad F(x) = x,$$

where  $F$  is a continuous operator defined on a nonempty open subset  $D$  of a Banach space  $X$  with values in itself.

Many problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using mathematical modelling [6, 9, 20, 22]. The solutions of these equations can be rarely be found in closed form. That is why most solution methods for these equations are usually iterative. The study about convergence of iterative procedures is usually based on two types: semilocal and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls.

Steffensen's method defined by

$$(1.2) \quad \begin{aligned} x_{n+1} &= x_n + A_n^{-1}(F(x_n) - x_n), \\ A_n &= I - [F(x_n), x_n; F] \quad \text{for each } n = 0, 1, 2, \dots, \end{aligned}$$

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Received September 27, 2015.

2010 *Mathematics Subject Classification.* 47H10, 47J25, 47J05, 49M15, 65G99.

*Key words and phrases.* Steffensen's method, Banach space, fixed point, local-semilocal convergence, divided difference.

where,  $x_0$  is an initial point, and  $[\cdot, \cdot; F] \in L(X)$  denotes a divided difference with the property

$$(1.3) \quad [x, y; F](x - y) = F(x) - F(y) \quad \text{for each } x, y \in D \text{ with } x \neq y,$$

has been used to generate a sequence approximating  $x^*$  [6, 22, 25]. Here,  $L(X)$  denotes the space of bounded linear operators from  $X$  into  $X$ .

It is known that if  $F$  is differentiable, then

$$[x, x; F] = F'(x) \quad \text{for all } x \in X.$$

There is a plethora on local as well as semi-local convergence results for Steffensen's method. We refer the reader to [1–26] and the references therein. The usual hypotheses include (1.3) and Lipschitz-type hypotheses of the form

$$\|[x, y; F] - [y, z; F]\| \leq l_1 \|x - z\|$$

(see e.g. [13]) or

$$\|[x, y; F] - [u, v; F]\| \leq l_2 (\|x - u\| + \|y - v\|)$$

(see e.g. [19]) or

$$\|[x, y; F] - [y, z; F]\| \leq l_3 \|x - z\| + l_4 \|x - y\| + l_5 \|y - z\|,$$

(see e.g. [1, 6, 9, 13, 19]) where  $l_i > 0$ ,  $i = 1, 2, 3, 4, 5$  and for each  $x, y, z, u, v \in D$  together with a contractive hypothesis (or not) of operator  $F$ .

In the present paper we provide a new semi-local and local convergence analysis for Steffensen's method. Our approach uses more general Lipschitz-type hypotheses and tighter majorizing sequences than the above listed references.

The paper is organized as follows: the semilocal convergence and local convergence are given in Section 2 and Section 3, respectively. The numerical examples are given in the concluding Section 4.

## 2. Semilocal convergence

We present the semilocal convergence analysis of Steffensen's method (1.2) in this section. It is convenient to introduce some parameters. Let  $a_0 > 0$ ,  $a > 0$ ,  $b > 0$ ,  $c \geq 0$ ,  $d \geq 0$ ,  $\eta \geq 0$ ,  $M_i \geq 0$ ,  $i = 1, 2, 3$  and  $K_j \geq 0$ ,  $j = 1, 2, 3, 4$  be given parameters. Define parameters  $\alpha, \beta, \gamma, \delta, \lambda, \mu, \xi_0, \xi$  and  $\xi_1$  by

$$(2.1) \quad \begin{aligned} \alpha &= a[(K_1 + K_2)d + K_3 + K_4], & \beta &= ac(K_2 + K_3), \\ \gamma &= (K_1 + K_2)d + K_3 + K_4, & \delta &= a_0 + (K_2 + K_3)c, \\ \lambda &= \gamma(M_1 + M_3), & \mu &= \delta(M_1 + M_3) + M_2 + M_3, \\ \xi_0 &= \frac{a\mu}{1-\beta}, & \xi &= \frac{a\lambda}{1-\beta}, & \xi_1 &= \frac{\alpha}{1-\beta} \quad \text{for } \beta \neq 1. \end{aligned}$$

Moreover, define a function  $h : [0, 1) \rightarrow \mathbb{R}$  by

$$h(t) = \xi_1 \eta t + \xi \eta^2 + \xi_0 \eta + \xi_1 \eta - 1 + \frac{[(\xi \eta + \xi_1)t^2 + (\xi \eta + \xi_0)t - (\xi_0 + \xi \eta)]\eta}{1 - t}.$$

Suppose that

$$(2.2) \quad \xi_1 \eta < 1.$$

Then, we have by (2.2) that  $h(0) = \xi_1\eta - 1 < 0$  and  $h(t) \rightarrow +\infty$  as  $t \rightarrow 1^-$ . It follows from the intermediate value theorem that function  $h$  has zeros in the interval  $(0, 1)$ . Denote by  $q$  the largest such zero. Then, we have that

$$(2.3) \quad h(t) \leq 0 \quad \text{for each } t \in [0, q].$$

We can show the following auxiliary result on majorizing sequences for the Steffensen's method (1.2) using the preceding notation.

**Lemma 2.1.** *Suppose that (2.2),*

$$(2.4) \quad \beta < 1$$

and

$$(2.5) \quad (\xi_0 + \xi_1 q)\eta \leq q$$

hold. Then, the scalar sequence  $\{t_n\}$  generated by

$$(2.6) \quad t_0 = 0, t_1 = \eta, t_{n+2} = t_{n+1} + \frac{(\xi t_n + \xi_0)(t_{n+1} - t_n)^2}{1 - \xi_1 t_{n+1}} \quad \text{for each } n = 0, 1, 2, \dots,$$

is well defined, nondecreasing, bounded from above by  $t^{**}$  defined by

$$(2.7) \quad t^{**} = \frac{\eta}{1 - q}$$

and converges to its unique least upper bound  $t^*$  which satisfies

$$(2.8) \quad \eta \leq t^* \leq t^{**}.$$

Moreover, the following estimates hold

$$(2.9) \quad 0 \leq t_{n+1} - t_n \leq q^n \eta$$

and

$$(2.10) \quad 0 \leq t^* - t_n \leq \frac{q^n \eta}{1 - q}.$$

*Proof.* We shall show estimates (2.9) and (2.10) using mathematical induction. Define a sequence  $\{q_n\}$  by

$$(2.11) \quad q_n = \frac{(\xi t_n + \xi_0)(t_{n+1} - t_n)}{1 - \xi_1 t_{n+1}} \quad \text{for each } n = 0, 1, 2, \dots$$

