J. Korean Math. Soc. **54** (2017), No. 1, pp. 1–16
https://doi.org/10.4134/JKMS.j140710
pISSN: 0304-9914 / eISSN: 2234-3008

PROFITABILITY AND SUSTAINABILITY OF A TOURISM-BASED SOCIAL-ECOLOGICAL DYNAMICAL SYSTEM BY BIFURCATION ANALYSIS

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Abstract. In this paper we study a four dimensional tourism-based social-ecological dynamical system. In fact we analyse tourism profitability, compatibility and sustainability by using bifurcation theory in terms of structural properties of attractors of system. For this purpose first we transformed it into a three dimensional system such that the reduced system is the extended and modified model of the previous three dimensional models suggested for tourism with the same dimension. Then we investigate transcritical, pitchfork and saddle-node bifurcation points of system. And numerically by finding some branches of stable equilibria for system show the profitability of tourism industry. Then by determining the Hopf bifurcation points of system we find a family of stable attractors for that by numerical techniques. Finally we conclude the existence of these stable limit cycles implies profitability and compatibility and then the sustainability of tourism.

1. Introduction

Tourism is an activity done by an individual or a group of individuals, which leads to a motion from a place to another. This industry has increased in recent decades and has become one of the main sources of income in many countries. Indeed it is an opportunity to promote social and economic development, but it could degrade natural non renewable resources. This shows the potentially paradoxical character of tourism, see [3, 6, 7]. In addition, using mathematical models in all sciences may help to analyse a system and to study the effects of different components, and to make predictions about its behaviour.

In [3] a detailed analysis of the three dimensional tourism-based social-ecological model was performed and authors achieved remarkable results. However as they recognized, the three dimensional model can not describe all social, cultural and political aspects of tourism dynamics. Furthermore, the model

Received November 14, 2014; Revised December 27, 2015.
2010 *Mathematics Subject Classification.* 34D45, 34K18.
Key words and phrases. tourism-based social-ecological dynamical systems, profitability, sustainability, attractors, bifurcation theory.

should be modified to have some more realistic features of tourism. Then in [6] to preserve the low dimensionality of the model just two main tourists, i.e., mass tourists and eco tourists were considered and therefore a four dimensional dynamical system was investigated. Such development is according to the recognition of the related role played in specific touristic sites by the different topologies of tourists on dynamical systems, which are distinguished by tourists' perception and behaviour, see [2].

On the other hand, profitability, compatibility and sustainability are very significant concepts in the field of resource management. Profitability means that the tourism industry can sustain indefinitely. Furthermore, the environmental impact of a policy in a given area is compatible when it avoids complete degradation of the environment. And sustainability means a chance of maintaining the tourism industry indefinitely without jeopardizing the environment. In fact sustainability refers to the possibility of keeping alive all meaning full social and natural compartments of an evolving system forever. For more information see [3]. In [3] profitability, compatibility and sustainability are interpreted in terms of structural properties of the attractors of system. Moreover, in [3] bifurcations of a three dimensional tourism-based social-ecological dynamical system were determined numerically. And in [6, 7] Hopf bifurcation of the following system just for one value of parameter ω was found. Furthermore, in [9] a region for the following system of different values of parameter ω for which a special type of Hopf bifurcation occurs was obtained.

$$
(1.1) \quad \begin{cases} \frac{dT_1}{dt} = T_1 \left(\frac{\mu_{1E} E^2}{E^2 + \varphi_{1E}^2} - \beta_1 \frac{C}{T_1 + T_2 + 1} - \alpha_1 T_1 - \alpha_1^* T2 - \gamma_1^* T_1 T_2 - \omega \right), \\ \frac{dT_2}{dt} = T_2 \big(\frac{\mu_{2E} E}{E + \varphi_{2E}} + \mu_{2C} \frac{C}{C + \varphi_{2C}(T_1 + T_2 + 1)} - \alpha_2 T_2 - \alpha_2^* T1 - \gamma_2^* T_1 T_2 - \omega \big), \\ \frac{dE}{dt} = \frac{rE^2}{k} - rE - E(\beta C + \gamma_2 T_2) + \gamma_1 T_1 E, \\ \frac{dC}{dt} = -\delta C + \epsilon_1 T_1 + \epsilon_2 T_2. \end{cases}
$$

In system (1.1) eco-tourists T_1 and mass-tourists T_2 are two state variables of the system (1.1) with environment E and capital C. In addition, $T_1(t), T_2(t), E(t), C(t) \geq 0$, $t \geq 0$ and all parameters are non negative, except ω which is negative if tourists like crowding. For more details about system (1.1) see [6, 7].

In this paper to study system (1.1) we first transform it into a three dimensional system. Such that in our presented model unlike the suggested model in [3], two different types of tourists, mass-tourists and eco-tourists, are considered. In fact, this model is the extended and modified model of the mentioned model in [3] with the same dimension. Then we analytically find transcritical, pitchfork and saddle-node bifurcations for the reduced system by means of bifurcation theory. And by using the notions in [3] we interpret the concepts profitability, compatibility and sustainability in terms of structural properties of attractors of our three dimensional tourism-based social-ecological dynamical system. Then we numerically find some branches of stable equilibria for our reduced system and show that the existence of these branches can result the profitability of tourism industry. In addition, we determine all Hopf bifurcation values of parameter ω. And to prove the tourism sustainability we find a family of stable attractors by numerical techniques. In addition, we show the existence of these stable limit cycles implies profitability and compatibility and then the sustainability of tourism.

The paper organizes as follows: Section two gives some mathematical concepts which are going to be used in other sections. The third section is devoted to analyse transcritical, pitchfork, saddle-node bifurcations and then the profitability of tourism industry is showed. And in section four all Hopf bifurcation values of parameter ω are determined and sustainability of tourism is concluded.

2. Preliminaries

In this section we shall state some mathematical concepts and basic theorems.

In dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behaviour. In general, at a bifurcation point, the local stability properties of equilibria, periodic orbits or other invariant sets change.

A saddle-node bifurcation or tangent bifurcation is a collision and disappearance of two branches of equilibria. One of these branches is stable, the other is unstable.

In a transcritical bifurcation, two families of fixed points collide and exchange their stability properties. The family that was stable before the bifurcation is unstable after it. The other family of fixed points goes from being unstable to being stable. But, in pitchfork bifurcation one family of fixed points transfers its stability properties to two families after or before the bifurcation point. If this occurs after the bifurcation point, then pitchfork bifurcation is called supercritical. Similarly, a pitchfork bifurcation is called subcritical if the nontrivial fixed points occur for values of the parameter lower than the bifurcation value. See [1, 4, 10].