Then, we show using mathematical induction that

$$(2.12) \quad q_m \leq q \quad \text{for each } m = 0, 1, 2, \dots$$

Estimate (2.12) holds for  $m = 0$  by (2.5), (2.6) and (2.11) (for  $n = 0$ ). It follows from (2.7), (2.6) and (2.12) (for  $n = 0$ ) that

$$0 \leq t_2 - t_1 \leq q\eta \Rightarrow t_2 \leq t_1 + q\eta = (1 + q)\eta = \frac{1 - q^2}{1 - q}\eta < t^{**}.$$

Let us assume that (2.12) holds for all positive integers  $m \leq n$ . Then, we have again by (2.12) that

$$(2.13) \quad 0 \leq t_{m+1} - t_m \leq q^m \eta$$

and

$$(2.14) \quad t_{m+1} \leq \frac{1 - q^{m+1}}{1 - q} \eta < t^{**}.$$

We shall show that (2.12) holds for  $m + 1$  replacing  $m$ . That is we must have that

$$(2.15) \quad \frac{(\xi_0 + \xi t_{m+1})(t_{m+2} - t_{m+1})}{1 - \xi_1 t_{m+2}} \leq q$$

or by (2.13) and (2.14) that

$$\frac{(\xi_0 + \xi \frac{1 - q^{m+1}}{1 - q} \eta) q^{m+1} \eta}{1 - \xi_1 \frac{1 - q^{m+2}}{1 - q} \eta} \leq q$$

or

$$(2.16) \quad \xi_0 \eta q^m + \xi \frac{1 - q^{m+1}}{1 - q} q^m \eta^2 + \xi_1 \frac{1 - q^{m+2}}{1 - q} \eta - 1 \leq 0.$$

Estimate (2.16) motivates us to define recurrent functions  $f_m$  on the interval  $[0, 1)$  by

$$(2.17) \quad f_m(t) = \xi_0 \eta t^m + \xi \frac{1 - t^{m+1}}{1 - t} \eta^2 t^m + \xi_1 \frac{1 - t^{m+2}}{1 - t} \eta - 1.$$

We need a relationship between two consecutive functions  $f_m$ . Using (2.17) and some straight forward algebraic manipulation, we get that

$$(2.18) \quad f_{m+1}(t) = f_m(t) + g_m(t) \eta t^m,$$

where

$$(2.19) \quad g_m(t) = \xi \eta (t^{m+2} + t^{m+1} - 1) + \xi_1 t^2 + \xi_0 t - \xi_0.$$

We also need a relationship between two consecutive functions  $g_m$ . Using (2.19), we get that

$$(2.20) \quad g_{m+1}(t) = g_m(t) + \xi \eta t^{m+1} (t^2 - 1) \leq g_m(t) \leq \dots \leq g_0(t).$$

In view of (2.18) and (2.20), we obtain in turn that

$$(2.21) \quad \begin{aligned} f_{m+1}(t) &\leq f_m(t) + g_0(t) t^m \eta \leq f_{m-1}(t) + g_0(t) t^{m-1} \eta + g_0(t) t^m \eta \\ &\leq \dots \leq f_0(t) + g_0(t) \eta \frac{1 - t^{m+1}}{1 - t} \leq f_0(t) + \frac{g_0(t) \eta}{1 - t} = h(t). \end{aligned}$$

Estimate (2.16) certainly holds, if

$$f_m(q) \leq 0$$

or by (2.21), if  $h(q) \leq 0$  which is true by (2.3) for  $t = q$ . Hence, the induction for (2.12) (i.e., for (2.13) and (2.14)) is complete.

It follows that the sequence  $\{t_m\}$  is nondecreasing, bounded from above by  $t^{**}$  defined by (2.7) and as such it converges to its unique least upper bound  $t^*$  which satisfies (2.8). Finally, estimate (2.10) follows from (2.9) by using standard majorizing techniques [6], [9], [20].  $\square$

Next, we present another auxiliary result on majorizing sequences for the Steffensen method (1.2).

**Lemma 2.2.** *Suppose that*

$$(2.22) \quad \xi_1 t_n < 1 \quad \text{for each } n = 0, 1, 2, \dots$$

*Then, the sequence  $\{t_n\}$  is well defined, nondecreasing and converges to its unique least upper bound  $t^*$  which satisfies*

$$(2.23) \quad \eta \leq t^* \leq \frac{1}{\xi_1} \quad \text{for } \xi_1 \neq 0.$$

*Proof.* The sequence  $\{t_n\}$  is nondecreasing and bounded above by  $\frac{1}{\xi_1}$ .  $\square$

*Remark 2.3.* The convergence criteria of Lemma 2.1 obviously imply criterion (2.22) but not necessarily vice versa. Clearly, (2.23) is the weakest convergence criterion for sequence  $\{t_n\}$ . This criterion can be verified, since sequence  $\{t_n\}$  is known in advance. Notice also that one can test to see if there exists a finite natural integer  $N$  such that

$$(2.24) \quad t_{N+n} = t_N < \frac{1}{\xi_1} \quad \text{for } \xi_1 \neq 0$$

for each  $n = 0, 1, 2, \dots$  which implies  $t_n \leq t_{n+1}$  for each  $n = 0, 1, 2, \dots$  and  $\lim_{n \rightarrow \infty} t_n = t^*$ .

Let  $U(x, \rho)$  and  $\overline{U}(x, \rho)$  stand, respectively for the open and closed balls in  $X$  with center  $x \in X$  and of radius  $\rho > 0$ . We shall show the semilocal convergence of the Steffensen method (1.2) under the conditions (C):

(C<sub>1</sub>)  $F : D \subset X \rightarrow X$  is a continuous operator;

(C<sub>2</sub>) There exists a divided difference  $[\cdot, \cdot; F] : D \times D \rightarrow L(X)$  satisfying (1.3);

There exist  $x_0 \in D$ ,  $a_0 > 0$ ,  $a > 0$ ,  $b > 0$ ,  $c \geq 0$ ,  $d \geq 0$ ,  $\eta \geq 0$ ,  $M_i \geq 0$ ,  $i = 1, 2, 3$  and  $K_j \geq 0$ ,  $j = 1, 2, 3, 4$  such that for each  $x, y, u \in D$

(C<sub>3</sub>)  $A_0^{-1} \in L(X)$ ,  $\|A_0\| \leq a_0$ ,  $\|A_0^{-1}\| \leq a$ ;

(C<sub>4</sub>)  $\|[F(x_0), x_0; F]\| \leq b$ ;

(C<sub>5</sub>)  $\|A_0^{-1}(F(x_0) - x_0)\| \leq \eta$ ;

(C<sub>6</sub>)  $\|F(x_0) - x_0\| \leq c$ ;

(C<sub>7</sub>)  $\|F(x) - F(x_0)\| \leq d\|x - x_0\|$ ;

(C<sub>8</sub>)  $\|[F(x), x; F] - [F(x_0), x_0; F]\| \leq K_1\|F(x) - F(x_0)\| + K_2\|F(x) - x_0\| + K_3\|F(x_0) - x\| + K_4\|x - x_0\|$ ;

(C<sub>9</sub>)  $\|[x, y; F] - [u, y; F]\| \leq M_1\|x - u\| + M_2\|x - y\| + M_3\|u - y\|$ ;

(C<sub>10</sub>)  $U_1 = \overline{U}(x_0, dt^* + c) \subseteq D$

and hypotheses of Lemma 2.1 (or Lemma 2.2) hold.

**Theorem 2.4.** *Suppose that the (C) conditions hold. Then, the sequence  $\{x_n\}$  generated for  $x_0 \in D$  by the Steffensen method (1.2) is well defined, remains in  $\overline{U}(x_0, t^*)$  for each  $n = 0, 1, 2, \dots$  and converges to a fixed point  $x^* \in \overline{U}(x_0, t^*)$  of operator  $F$ . Moreover, the following estimates hold*

$$(2.25) \quad \|x_{n+1} - x_n\| \leq t_{n+1} - t_n$$

and

$$(2.26) \quad \|x_n - x^*\| \leq t^* - t_n.$$

*Proof.* We shall show using mathematical induction that the following hold

$$(2.27) \quad F(x_m) \in U_1,$$

$$(2.28) \quad \|x_{m+1} - x_m\| \leq t_{m+1} - t_m$$

and

$$(2.29) \quad \overline{U}(x_{m+1}, t^* - t_{m+1}) \subseteq \overline{U}(x_m, t^* - t_m).$$

By (C<sub>6</sub>) and (C<sub>10</sub>) we get that  $F(x_0) \in U_1$ . We have by (1.2), (2.6) and (C<sub>5</sub>) that

$$\|x_1 - x_0\| = \|A_0^{-1}(F(x_0) - x_0)\| \leq \eta = t_1 - t_0,$$

which shows (2.25) for  $n = 0$ . For every  $z \in \overline{U}(x_1, t^* - t_1)$ , we get that

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^* - t_1 + t_1 - t_0 = t^* - t_0,$$

which shows  $z \in \overline{U}(x_0, t^* - t_0)$ . Hence, estimates (2.27)-(2.29) hold for  $m = 0$ . Let us assume these estimates hold for all positive integers  $n \leq m$ . Then, we get that

$$\|x_{m+1} - x_0\| \leq \sum_{i=1}^{m+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{m+1} (t_i - t_{i-1}) = t_{m+1} - t_0 = t_{m+1} \leq t^*,$$

and by (C<sub>6</sub>), (C<sub>7</sub>) and (C<sub>10</sub>)

$$\begin{aligned} \|F(x_{m+1}) - x_0\| &\leq \|F(x_{m+1}) - F(x_0)\| + \|F(x_0) - x_0\| \\ &\leq d\|x_{m+1} - x_0\| + c \leq d(t_{m+1} - t_0) + c \leq dt^* + c, \end{aligned}$$

which implies  $x_{m+1} \in \overline{U}(x_0, t^*)$  and  $F(x_{m+1}) \in \overline{U}(x_0, dt^* + c)$ .

We must show that  $A_{m+1}^{-1} \in L(X)$ . Using the induction hypotheses, the proof of Lemma 2.1 (or Lemma 2.2), (2.1), (C<sub>3</sub>) and (C<sub>6</sub>)-(C<sub>8</sub>), we get in turn that

$$\begin{aligned} (2.30) \quad &\|A_0^{-1}\| \|A_{m+1} - A_0\| \\ &\leq a\|[F(x_{m+1}), x_{m+1}; F] - [F(x_0), x_0; F]\| \\ &\leq a[K_1\|F(x_{m+1}) - F(x_0)\| + K_2(\|F(x_{m+1}) - F(x_0)\| + \|F(x_0) - x_0\|) \\ &\quad + K_3(\|x_{m+1} - x_0\| + \|F(x_0) - x_0\|) + K_4\|x_{m+1} - x_0\|] \\ &\leq a[K_1d\|x_{m+1} - x_0\| + K_2(d\|x_{m+1} - x_0\| + c) + K_3(\|x_{m+1} - x_0\| + c) \\ &\quad + K_4\|x_{m+1} - x_0\|] \end{aligned}$$

$$\leq \alpha \|x_{m+1} - x_0\| + \beta \leq \alpha t_{m+1} + \beta < 1.$$

It follows from (2.30) and the Banach lemma on invertible operators [6], [9], [20] that  $A_{m+1}^{-1} \in L(X)$  and

$$(2.31) \quad \|A_{m+1}^{-1}\| \leq \frac{\|A_0^{-1}\|}{1 - \|A_0^{-1}\| \|A_{m+1} - A_0\|} \leq \frac{a}{1 - (\alpha \|x_{m+1} - x_0\| + \beta)} \leq \frac{a}{1 - (\alpha t_{m+1} + \beta)}.$$