Consider the following system

$$\dot{x} = f(x, \mu),$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, and smooth function f. Assume that at $\mu = \mu_0$, $x = x_0$, the above system has an equilibrium for which there exists a simple zero eigenvalue. The following theorem states the sufficient conditions for existence of saddle-node, transcritical and pitchfork bifurcations.

Theorem 2.1. *Let $\dot{x} = f(x, \mu)$ be a system of differential equations in $\mathbb{R}^n$ depending on the single parameter μ. When $\mu = \mu_0$, assume that there is an equilibrium p for which the following hypotheses are satisfied:*

(1) $D_x f_{\mu_0}(p)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w. $D_x f_{\mu_0}(p)$ has k eigenvalues with negative real parts and $(n-k-1)$ eigenvalues with positive real parts (counting multiplicity).

(2) $w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) \neq 0$,

(3) $w^T[D_x^2 f(p, \mu_0)(v, v)] \neq 0$.

Then the system $\dot{x} = f(x, \mu)$ experiences a saddle-node bifurcation at the equilibrium point p as the parameter μ varies through the bifurcation value $\mu = \mu_0$.

Moreover, if the condition (2) are changed to

(2′) $w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) = 0$, $w^T[\frac{\partial^2 f}{\partial \mu \partial x}(p, \mu_0)v] \neq 0$,

the system has a transcritical bifurcation at the equilibrium point p. And if the conditions (2), (3) are changed to

(2′) $w^T(\frac{\partial f}{\partial \mu}(p, \mu_0)) = 0$, $w^T[\frac{\partial^2 f}{\partial \mu \partial x}(p, \mu_0)v] \neq 0$,

(3′) $w^T[D_x^2 f(p, \mu_0)(v, v)] = 0, w^T[D_x^3 f(p, \mu_0)(v, v, v)] \neq 0$,

the system has a pitchfork bifurcation at p. See [4, 10].

Proof. See [4]. □

A Hopf or Poincare-Andronov-Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues of linearization around the fixed point cross the imaginary axis of the complex plane. The importance of the Hopf bifurcation is that it provides a means of proving the existence of the local birth or death of a periodic solution from a fixed point as a parameter crosses a critical value. See [1, 4, 10].

Now, consider a system $\dot{x} = f(x, \mu)$ with a parameter value μ_0 and the equilibrium $p(\mu_0)$ at which $D_x f_{\mu_0}(p)$ has a simple pair of pure imaginary eigenvalues $\pm i\omega, \omega > 0$, and no other eigenvalues with zero real part. The implicit function theorem guarantees (since $D_x f_{\mu_0}(p)$ is invertible) that for each μ near μ_0 there will be an equilibrium $p(\mu)$ near $p(\mu_0)$ which varies smoothly with μ. The dimension of stable and unstable manifolds of $p(\mu)$ change if the eigenvalues of $Df(p(\mu))$ cross the imaginary axis at μ_0. We mention the following theorem which states the necessary conditions for existence of simple Hopf bifurcation.

Theorem 2.2. *Suppose that the system $\dot{x} = f(x, \mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, has an equilibrium (x_0, μ_0) at which the following properties are satisfied:*

(1) $D_x f_{\mu_0}(x_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. Therefore, there exists a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu_0) = x_0$. The eigenvalues $\lambda(\mu), \bar{\lambda}(\mu)$ of $D_x f_{\mu_0}(x(\mu))$ which are imaginary at $\mu = \mu_0$ vary smoothly with μ. Furthermore, if,

(2) $\frac{d}{d\mu}(Re\lambda(\mu)) \mid_{\mu=\mu_0} = d \neq 0$,

then simple Hopf bifurcation will occur (see [4, 5, 8]).

Proof. See [4]. □

3. Transcritical, pitchfork and saddle-node bifurcations of system (3.1)

Consider the four-dimensional nonlinear tourism-based social-ecological dynamical system (1.1). Let $T_1 + T_2 + 1 = T$ and assume that $\gamma_1^* = \gamma_2^* = \gamma$, $\alpha_1 = \alpha_2 = \alpha_1^* = \alpha_2^* = \alpha$ and $\epsilon_1 = \epsilon_2 = \epsilon$. Moreover we suppose that $T_1 = lT_2$, where l is a nonnegative constant. In this case $T_2 = \frac{1}{l+1}(T-1)$ and $T_1 = \frac{l}{l+1}(T-1)$. Therefore, we can change system (1.1) in the following three-dimensional form

$$\begin{cases} \frac{dT}{dt} = (T-1)[\frac{l\frac{\mu_{1E}E^2}{E^2+\varphi_{1E}^2}+\frac{\mu_{2E}E}{E+\varphi_{2E}}}{l+1} - \frac{l\beta_1 C}{(l+1)T} + \frac{\mu_{2C}\cdot C}{(l+1)(C+\varphi_{2C}T)} \\ \qquad -\alpha(T-1) - \frac{\gamma l}{(l+1)^2}(T-1)^2 - \omega], \\ \frac{dE}{dt} = \frac{rE^2}{k} - rE - \beta CE + \frac{\gamma_1 l - \gamma_2}{l+1}(T-1)E, \\ \frac{dC}{dt} = -\delta C + \epsilon(T-1). \end{cases} \tag{3.1}$$

This model is an extension of the three-dimensional system mentioned in [3] with the same dimension. Such that in our model two different types of tourists, mass-tourists and eco-tourists are considered. In fact system (3.1) is a modified model of the suggested model in [3] such that it has some more realistic features of tourism.

Furthermore, in system (3.1) since $T_1(t), T_2(t) \geq 0$ for $t \geq 0$, we have $T(t) \geq 1$ for all $t \geq 0$. Moreover, as it mentioned in [6, 7] parameter ω can be considered as a measure of competition among all alternative tourist sites in the region. In this section we consider parameter ω as a bifurcation parameter and investigate transcritical and pitchfork bifurcation points for system (3.1). By determining bifurcation points we can determine the profitability of tourism industry.