Hence,  $x_{m+2}$  is well defined. Using (1.2) we obtain in turn the identities

$$(2.32) \quad \begin{aligned} & F(x_{m+1}) - x_{m+1} \\ &= F(x_{m+1}) - x_{m+1} - F(x_m) + x_m + (I - [F(x_m), x_m; F])(x_{m+1} - x_m) \\ &= F(x_{m+1}) - F(x_m) - [F(x_m), x_m; F](x_{m+1} - x_m) \\ &= ([x_{m+1}, x_m; F] - [F(x_m), x_m; F])(x_{m+1} - x_m) \end{aligned}$$

and

$$x_{m+1} - x_m = A_m^{-1}(F(x_m) - x_m)$$

or

$$A_m(x_{m+1} - x_m) = F(x_m) - x_m$$

or

$$x_{m+1} - x_m - [F(x_m), x_m; F](x_{m+1} - x_m) = F(x_m) - x_m$$

or

$$(2.33) \quad x_{m+1} = F(x_m) + [F(x_m), x_m; F](x_{m+1} - x_m).$$

Using (2.1), (C<sub>3</sub>), (C<sub>6</sub>)-(C<sub>8</sub>) and (2.33), we get in turn that

$$(2.34) \quad \begin{aligned} & \|x_{m+1} - F(x_m)\| \\ &\leq \|[F(x_m), x_m; F]\| \|x_{m+1} - x_m\| \\ &\leq [\|[F(x_m), x_m; F] - [F(x_0), x_0; F]\| + \|[F(x_0), x_0; F]\|] \|x_{m+1} - x_m\| \\ &\leq [(K_1 + K_2)d + K_3 + K_4] \|x_m - x_0\| + a_0 + (K_2 + K_3)c \|x_{m+1} - x_m\| \\ &= (\gamma \|x_m - x_0\| + \delta) \|x_{m+1} - x_m\| \leq (\gamma t_m + \delta)(t_{m+1} - t_m). \end{aligned}$$

In view of (2.1), (C<sub>9</sub>) and (2.32)-(2.34) we also have that

$$(2.35) \quad \begin{aligned} & \|F(x_{m+1}) - x_{m+1}\| \\ &= \|([x_{m+1}, x_m; F] - [F(x_m), x_m; F])(x_{m+1} - x_m)\| \\ &\leq [M_1 \|x_{m+1} - F(x_m)\| + M_2 \|x_{m+1} - x_m\| \\ &\quad + M_3 (\|x_m - x_{m+1}\| + \|F(x_m) - x_{m+1}\|)] \|x_{m+1} - x_m\| \\ &\leq [M_1 (\gamma \|x_m - x_0\| + \delta) \|x_{m+1} - x_m\| + M_2 \|x_{m+1} - x_m\| \\ &\quad + M_3 \|x_{m+1} - x_m\| + M_3 \|x_{m+1} - F(x_m)\|] \|x_{m+1} - x_m\| \\ &\leq [(M_1 + M_3)(\gamma \|x_m - x_0\| + \delta) + M_2 + M_3] \|x_{m+1} - x_m\|^2 \\ &\leq (\lambda \|x_m - x_0\| + \mu) \|x_{m+1} - x_m\|^2 \\ &\leq (\lambda t_m + \mu)(t_{m+1} - t_m)^2. \end{aligned}$$

Using (1.2), (2.1), (2.6), (2.31) and (2.35) we get that

$$\begin{aligned}
\|x_{m+2} - x_{m+1}\| &\leq \|A_{m+1}^{-1}\| \|F(x_{m+1}) - x_{m+1}\| \\
&\leq \frac{a(\lambda\|x_m - x_0\| + \mu)\|x_{m+1} - x_m\|^2}{1 - (\alpha\|x_{m+1} - x_0\| + \beta)} \\
&= \frac{(\xi\|x_m - x_0\| + \xi_0)\|x_{m+1} - x_m\|^2}{1 - \xi_1\|x_{m+1} - x_0\|} \\
&\leq \frac{(\xi t_m + \xi_0)(t_{m+1} - t_m)^2}{1 - \xi_1 t_{m+1}} = t_{m+2} - t_{m+1},
\end{aligned}$$

which completes the induction for (2.28).

We have that for each  $w \in \overline{U}(x_{m+2}, t^* - t_{m+2})$ ,

$$\begin{aligned}
\|w - x_{m+1}\| &\leq \|w - x_{m+2}\| + \|x_{m+2} - x_{m+1}\| \\
&\leq t^* - t_{m+2} + t_{m+2} - t_{m+1} = t^* - t_{m+1},
\end{aligned}$$

which completes the induction for (2.29). Hence, it follows from (2.27)-(2.29) and Lemma 2.1 (or Lemma 2.2) that sequence  $\{x_m\}$  is complete in a Banach space  $X$  and as such it converges to some  $x^* \in \overline{U}(x_0, t^*)$  (since  $\overline{U}(x_0, t^*)$  is a closed set). By letting  $m \rightarrow \infty$  in (2.34), we get that  $F(x^*) = x^*$ . Finally, estimate (2.26) follows from (2.25) by using standard majorizing techniques [6], [9], [20].  $\square$

Next, we present a result concerning the uniqueness of the fixed point  $x^*$ .

**Proposition 2.5.** *Suppose that the hypotheses of Theorem 2.4 hold and there exists  $R \geq t^*$  such that*

$$(2.36) \quad \overline{U}(x_0, R) \subseteq D$$

and

$$(2.37) \quad a[(K_1d + K_2)t^* + (K_3 + K_4)R + K_3c] < 1.$$

Then, the limit point  $x^*$  is the only fixed point of operator  $F$  in  $\overline{U}(x_0, R)$ .

*Proof.* Let  $y^* \in \overline{U}(x_0, R)$  be such that  $F(y^*) = y^*$ . The existence of  $x^*$  has been established in Theorem 2.4. Set  $Q = I - [F(x^*), y^*; F]$ . Then, using  $(C_6)$ - $(C_8)$ , (2.36) and (2.37) we get in turn that

$$\begin{aligned}
\|A_0^{-1}(Q - A_0)\| &\leq a[K_1\|F(x^*) - F(x_0)\| + K_2\|F(x^*) - x_0\| \\
&\quad + K_3(\|F(x_0) - x_0\| + \|x_0 - y^*\|) + K_4\|y^* - x_0\|] \\
&\leq a[K_1d\|x^* - x_0\| + K_2\|x^* - x_0\| \\
&\quad + K_3(c + \|x_0 - y^*\|) + K_4\|y^* - x_0\|] \\
&\leq a[(K_1d + K_2)t^* + (K_3 + K_4)R + K_3c] < 1.
\end{aligned}$$

It follows that  $Q^{-1} \in L(X)$ . Then, using the identity

$$0 = x^* - y^* - F(x^*) + F(y^*) = (I - [x^*, y^*; F])(x^* - y^*) = Q(x^* - y^*),$$



we deduce that  $x^* = y^*$ .  $\square$

*Remark 2.6.* (a) The limit point  $t^*$  can be replaced by  $t^{**}$  (given in closed form by (2.7)) in Theorem 2.3.

(b) Hypothesis  $(C_7)$  does not necessarily imply that  $F$  is a contraction operator on  $D$ . This is an important observation, since in studies involving the convergence of Steffensen's method (1.2)  $F$  is usually a contraction operator [2–4, 13, 21].

(c) Condition

$$(2.38) \quad \|[x, y; F] - [z, w; F]\| \leq L(\|x - z\| + \|y - w\|)$$

for each  $x, y, z, w \in D$  is the popular hypothesis for iterative methods using divided differences [2–4, 13, 21]. Clearly, conditions  $(C_8)$  and  $(C_9)$  are more general and weaker than (2.38). Hence, our results can be used in cases the earlier results cannot. Notice that if e.g. we set  $K_2 = K_3 = 0$  in  $(C_8)$ , then  $K_1 \leq L$  and  $K_4 \leq L$ . In case any of these inequalities is strict, then our estimates on the distances are tighter.

### 3. Local convergence

We present the local convergence analysis of the Steffensen method (1.2).

**Theorem 3.1.** *Let  $F : D \subseteq X \rightarrow Y$  be a continuous Fréchet-differentiable operator. Suppose that there exist a divided difference  $[\cdot, \cdot; F] : D \times D \rightarrow L(X)$  of order one for operator  $F$  on  $D$  satisfying (1.3) and  $x^* \in D$ ,  $p > 0$ ,  $p_i > 0$ ,  $i = 1, 2, 3, 4$  such that for each  $x, y \in D$*

$$(3.1) \quad F(x^*) = x^*, \quad A_*^{-1} = (I - F'(x^*))^{-1} \in L(X),$$

$$(3.2) \quad \|F(x) - F(x^*)\| \leq p\|x - x^*\|,$$

$$(3.3) \quad \|A_*^{-1}([x, x^*; F] - [y, x; F])\| \leq p_1\|x - y\| + p_2\|x^* - x\|,$$

$$(3.4) \quad \|A_*^{-1}([F(x), x; F] - F'(x^*))\| \leq p_3\|F(x) - x\| + p_4\|x - x^*\|$$

and

$$(3.5) \quad \overline{U}(x^*, pr) \subseteq D,$$

where

$$(3.6) \quad r = \frac{1}{p_3p + p_4 + p_2 + (p + 1)p_1}.$$

Then, the sequence  $\{x_n\}$  generated for  $x_0 \in U(x^*, r) - \{x^*\}$  by the Steffensen method (1.2) is well defined, remains in  $U(x^*, r)$  for each  $n = 0, 1, 2, \dots$  and converges to  $x^*$ . Moreover, the following estimates hold for  $n \geq 0$ :

$$(3.7) \quad \|x_{n+1} - x^*\| \leq \frac{(p_2 + (p + 1)p_1)\|x_n - x^*\|^2}{1 - (p_3p + p_4)\|x_n - x^*\|}.$$

Furthermore, if there exists  $R \geq \max\{r, \frac{1}{p_4}\}$  such that  $U(x^*, R) \subseteq D$ , then  $x^*$  is the only fixed point of operator  $F$  on  $U(x^*, R)$ .

*Proof.* Let  $x \in \overline{U}(x^*, r)$ . Then, we have by (3.1), (3.2) and (3.5) that

$$\|F(x) - x^*\| = \|F(x) - F(x^*)\| \leq p\|x - x^*\| \leq pr.$$

Hence,  $F(x) \in \overline{U}(x^*, pr)$ . Next, we shall show that

$$A(x)^{-1} = (I - [F(x), x; F])^{-1} \in L(X).$$

Using (3.2), (3.4), (3.5) and (3.6), we get in turn that

$$\begin{aligned} (3.8) \quad \|A_\star^{-1}(A(x) - A_\star)\| &= \|A_\star^{-1}([F(x), x; F] - F'(x^*))\| \\ &\leq p_3\|F(x) - F(x^*)\| + p_4\|x - x^*\| \\ &\leq (p_3p + p_4)\|x - x^*\| < (p_3p + p_4)r < 1. \end{aligned}$$

It follows from (3.8) that

$$\begin{aligned} (3.9) \quad \|A(x)^{-1}A_\star\| &\leq \frac{1}{1 - (p_3p + p_4)\|x - x^*\|} \\ &\leq \frac{1}{1 - (p_3p + p_4)r}. \end{aligned}$$

In particular (3.9) holds for  $x = x_0$ . Hence,  $x_1$  is well defined. We have the estimate

$$\begin{aligned} (3.10) \quad F(x_0) - x^* - [F(x_0), x_0; F](x_0 - x^*) \\ &= F(x_0) - F(x^*) - [F(x_0), x_0; F](x_0 - x^*) \\ &= ([x_0, x^*; F] - [F(x_0), x_0; F])(x_0 - x^*). \end{aligned}$$

Then, using (3.1), (3.2), (3.3), (3.9) and (3.10), we get in turn that

$$\begin{aligned} (3.11) \quad \|A_\star^{-1}([x_0, x^*; F] - [F(x_0), x_0; F])(x_0 - x^*)\| \\ &\leq (p_1\|x_0 - F(x_0)\| + p_2\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq (p_1(\|x_0 - x^*\| + \|F(x^*) - F(x_0)\|) + p_2\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq (p_1(1 + p) + p_2)\|x_0 - x^*\|^2. \end{aligned}$$

Moreover, by (1.2) we can write that

$$\begin{aligned} (3.12) \quad x_1 - x^* &= x_0 - x^* + A_0^{-1}(F(x_0) - x_0) \\ &= A_0^{-1}(F(x_0) - F(x^*) - [F(x_0), x_0; F])(x_0 - x^*) \\ &= A_0^{-1}([x_0, x^*; F] - [F(x_0), x_0; F])(x_0 - x^*). \end{aligned}$$