3.1. Transcritical and pitchfork bifurcations

Consider the system (3.1) with the following Jacobian matrix at an equilibrium point

$$J = \begin{pmatrix} a_{11}-\omega & a_{12} & a_{13} \\ \frac{\gamma_1 l-\gamma_2}{l+1}E & \frac{2rE}{k}-r-\beta C+\frac{\gamma_1 l-\gamma_2}{l+1}(T-1) & -\beta E \\ \epsilon & 0 & -\delta \end{pmatrix}, \tag{3.2}$$

where

$$a_{11} = \frac{l\frac{\mu_{1E}E^2}{E^2+\varphi_{1E}^2}+\frac{\mu_{2E}E}{E+\varphi_{2E}}}{l+1} - 2\alpha(T-1) - \frac{3l\gamma(T-1)^2}{(l+1)^2} - \frac{\beta_1 Cl}{(l+1)T^2} + \frac{C\cdot\mu_{2C}}{l+1}[\frac{C+\varphi_{2C}}{(C+\varphi_{2C}\cdot T)^2}],$$

$$a_{12} = \frac{T-1}{l+1}[l(\frac{2E\cdot\mu_{1E}\cdot\varphi_{1E}^2}{(E^2+\varphi_{1E}^2)^2}) + (\frac{\mu_{2E}\cdot\varphi_{2E}}{(E+\varphi_{2E})^2})],$$

$$a_{13} = \frac{1-T}{l+1}[\frac{\beta_1 l}{T} - \frac{T\cdot\varphi_{2C}\cdot\mu_{2C}}{(C+T\cdot\varphi_{2C})^2}].$$

The equilibrium $P_0 = (1, k, 0)$ is a fixed point where ecosystem quality E is at its carrying capacity, with no tourists ($T = 1$ implies that $T_1 = T_2 = 0$, since $T_1, T_2 \geq 0$) and no accommodation and entertainment facilities ($C = 0$), see [6]. The eigenvalues of (3.2) at equilibrium $P_0 = (1, k, 0)$ are:

$$\lambda_1 = h - \omega,\ \lambda_2 = r,\ \lambda_3 = -\delta,$$

where $h = \frac{l(\frac{\mu_{1E}\cdot k^2}{k^2+\varphi_{1E}^2})+(\frac{\mu_{2E}\cdot k}{k+\varphi_{2E}})}{l+1}$.

The parameter ω which is considered as loss of attractiveness of the tourist location due to environmental degradation in terms of ecosystem goods and services identifies as a bifurcation parameter. Suppose that all the other parameters are fixed and positive. Then we state the following theorem.

Theorem 3.1. *Consider the three-dimensional nonlinear tourism-based social-ecological dynamical system* (3.1)*,*

(i) *If* $N = \frac{-2}{l+1}[\alpha(l+1) - \Gamma(l(\frac{2k\mu_{1E}\cdot\varphi_{1E}^2}{(k^2+\varphi_{1E}^2)^2}) + (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^2})) + (\frac{\beta_1\cdot l\cdot\varphi_{2C}-\mu_{2C}}{\delta\cdot\varphi_{2C}})\epsilon] \neq 0$, *then* $\omega = h$ *is a transcritical bifurcation value for system* (3.1) *at the equilibrium point* $(1, k, 0)$*. Note that* $\Gamma = \frac{\delta(\gamma_2-\gamma_1 l)k+\beta k\epsilon(l+1)}{\delta r(l+1)}$.

(ii) *If* $N = 0$ *and* $M = \frac{6}{l+1}[\frac{-\gamma l}{(l+1)} + (\beta_1\cdot l - \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} + \Gamma^2(l\cdot\mu_{1E}\cdot\varphi_{1E}^2(\frac{\varphi_{1E}^2-3k^2}{(k^2+\varphi_{1E}^2)^3}) - (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^3})) - \frac{\epsilon^2\cdot\mu_{2C}}{\delta^2\cdot\varphi_{2C}^2}] \neq 0$, *then* $\omega = h$ *is a pitchfork bifurcation value for system* (3.1) *at the equilibrium point* $(1, k, 0)$*. Moreover,* $(1, k, 0, h)$ *is a supercritical pitchfork bifurcation point, if* $M < 0$ *and it is subcritical if* $M > 0$*.*

Proof. (i) If $\omega = h$, then the corresponding right and left eigenvectors of $\lambda_1 = 0$ are $v = \left(1, \Gamma, \frac{\epsilon}{\delta}\right)^T$, and $w = (1, 0, 0)^T$ respectively. Now we check the conditions of Theorem 2.1:

1) $w^T \cdot \frac{\partial f}{\partial\omega} |_{(1,k,0,h)} = 0$,

2) $w^T[\frac{\partial^2 f}{\partial x\partial\omega} |_{(1,k,0,h)} \cdot v] = -1 \neq 0$ and since

$$\begin{aligned}
&D_x^2 f(1, k, 0, h)(v, v)\\
&= \begin{pmatrix} \sum_{j_1,j_2=1}^3 \frac{\partial^2 f_1(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}} v_{j_1} v_{j_2} \\ \sum_{j_1,j_2=1}^3 \frac{\partial^2 f_2(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}} v_{j_1} v_{j_2} \\ \sum_{j_1,j_2=1}^3 \frac{\partial^2 f_3(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}} v_{j_1} v_{j_2} \end{pmatrix}\\
&= \begin{pmatrix} \frac{-2}{l+1}[\alpha(l+1) - \Gamma(l(\frac{2k\mu_{1E}\cdot\varphi_{1E}^2}{(k^2+\varphi_{1E}^2)^2}) + (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^2})) + (\frac{\beta_1\cdot l\cdot\varphi_{2C}-\mu_{2C}}{\delta\cdot\varphi_{2C}})\epsilon] \\ 2\Gamma(\frac{r}{k}\Gamma - \frac{\beta\epsilon}{\delta} + \frac{\gamma_1 l-\gamma_2}{l+1}) \\ 0 \end{pmatrix},
\end{aligned}$$

where $x = (x_1, x_2, x_3) = (T, E, C)$ and $v = (v_1, v_2, v_3)$, so

3) $w^T[\ D_x^2 f(1, k, 0, h)(v, v)] = \frac{-2}{l+1}[\alpha(l+1) - \Gamma(l(\frac{2k\mu_{1E}\cdot\varphi_{1E}^2}{(k^2+\varphi_{1E}^2)^2}) + (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^2})) + (\frac{\beta_1\cdot l\cdot\varphi_{2C}-\mu_{2C}}{\delta\cdot\varphi_{2C}})\epsilon]$.

Hence if $N \neq 0$, then by Theorem 2.1, the point $(1, k, 0, h)$ is a transcritical bifurcation point which completes the proof of (2).