Furthermore, using (1.2), (3.6), (3.9) and (3.12), we obtain in turn that

$$\begin{aligned} (3.13) \quad \|x_1 - x^*\| &\leq \|A(x_0)^{-1}A_\star\| \|A_\star^{-1}([x_0, x^*; F] - [F(x_0), x_0; F])(x_0 - x^*)\| \\ &\leq \frac{(p_2 + (1 + p)p_1)\|x_0 - x^*\|^2}{1 - (p_3p + p_4)\|x_0 - x^*\|} < \|x_0 - x^*\| < r, \end{aligned}$$

which shows (3.7) for  $n = 0$  and  $x_1 \in U(x^*, r)$ . By simply replacing  $x_0, F(x_0), x_1$  by  $x_k, F(x_k), x_{k+1}$  in the preceding estimates, we arrive at estimate (3.7). It then follows from the estimate  $\|x_{k+1} - x_k\| < \|x_k - x^*\| < r^*$  that  $\lim_{k \rightarrow \infty} x_k = x^*$  and  $x_{k+1} \in U(x^*, r)$ . Finally, to show the uniqueness part, let  $y^* \in U(x^*, R)$  such that  $F(y^*) = y^*$ . Let  $Q = I - [x^*, y^*; F]$ . Then, using (3.4), we have in turn that

$$\begin{aligned} \|A_{x^*}^{-1}([x^*, y^*; F] - F'(x^*))\| &\leq p_3 \|F(x^*) - x^*\| + p_4 \|y^* - x^*\| \\ &= p_4 \|y^* - x^*\| < p_4 R < 1. \end{aligned}$$

Hence,  $Q^{-1} \in L(X)$ . Then, in view of the identity

$$0 = Q(y^* - x^*),$$

we deduce that  $x^* = y^*$ .  $\square$

A remark similar to Remark 2.6 can be follow for the local convergence.

#### 4. Numerical examples

In the next two examples, we define

$$[x, y; F] = \int_0^1 F'(y + t(x - y)) dt.$$

**Example 4.1.** Let  $X = D = \overline{U(0, 1)}$ , and define  $F$  on  $D$  by

$$F(x) = \rho e^x - \rho + \tau x,$$

where  $\rho, \tau$  are given parameters with  $\rho + \tau \neq 1$ . Then,  $x^* = 0$  is a fixed point of  $F$ .

Note that for any  $x, y \in D$  we have in turn that

$$\begin{aligned} |F(x) - F(x^*)| &= |\rho(e^x - 1) + \tau x| = |\rho(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots) + \tau x| \\ &= |\rho(1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots) + \tau||x| \\ &\leq (|\rho(1 + \frac{1}{2!} + \frac{1}{3!} + \dots)| + |\tau|)|x| = (|\rho|(e - 1) + |\tau|)|x|, \end{aligned}$$

$$\begin{aligned} |[x, x^*; F] - [y, x; F]| &= \left| \int_0^1 (F'(tx + (1-t)x^*) - F'(ty + (1-t)x)) dt \right| \\ &= \left| \int_0^1 \int_0^1 F''(\theta(tx) + (1-\theta)(ty + (1-t)x))(t(x-y) - (1-t)x) d\theta dt \right| \\ &= \left| \int_0^1 \int_0^1 \rho e^{\theta(tx) + (1-\theta)(ty + (1-t)x)} (t(x-y) - (1-t)x) d\theta dt \right| \\ &\leq \int_0^1 |\rho| e^{t|x-y| + (1-t)|x|} dt = \frac{1}{2} |\rho| e^{(|x-y| + |x|)} \end{aligned}$$

and

$$\begin{aligned}
& |[F(x), x; F] - F'(x^*)| \\
&= \left| \int_0^1 (F'(tF(x) + (1-t)x) - (\rho + \tau)) dt \right| \\
&= \left| \int_0^1 \rho(e^{tF(x) + (1-t)x} - 1) dt \right| \\
&= \left| \int_0^1 \rho \left( tF(x) + (1-t)x + \frac{(tF(x) + (1-t)x)^2}{2!} + \dots \right) dt \right| \\
&\leq \int_0^1 |\rho| |tF(x) + (1-t)x| \left| 1 + \frac{(tF(x) + (1-t)x)}{2!} + \dots \right| dt \\
&\leq |\rho| \int_0^1 |t(F(x) - x) + x| \left( 1 + \frac{1}{2!} + \dots \right) dt \\
&\leq \frac{1}{2} |\rho| (e-1) (|F(x) - x| + 2|x|).
\end{aligned}$$

We also have

$$A_\star = 1 - \rho - \tau,$$

then, we can choose  $p, p_1, p_2, p_3$  and  $p_4$  in Theorem 3.1 as follow

$$p = |\rho|(e-1) + |\tau|, \quad p_1 = p_2 = \frac{|\rho|e}{2|1 - \rho - \tau|}, \quad p_3 = \frac{|\rho|(e-1)}{2|1 - \rho - \tau|}, \quad p_4 = \frac{|\rho|(e-1)}{|1 - \rho - \tau|}.$$

Using (3.6), we have

$$r = \frac{2|1 - \rho - \tau|}{|\rho|(2e-1)(|\rho|(e-1) + |\tau| + 2)}$$

and

$$pr = \frac{2|1 - \rho - \tau| (|\rho|(e-1) + |\tau|)}{|\rho|(2e-1)(|\rho|(e-1) + |\tau| + 2)}.$$

We can choose suitable values of  $\rho$  and  $\tau$  such that condition (3.5) is satisfied. For example, let  $\rho = 0.4$  and  $\tau = -0.05$ , then we have that  $pr \approx 0.197316753 < 1$  and all conditions in Theorem 3.1 are satisfied. Hence, Theorem 3.1 applies for this example.

Note that for any  $x, y, u, v \in D$ , we have

$$\begin{aligned}
& |A_\star^{-1}([x, y; F] - [u, v; F])| \\
&= |A_\star^{-1} \int_0^1 (F'(tx + (1-t)y) - F'(tu + (1-t)v)) dt| \\
&= |A_\star^{-1} \int_0^1 \int_0^1 F''(\theta(tx + (1-t)y) + (1-\theta)(tu + (1-t)v)) (tx + (1-t)y \\
&\quad - tu - (1-t)v) d\theta dt|
\end{aligned}$$

$$\begin{aligned}
&= |A_{\star}^{-1} \int_0^1 \int_0^1 (\rho e^{\theta(tx+(1-t)y)+(1-\theta)(tu+(1-t)v)}(tx + (1-t)y \\
&\quad - tu - (1-t)v))d\theta dt| \\
&\leq \int_0^1 e|t(x-u) + (1-t)(y-v)|dt \\
&\leq \frac{|\rho|e}{2|1-\rho-\tau|}(|x-u| + |y-v|).
\end{aligned}$$

Then, if we replace conditions (3.3) and (3.4) by the classical condition for divided difference of order one

$$\|A_{\star}^{-1}([x, y; F] - [z, w; F])\| \leq l(\|x-z\| + \|y-w\|),$$

we can set

$$l = \frac{|\rho|e}{2|1-\rho-\tau|}.$$

That is to say, we obtain the radius  $r'$  of convergence ball for the Steffensen method as follows

$$r' = \frac{1}{(2p+3)l} = \frac{2|1-\rho-\tau|}{|\rho|e(2|\rho|(e-1) + 2|\tau| + 3)}.$$

Hence, we only get smaller radius of convergence ball for the Steffensen method provided that

$$|\rho|(e-1) + |\tau| + 2 - e > 0,$$

since

$$\frac{r'}{r} = \frac{(2e-1)(|\rho|(e-1) + |\tau| + 2)}{e(2|\rho|(e-1) + 2|\tau| + 3)} < 1 \Leftrightarrow |\rho|(e-1) + |\tau| + 2 - e > 0.$$

In the next, suppose  $\rho = 0.4$ ,  $\tau = -0.05$  and  $x_0 = 0.2$ , we will verify that all conditions of Theorem 2.4 hold. It is obvious that  $(C_1)$  and  $(C_2)$  are satisfied. We also have  $A_0 \approx 0.589938812 \neq 0$ , and  $(C_3)$  is true for  $a_0 \approx 0.589938812$  and  $a \approx 1.69509105$ . It is easy to see that  $(C_4)$ - $(C_6)$  are satisfied if we set  $b \approx 0.410061188$ ,  $\eta \approx 0.205849987$  and  $c \approx 0.121438897$ . Noting that  $F'(x) = 0.4e^x - 0.05$  and  $F''(x) = 0.4e^x$ , we deduce that  $F'(x)$  increases monotonically in  $D = [-1, 1]$  and it has the biggest value  $0.4e - 0.05$  in  $[-1, 1]$ . Then,  $(C_7)$  is true for  $d = 0.4e - 0.5 \approx 1.037312731$ . In view of

$$\begin{aligned}
&|[F(x), x; F] - [F(x_0), x_0; F]| \\
&= \left| \int_0^1 (F'(tF(x) + (1-t)x) - F'(tF(x_0) + (1-t)x_0))dt \right| \\
&= \left| \int_0^1 \int_0^1 F''(\theta(tF(x) + (1-t)x) + (1-\theta)(tF(x_0) + (1-t)x_0)) \right. \\
&\quad \left. (t(F(x) - F(x_0)) + (1-t)(x - x_0))d\theta dt \right| \\
&= \left| \int_0^1 \int_0^1 0.4e^{\theta(tF(x)+(1-t)x)+(1-\theta)(tF(x_0)+(1-t)x_0)} \right.
\end{aligned}$$

$$\begin{aligned} & |(t(F(x) - F(x_0) + (1-t)(x - x_0)))d\theta dt| \\ & \leq 0.2e(|F(x) - F(x_0)| + |x - x_0|) \quad \text{for any } x \in D \end{aligned}$$

and

$$\begin{aligned} & |[x, y; F] - [u, y; F]| \\ & = \left| \int_0^1 (F'(tx + (1-t)y) - F'(tu + (1-t)y))dt \right| \\ & = \left| \int_0^1 \int_0^1 F''(\theta(tx + (1-t)y) + (1-\theta)(tu + (1-t)y))(tx - tu)d\theta dt \right| \\ & = \left| \int_0^1 \int_0^1 0.4e^{\theta(tx+(1-t)y)+(1-\theta)(tu+(1-t)y)}t(x-u)d\theta dt \right| \\ & \leq 0.2e|x - u| \quad \text{for any } x, y, u \in D, \end{aligned}$$

we deduce that  $(C_8)$  and  $(C_9)$  are true if we set  $K_1 = 0.2e$ ,  $K_2 = 0$ ,  $K_3 = 0$ ,  $K_4 = 0.2e$ ,  $M_1 = 0.2e$ ,  $M_2 = 0$  and  $M_3 = 0$ . Using (2.1) and the definition of  $q$ , we obtain that

$$\begin{aligned} \alpha & \approx 1.8775, \quad \beta = 0, \quad \gamma \approx 1.1076, \quad \delta \approx 0.5899, \quad \lambda \approx 0.6022, \\ \mu & \approx 0.3207, \quad \xi_0 \approx 0.5437, \quad \xi \approx 1.0207, \quad \xi_1 \approx 1.8775, \quad q \approx 0.6449. \end{aligned}$$

Then, we have that  $\xi_1\eta \approx 0.3865 < 1$ ,  $\beta < 1$  and  $(\xi_0 + \xi_1q)\eta \approx 0.3612 < q$ . That is to say, all conditions in Lemma 2.1 hold, so the sequence  $\{t_n\}$  given by (2.6) is well defined and converges to  $t^*$ . By a simple computation, we get  $t^* \approx 0.2622$ , and then we deduce that  $U_1 = \overline{U}(x_0, dt^* + c) \approx [-0.1934, 0.5934] \subseteq D$ , which means  $(C_{10})$  is satisfied. Hence, all conditions in Theorem 2.4 are satisfied, and thus all conclusions of this theorem are true.

**Example 4.2.** Let  $X = D = \overline{U}(0, \frac{\pi}{2})$ , and define  $F$  on  $D$  by

$$F(x) = \rho \sin x + \tau x,$$

where  $\rho, \tau$  are given parameters with  $\rho + \tau \neq 1$ . Then,  $x^* = 0$  is a fixed point of  $F$ .