(ii) Since we have

$$D_x^3 f(1,k,0,h)(v,v,v)$$
$$= \begin{pmatrix} \sum_{j_1,j_2,j_3=1}^{3} \frac{\partial^3 f_1(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}} v_{j_1}v_{j_2}v_{j_3} \\ \sum_{j_1,j_2,j_3=1}^{3} \frac{\partial^3 f_2(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}} v_{j_1}v_{j_2}v_{j_3} \\ \sum_{j_1,j_2,j_3=1}^{3} \frac{\partial^3 f_3(1,k,0,h)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}} v_{j_1}v_{j_2}v_{j_3} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{6}{l+1}[\frac{-\gamma l}{(l+1)} + (\beta_1 \cdot l - \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} + \Gamma^2(l \cdot \mu_{1E} \cdot \varphi_{1E}^2(\frac{\varphi_{1E}^2 - 3k^2}{(k^2+\varphi_{1E}^2)^3}) - (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^3})) - \frac{\epsilon^2\cdot\mu_{2C}}{\delta^2\cdot\varphi_{2C}^2}] \\ 0 \\ 0 \end{pmatrix},$$

in which $x = (x_1, x_2, x_3) = (T, E, C)$ and $v = (v_1, v_2, v_3)$, thus

$$w^T[\ D_x^3 f(1,k,0,h)(v,v,v)]$$
$$= \frac{6}{l+1}[(\beta_1 \cdot l + \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} + \Gamma^2(l \cdot \mu_{1E} \cdot \varphi_{1E}^2(\frac{\varphi_{1E}^2 - 3k^2}{(k^2+\varphi_{1E}^2)^3}) - (\frac{\mu_{2E}\cdot\varphi_{2E}}{(k+\varphi_{2E})^3})) + \frac{\epsilon^2\cdot\mu_{2C}}{\delta^2\cdot\varphi_{2C}^2}].$$

Therefore, if $N = 0$ and $M \neq 0$, then by Theorem 2.1, the point $(1, k, 0, h)$ is a pitchfork bifurcation point for (3.1). □

Now, we investigate the equilibrium point $O = (1, 0, 0)$ which is called the extinction equilibrium for (3.1) at which there is no tourists ($T = 1$ implies that $T_1 = T_2 = 0$, since $T_1, T_2 \geq 0$), no ecosystem quality, no accommodation and entertainment facilities.

The eigenvalues of (3.2) at $O = (1, 0, 0)$ are: $\lambda_1 = -\omega$, $\lambda_2 = -r$, $\lambda_3 = -\delta$.

We again consider parameter ω as a bifurcation parameter and suppose that all the other parameters are fixed and positive.

Theorem 3.2. *Consider the system* (3.1)*,*

(i) *If* $N' = \frac{-2}{l+1}[\alpha(l+1) + (\frac{\beta_1 \cdot l \cdot \varphi_{2C} - \mu_{2C}}{\delta \cdot \varphi_{2C}})\epsilon] \neq 0$*, then* $\omega = 0$ *is a transcritical bifurcation value for system* (3.1) *at the equilibrium point* $(1, 0, 0)$*.*

(ii) *If* $N' = 0$ *and* $M' = \frac{6}{l+1}[(\beta_1 \cdot l - \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} - \frac{\epsilon^2 \cdot \mu_{2C}}{\delta^2 \cdot \varphi_{2C}^2} - \frac{\gamma l}{(l+1)}] \neq 0$*, then* $\omega = 0$ *is a pitchfork bifurcation value for system* (3.1) *at the equilibrium point* $(1, 0, 0)$*. Moreover,* $(1, 0, 0, 0)$ *is a supercritical pitchfork bifurcation point, if* $M' < 0$ *and it is subcritical if* $M' > 0$*.*

Proof. (i) If $\omega = 0$, then $\lambda_1 = 0$ and corresponding right and left eigenvalues are: $v = \left(1, 0, \frac{\epsilon}{\delta}\right)^T$, and $w = (1, 0, 0)^T$. Furthermore, the conditions of Theorem 2.1, are:

1) $w^T \cdot \frac{\partial f}{\partial \omega} \mid_{(1,0,0,0)} = 0$,

2) $w^T[\frac{\partial^2 f}{\partial x \partial \omega} \mid_{(1,0,0,0)} \cdot v] = -1 \neq 0$ and since

$$D_x^2 f(1,0,0,0)(v,v) = \begin{pmatrix} \frac{-2}{l+1}[\alpha(l+1) + (\frac{\beta_1 \cdot l \cdot \varphi_{2C} - \mu_{2C}}{\delta \cdot \varphi_{2C}})\epsilon] \\ 0 \\ 0 \end{pmatrix},$$

where $x = (x_1, x_2, x_3) = (T, E, C)$ and $v = (v_1, v_2, v_3)$, so

3) $w^T[\ D_x^2 f(1,0,0,0)(v,v)] = \frac{-2}{l+1}[\alpha(l+1) + (\frac{\beta_1 \cdot l \cdot \varphi_{2C} - \mu_{2C}}{\delta \cdot \varphi_{2C}})\epsilon]$.

Hence if $N' \neq 0$, then by Theorem 2.1, the point (1,0,0,0) is a transcritical bifurcation point for (3.1).

(ii) Since we have

$$D_x^3 f(1,0,0,0)(v,v,v) = \begin{pmatrix} \frac{6}{l+1}[(\beta_1 \cdot l - \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} - \frac{\epsilon^2 \cdot \mu_{2C}}{\delta^2 \cdot \varphi_{2C}^2} - \frac{\gamma l}{(l+1)}] \\ 0 \\ 0 \end{pmatrix},$$

thus

$$w^T[\ D_x^3 f(1,0,0,0)(v,v,v)] = \frac{6}{l+1}[(\beta_1 \cdot l - \frac{\mu_{2C}}{\varphi_{2C}})\frac{\epsilon}{\delta} - \frac{\epsilon^2 \cdot \mu_{2C}}{\delta^2 \cdot \varphi_{2C}^2} - \frac{\gamma l}{(l+1)}].$$

Therefore, if $N' = 0$ and $M' \neq 0$, then by Theorem 2.1, the point (1,0,0,0) is a pitchfork bifurcation point for (3.1). ☐

Example 3.3. Consider the three-dimensional system (3.1), and let the model's parameters assume the same numerical values as in [6] and [7], i.e., let

$$\begin{aligned} &\beta_1 = 1,\ \alpha = 0.5,\ \gamma = 0.4,\ \delta = 0.5,\ \epsilon = 1.5,\ \varphi_{2C} = 1, \\ &\mu_{1E} = 3.5,\ \mu_{2E} = 2,\ \mu_{2C} = 3,\ \varphi_{1E} = 1.5,\ \varphi_{2E} = 0.5, \\ &r = 0.005,\ \beta = 1,\ k = 5,\ \gamma_1 = 3.65,\ \gamma_2 = 1.5. \end{aligned}$$

These values of parameters are gathered from different research activities on tourism dynamics by using questionnaires, administrated through personal interviews. In addition, these researches are performed in a region located in southern Italy. Further details can be found in [6, 7, 11].

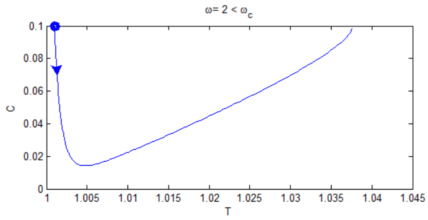
In addition, suppose that $l = \frac{\gamma_2}{\gamma_1} \simeq 0.4109$. In this case $h \simeq 2.223859$, and the point $(1,5,0)$ is an equilibrium point for (3.1). For $\omega_c = 2.223859$ the Jacobian matrix of (3.1) at $(1,5,0)$ has eigenvalues:

$$\lambda_1 = 0,\ \lambda_2 = 0.005,\ \lambda_3 = -0.5.$$

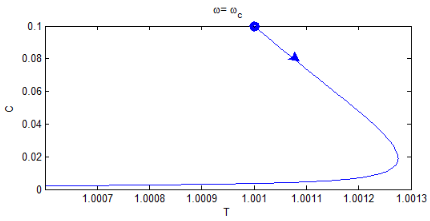
Furthermore $N \simeq 335.9188 > 0$, so by Theorem 3.1, $\omega_c = 2.223859$ is a transcritical bifurcation value for (3.1) at the equilibrium point $(1,5,0)$.

Trajectories of system (3.1) are shown in Figure 1. Here numerical simulations are performed by the aid of computer language Matlab.

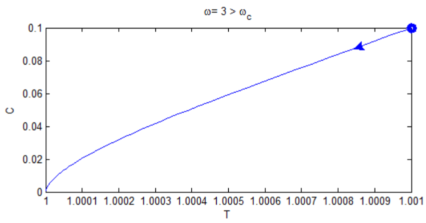
Note 1. By using [3] we can say for system (3.1) the economic impact of a policy applied to a given site is profitable if at least one of the associated attractors of model is characterized by $T(t) \geq 1$ for all t. If attractors characterized by $T(t) > 1$ for all t, the policy is called profitable and safe and if they characterized by the absence of the tourism industry (i.e., $T(t) = 1$) we say that the policy is profitable but risky. Moreover, the policy is called compatible if at least one of the associated attractors has $E(t) > 0$ for all t. Finally a policy is sustainable when one of its associated attractors is characterized by$E(t) > 0$ and $T(t) > 1$ for all t, i.e., when one of its attractors is strictly positive. For



(a)



(b)



(c)

FIGURE 1. The occurring of the transcritical bifurcation for the equilibrium point $(T_0, E_0, C_0) = (1, 5, 0)$ in the phase space TC. Trajectories are drawn with the initial value (1.001, 5.3, 0.1). As it is shown, for $\omega \geq \omega_c$ the trajectory approaches to (T_0, C_0). While for $\omega < \omega_c$, the trajectory moves away from the point (T_0, C_0).

more details see [3]. Notice that since system (3.1) is a third order model, its attractors can be stable equilibria and limit cycles.

Now we give an example to investigate the profitability of system (3.1).

Example 3.4. Consider system (3.1), with the same parameter values as in the previous example. For $\omega_c = 0$ at equilibrium $(1, 0, 0)$ we have $N' \simeq 10.0097 > 0$. So by Theorem 3.2, $\omega_c = 0$ is a transcritical bifurcation value for (3.1) at $(1, 0, 0)$.

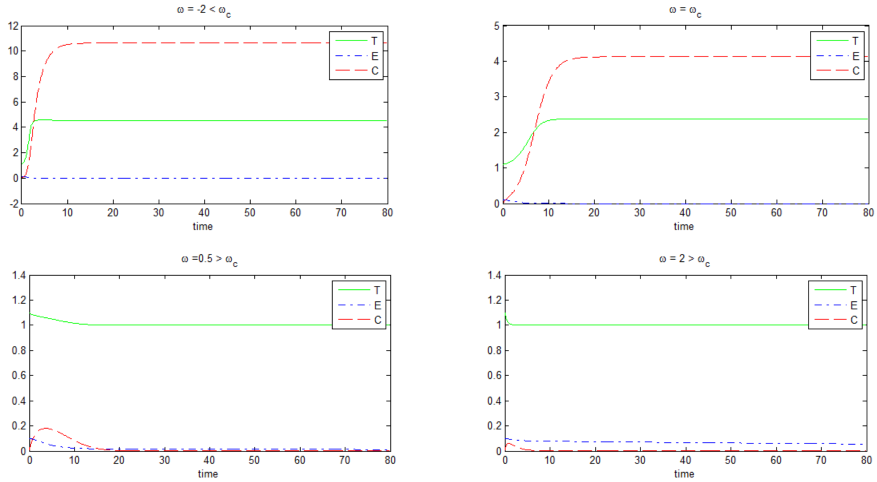


FIGURE 2. time dependent behaviour of the model. Trajectories are drawn with the initial value (1.1, 0.1, 0.01).

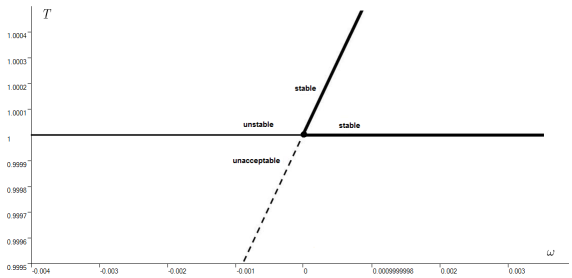


FIGURE 3. Transcritical bifurcation diagram for system (3.1) in the space $T\omega$.