Note that for any  $x, y \in D$  we have in turn that

$$|F(x) - F(x^*)| = |\rho \sin x + \tau x| \leq (|\rho| + |\tau|)|x|,$$

$$\begin{aligned} & |[x, x^*; F] - [y, x; F]| \\ & = \left| \int_0^1 (F'(tx + (1-t)x^*) - F'(ty + (1-t)x))dt \right| \\ & = \left| \int_0^1 \int_0^1 F''(\theta(tx) + (1-\theta)(ty + (1-t)x))(t(x - y) - (1-t)x)d\theta dt \right| \\ & = \left| \int_0^1 \int_0^1 \rho \sin(\theta(tx) + (1-\theta)(ty + (1-t)x))(t(x - y) - (1-t)x)d\theta dt \right| \end{aligned}$$

$$\leq \int_0^1 |\rho|(t|x-y| + (1-t)|x|)dt = \frac{1}{2}|\rho|(|x-y| + |x|)$$

and

$$\begin{aligned} & |[F(x), x; F] - F'(x^*)| \\ &= \left| \int_0^1 (F'(tF(x) + (1-t)x) - F'(0))dt \right| \\ &= \left| \int_0^1 \int_0^1 F''(\theta(tF(x) + (1-t)x))(tF(x) + (1-t)x)d\theta dt \right| \\ &= \left| \int_0^1 \int_0^1 \rho \sin(\theta(tF(x) + (1-t)x))(tF(x) + (1-t)x)d\theta dt \right| \\ &\leq \frac{1}{2}|\rho|(|F(x) - x| + 2|x|). \end{aligned}$$

We also have

$$A_* = 1 - \rho - \tau,$$

then, we can choose  $p, p_1, p_2, p_3$  and  $p_4$  in Theorem 3.1 as follow

$$p = |\rho| + |\tau|, \quad p_1 = p_2 = \frac{|\rho|}{2|1 - \rho - \tau|}, \quad p_3 = \frac{|\rho|}{2|1 - \rho - \tau|}, \quad p_4 = \frac{|\rho|}{|1 - \rho - \tau|}.$$

Using (3.6), we have

$$r = \frac{|1 - \rho - \tau|}{|\rho|(|\rho| + |\tau| + 2)}$$

and

$$pr = \frac{|1 - \rho - \tau|(|\rho| + |\tau|)}{|\rho|(|\rho| + |\tau| + 2)}.$$

We can choose suitable values of  $\rho$  and  $\tau$  such that condition (3.5) is satisfied. For example, let  $\rho = 0.5$  and  $\tau = 0$ , then we have that  $pr = 0.2 < 1$  and all conditions in Theorem 3.1 are satisfied. Hence, Theorem 3.1 applies for this example.

In the next, suppose  $\rho = 0.5$ ,  $\tau = 0$  and  $x_0 = 0.39$ , we will verify that all conditions of Theorem 2.4 hold. It is obvious that  $(C_1)$  and  $(C_2)$  are satisfied. We also have  $A_0 \approx 0.521682172 \neq 0$ , and  $(C_3)$  is true for  $a_0 \approx 0.521682172$  and  $a \approx 1.916875931$ . It is easy to see that  $(C_4)$ - $(C_6)$  are satisfied if we set  $b \approx 0.478317828$ ,  $\eta \approx 0.383194602$  and  $c \approx 0.199905792$ . Noting that  $F'(x) = 0.5 \cos x$ , we deduce easily that  $(C_7)$  is true for  $d = 0.5$ . In view of

$$\begin{aligned} & |[F(x), x; F] - [F(x_0), x_0; F]| \\ &= \left| \int_0^1 (F'(tF(x) + (1-t)x) - F'(tF(x_0) + (1-t)x_0))dt \right| \\ &= \left| \int_0^1 \int_0^1 F''(\theta(tF(x) + (1-t)x) + (1-\theta)(tF(x_0) + (1-t)x_0)) \right. \\ &\quad \left. (t(F(x) - F(x_0)) + (1-t)(x - x_0))d\theta dt \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \int_0^1 \int_0^1 0.5 \sin(\theta(tF(x) + (1-t)x) + (1-\theta)(tF(x_0) + (1-t)x_0)) \right. \\
&\quad \left. (t(F(x) - F(x_0)) + (1-t)(x - x_0)) d\theta dt \right| \\
&\leq 0.25(|F(x) - F(x_0)| + |x - x_0|) \quad \text{for any } x \in D
\end{aligned}$$

and

$$\begin{aligned}
&|[x, y; F] - [u, y; F]| \\
&= \left| \int_0^1 (F'(tx + (1-t)y) - F'(tu + (1-t)y)) dt \right| \\
&= \left| \int_0^1 \int_0^1 F''(\theta(tx + (1-t)y) + (1-\theta)(tu + (1-t)y))(tx - tu) d\theta dt \right| \\
&= \left| \int_0^1 \int_0^1 0.5 \sin(\theta(tx + (1-t)y) + (1-\theta)(tu + (1-t)y)) t(x - u) d\theta dt \right| \\
&\leq 0.25|x - u| \quad \text{for any } x, y, u \in D,
\end{aligned}$$

we deduce that  $(C_8)$  and  $(C_9)$  are true if we set  $K_1 = 0.25$ ,  $K_2 = 0$ ,  $K_3 = 0$ ,  $K_4 = 0.25$ ,  $M_1 = 0.25$ ,  $M_2 = 0$  and  $M_3 = 0$ . Using (2.1) and the definition of  $q$ , we obtain that

$$\alpha \approx 0.7188, \quad \beta = 0, \quad \gamma = 0.375, \quad \delta \approx 0.5217, \quad \lambda \approx 0.09375,$$

$$\mu \approx 0.1304, \quad \xi_0 = 0.25, \quad \xi \approx 0.1797, \quad \xi_1 \approx 0.7188, \quad q \approx 0.7773.$$

Then, we have that  $\xi_1 \eta \approx 0.2755 < 1$ ,  $\beta < 1$  and  $(\xi_0 + \xi_1 q) \eta \approx 0.3099 < q$ . That is to say, all conditions in Lemma 2.1 hold, so the sequence  $\{t_n\}$  given by (2.6) is well defined and converges to  $t^*$ . By a simple computation, we get  $t^* \approx 0.4499$ , and then we deduce that  $U_1 = \overline{U}(x_0, dt^* + c) \approx [-0.0348, 0.8148] \subseteq D$ , which means  $(C_{10})$  is satisfied. Hence, all conditions in Theorem 2.4 are satisfied, and thus all conclusions of this theorem are true.

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