Time dependent behaviour of system (3.1) is illustrated in Figure 2 by means of software Matlab. In addition by using software Oscill, transcritical bifurcation diagram is drawn in Figure 3. Since we have assumed that $T_1, T_2 > 0$, therefore T is equal or greater than 1 and in Figure 3 the branch $T < 1$, is not acceptable. Thus by the above results, in Figure 3 we have two branches: $T = 1$ and $T > 1$. $T = 1$ means the absence of tourism industry, because of an unexpected accidental shock, like a war or epidemics.

Furthermore by Note 1, in Figure 3 the branch $\omega > 0$, and $T > 1$ of stable equilibria of system (3.1) means the tourism industry is profitable and safe. And the brabch $\omega > 0$, $T = 1$ which means the absence of tourism industry implies that the policy is profitable but risky.

3.2. Saddle-node bifurcation

Here we study saddle-node bifurcation for system (3.1) analytically and show that there is a bridge between profitability and bifurcation theory.

Let $\bar{p} = (T, E, C)$ be an equilibrium point for system (3.1) and A be the Jacobian matrix of that at $\bar{p}$. In addition, suppose that for $\omega = \omega_0$, $\lambda = 0$ is a simple root of the characteristic polynomial of A. Hence $\bar{w} = \left(1, -D, \frac{a_{13}+\beta ED}{\delta}\right)^T$ is the left eigenvector corresponding to $\lambda = 0$ and $\bar{v} = \left(1, \frac{(\omega_0 - a_{11}) - a_{13}\frac{\epsilon}{\delta}}{a_{12}}, \frac{\epsilon}{\delta}\right)^T$ is its right eigenvector where $D = \frac{a_{12}}{[\frac{2r}{k}E - r - \beta C + \frac{\gamma_1 l - \gamma_2}{l+1}(T-1)]}$.

Furthermore, we have:

1) $\bar{w}^T(\frac{\partial f}{\partial \omega}|_{(T,E,C,\omega_0)}) = \bar{w}^T \begin{pmatrix} -(T-1) \\ 0 \\ 0 \end{pmatrix} = 1 - T,$

2) $\bar{w}^T[D^2 f(T, E, C, \omega_0)(\bar{v}, \bar{v})] = -2\alpha - 6\gamma\frac{l}{(l+1)^2}(T-1) - \frac{\mu_{2C}}{l+1} \cdot \frac{C(C+\varphi_{2C})2\varphi_{2C}}{(C+\varphi_{2C}T)^3} + \frac{2\beta_1 lC}{(l+1)T^3} + \frac{2}{l+1}(\frac{(\omega_0 - a_{11}) - a_{13}\frac{\epsilon}{\delta}}{a_{12}})[l(\frac{2E\cdot\mu_{1E}\cdot\varphi_{1E}^2}{(E^2+\varphi_{1E}^2)^2}) + (\frac{\mu_{2E}\cdot\varphi_{2E}}{(E+\varphi_{2E})^2}) - D(\gamma_1 l - \gamma_2) + D(l+1)(\frac{\beta\epsilon}{\delta})] + 2\frac{\epsilon}{\delta}[\frac{-\beta_1 l}{(l+1)T^2} + \frac{\mu_{2C}\cdot\varphi_{2C}}{l+1}(\frac{2CT - C + \varphi_{2C}T}{(C+\varphi_{2C}T)^3})] + (\frac{T-1}{l+1}[2l\mu_{1E}\varphi_{1E}^2(\frac{\varphi_{1E}^2 - 3E^2}{(E^2+\varphi_{1E}^2)^3}) - 2\mu_{2E}\cdot\varphi_{2E}(\frac{1}{(E\varphi_{2E})^3})] - \frac{2Dr}{k})(\frac{(\omega_0 - a_{11}) - a_{13}\frac{\epsilon}{\delta}}{a_{12}})^2 + \frac{\epsilon^2}{\delta^2}[\frac{1-T}{l+1} \cdot \frac{2T\varphi_{2C}\mu_{2C}}{(C+\varphi_{2C}T)^3}] = \bar{N}.$

Now we state the sufficient conditions to happen the saddle-node bifurcation for (3.1) in the following theorem.

Theorem 3.5. *Suppose that $\bar{p} = (T, E, C)$ is an equilibrium point for system (3.1). If $T > 1$ and $\bar{N} \neq 0$, then $\omega = \omega_0$ is a saddle-node bifurcation value for (3.1) at $\bar{p}$.*

Proof. By above discussion and Theorem 2.1, $\omega = \omega_0$ is a saddle-node bifurcation value for (3.1) at $\bar{p}$ if $1 - T \neq 0$ and $\bar{N} \neq 0$. Since $T > 0$ it means that $T > 1$ and $\bar{N} \neq 0$. □

Example 3.6. Consider system (3.1) with the same parameter values as in the Example 3.3. For $\omega \simeq 0.01$ the equilibrium point $\bar{p} \simeq (1.116914, 0.01, 0.321)$ satisfies in the conditions of Theorem 3.5, Since for $\bar{p}$ we have $T = 1.116914 > 1$ and $\bar{N} \neq 0$. Thus $\omega_0 \simeq 0.01$ is a saddle-node bifurcation value for (3.1) at $\bar{p}$. For ω_0 bifurcation diagrams of (3.1) at $\bar{p}$ are illustrated in Figures 4 and 5, by the aid of software Oscill. In addition in Figure 4, there is a branch $T > 1$ of stable equilibria for system (3.1) which means the tourism industry is profitable and safe.

4. Hopf bifurcation

From an applied perspective, the importance of the Hopf bifurcation is that it provides a means of proving the existence of the local birth or death of a periodic solution from a fixed point as a parameter crosses a critical value. Here we identify Hopf bifurcation points of (3.1). And we show that the existence of a family of stable attractors implies that tourism industry can maintain for

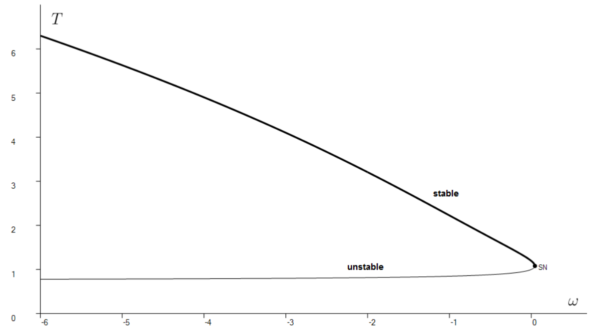


FIGURE 4. Saddle-node bifurcation diagram for system (3.1) in the space $T\omega$.

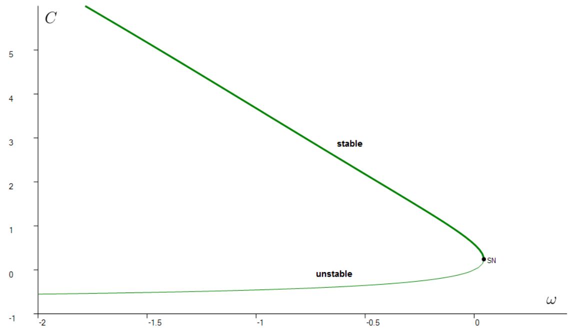


FIGURE 5. Saddle-node bifurcation diagram for system (3.1) in the space $C\omega$.

ever without jeopardizing the environment. That means we show sustainability of system (3.1) by using bifurcation tool.

Let $P^* = (T^*, E^*, C^*)$ be an equilibrium point of system (3.1) and parameter ω be considered as a bifurcation parameter. Moreover, suppose that all the other parameters are fixed and positive. Furthermore, the characteristic polynomial of the Jacobian matrix (3.2) at $P^* = (T^*, E^*, C^*)$ can be written as

$$P(\lambda, \omega) = p_0(\omega) + p_1(\omega)\lambda + p_2(\omega)\lambda^2 + p_3(\omega)\lambda^3, \tag{4.1}$$

in which

$p_3(\omega) = 1,$

$p_2(\omega) = \omega + \delta + r + \beta C^* - a_{11}^* - \frac{2r}{k}E^* - \frac{\gamma_1 l - \gamma_2}{l+1}(T^* - 1),$

$p_1(\omega) = (\omega + r)\delta + (\delta + \omega - a_{11}^*)\beta C^* + (a_{11}^* - \delta - \omega)\frac{2r}{k}E^*$

$$+ (a_{11}^* - \delta - \omega)\tfrac{\gamma_1 l - \gamma_2}{l+1}(T^* - 1) + \omega r - a_{13}^*\epsilon - a_{11}(\delta + r) - a_{12}^*\tfrac{\gamma_1 l - \gamma_2}{l+1}E^*,$$

$$p_0(\omega) = a_{12}^* E^*(\beta\epsilon - \delta\tfrac{\gamma_1 l - \gamma_2}{l+1}) + \tfrac{2r}{k}E^*(a_{13}^*\epsilon + \delta(a_{11}^* - \omega))$$
$$+ (a_{13}^*\epsilon + a_{11}^*\delta - \delta\omega)\tfrac{\gamma_1 l - \gamma_2}{l+1}(T^* - 1) + (\omega - a_{11}^*)\delta r$$
$$+ (\delta(\omega - a_{11}^*) - a_{13}^*\epsilon)\beta C^* - a_{13}^*\epsilon r.$$

Where

$$a_{11}^* = \frac{l\frac{\mu_{1E}(E^*)^2}{(E^*)^2+\varphi_{1E}^2} + \frac{\mu_{2E}E^*}{E^*+\varphi_{2E}}}{l+1} - 2\alpha(T^* - 1) - \tfrac{2l\gamma(T^*-1)}{(l+1)^2} - \tfrac{\beta_1 C^* l}{(l+1)(T^*)^2}$$
$$- \tfrac{C^*\cdot\mu_{2C}}{l+1}[\tfrac{C^*+\varphi_{2C}}{(C^*+\varphi_{2C}\cdot T^*)^2}],$$
$$a_{12}^* = \tfrac{T^*-1}{l+1}[l(\tfrac{2E^*\cdot\mu_{1E}\cdot\varphi_{1E}^2}{((E^*)^2+\varphi_{1E}^2)^2} + (\tfrac{\mu_{2E}\cdot\varphi_{2E}}{(E^*+\varphi_{2E})^2})],$$
$$a_{13}^* = \tfrac{1-T^*}{l+1}[\tfrac{\beta_1 l}{T^*} + \tfrac{T^*\cdot\varphi_{2C}\cdot\mu_{2C}}{(C^*+T^*\cdot\varphi_{2C})^2}].$$

If we assume $p_1(\omega) > 0$ and $p_0(\omega) - p_1(\omega)p_2(\omega) = 0$, then $\lambda = i\theta(\theta \neq 0)$ is a root of (4.1) where $\theta^2 = p_1(\omega)$. By Theorem 2.2, we need only the following transversality condition in order to have Hopf bifurcation.

(4.2)
$$Re(\frac{d\lambda}{d\omega}\mid_{\lambda=i\theta}) = \frac{[p_1 - \delta r - \delta\beta C^* + \frac{2r}{k}\delta E^* + \delta\frac{\gamma_1 l - \gamma_2}{l+1}(T^* - 1)] + p_2(\omega)[\delta + r + \beta C^* - \frac{2r}{k}E^*]}{2p_1(1 + p_2^2(\omega))} \neq 0.$$

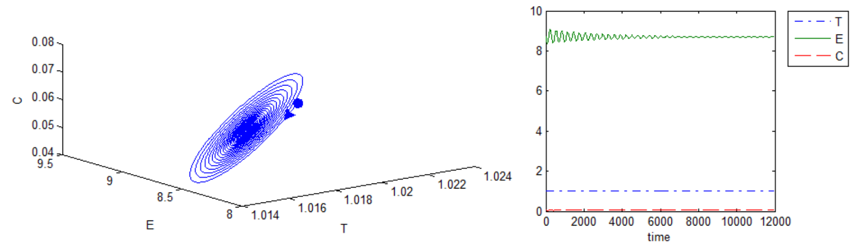
Theorem 4.1. *If* $p_1(\omega) > 0$, $p_0(\omega) - p_1(\omega)p_2(\omega) = 0$ *and* $Q = Re(\frac{d\lambda}{d\omega}\mid_{\lambda=i\theta})$ $\neq 0$, *then Hopf bifurcation for system* (3.1) *at the equilibrium point* $P^* = (T^*, E^*, C^*)$ *occurs.*

Proof. By above discussion it is clear. □

Example 4.2. Consider system (3.1) and suppose that all the parameters except l have the same values as in the Example 3.3, but $l = 5$. For $\omega \simeq 3.111$ and $p^* = (1.02, 8.5613, 0.06)$ we have:

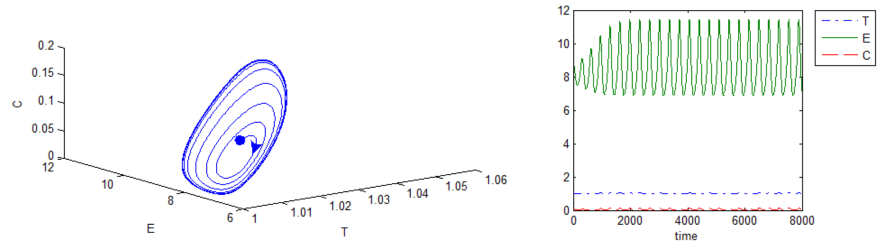
$$p_0(\omega) \simeq 0.000268409,\ p_1(\omega) \simeq 0.0005376,\ p_2(\omega) \simeq 0.4991668$$
$$\text{and } Q \simeq 213.341343 \neq 0.$$

Hence by Theorem 4.1, $\omega = 3.111$ is a Hopf bifurcation value at $p^* = (1.02, 8.5613, 0.06)$ for system (3.1). Furthermore, for $\omega_{Hopf} \simeq 3.111$ the Jacobian matrix J at p^* has a pair of pure imaginary eigenvalues $\lambda_1, \lambda_2 \simeq \pm 0.023188i$ (such that $(0.023188)^2 = p_1(\omega)$), and a real eigenvalue $\lambda_3 \simeq -0.49916693$. Moreover, for some ω_1 next to ω_{Hopf} and $\omega_1 < \omega_{Hopf}$ the equilibrium point p^* is an unstable focus. Such that as ω passes through ω_{Hopf} ($\omega \geq \omega_{Hopf}$), it is a stable focus. So p^* changes its stability. Furthermore, there exists a value of ω, say $\omega = \omega_u$ such that for all $\omega_u \leq \omega < \omega_{Hopf}$, a stable limit cycle O_{p^*} is found around the unstable focus p^*. Additionally, initial conditions near p^* lead to O_{p^*} and the periodic orbit O_{p^*} in the neighbourhood of p^* collapses as the stability of the equilibrium point p^* changes.



(a) (b)

FIGURE 6. $\omega = 3.12 > \omega_{Hopf}$: (a) related trajectories in the phase space TEC; (b) time dependent behaviour of the model. Initial conditions in both parts are $(1.02, 8.7, 0.06)$.

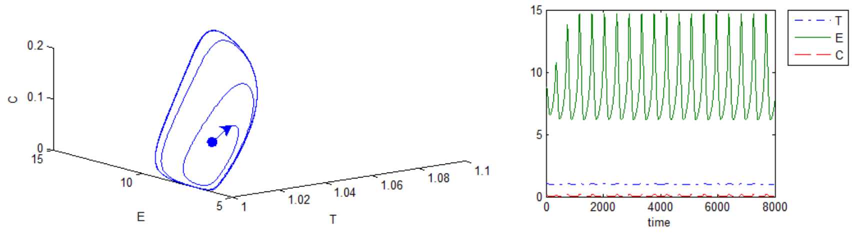


(a) (b)

FIGURE 7. $\omega = 3.11 < \omega_{Hopf}$: (a) the related trajectories in the phase space TEC; (b) time dependent behaviour of the model. Initial conditions in both parts are $(1.02, 8.7, 0.06)$.

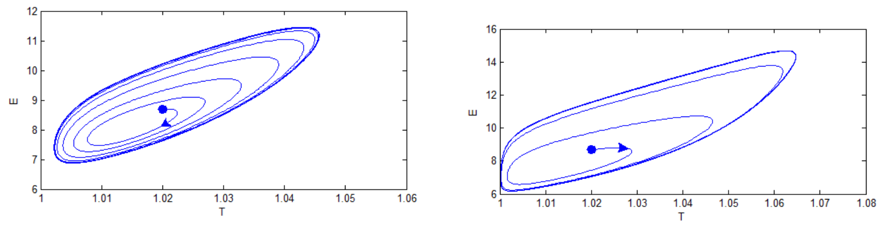
To investigate the behaviour of the system (3.1) in the neighbourhood of p^* we illustrate the mentioned Hopf bifurcation. For this purpose we solved the equation (3.1) by numerical techniques and carried out some numerical simulation by means of software Matlab. These numerical simulations are as follows.

For $\omega = 3.12 > \omega_{Hopf}$ and $\omega = 3.11, 3.1 < \omega_{Hopf}$ the related trajectories in the phase space TEC and time dependent behaviour of the model near the equilibrium p^* are drawn in Figures 6-8. As it is shown in Figures 7 and 8, for $\omega = 3.11, 3.1 < \omega_{Hopf}$ there exist stable limit cycles around the unstable focus p^*. Moreover for these attractors $T(t) \geq 1$ and $E(t) \geq 0$ for all t. So tourism is profitable and safe and compatible which implies the sustainability of tourism for system (3.1). In Figure 9 these attractors are illustrated in TE space.



(a) (b)

FIGURE 8. $\omega = 3.1 < \omega_{Hopf}$: (a) the related trajectories in the phase space TEC; (b) time dependent behaviour of the model. Initial conditions in both parts are $(1.02, 8.7, 0.06)$.



(a) (b)

FIGURE 9. (a) A stable attractor for $\omega = 3.11$ in the phase space TE; (b) A stable attractor for $\omega = 3.1$ in the phase space TE. Initial conditions in both parts are $(1.02, 8.7, 0.06)$.

5. Conclusion

In this paper to study the four dimensional system (1.1). Indeed, we transformed it into a three dimensional system (3.1) such that system (3.1) is the extended and modified model of the suggested model in [3] for tourism without increasing its dimension. Then, we analysed system(3.1) by means of bifurcation theory and showed that there is a bridge between profitability, compatibility and sustainability of system (3.1) and bifurcation theory. In other word by using [3] we interpreted the concepts profitability, compatibility and sustainability in terms of structural properties of attractors of system (3.1). For this purpose we firstly found transcritical, pitchfork and saddle-node bifurcation points for system (3.1). Then we numerically find some branches of stable equilibria for system (3.1) and showed that the existence of these branches can result the profitability of tourism industry. Then to prove the tourism sustainability of our system we investigated it from Hopf bifurcation point of view.

We determine all Hopf bifurcation values of parameter ω for system (3.1) and then find a family of stable attractors for that by numerical techniques. Finally we showed the existence of these stable limit cycles implies profitability and compatibility and then the sustainability of tourism.

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