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NEW CONCEPTS OF REGULAR INTERVAL-VALUED FUZZY GRAPHS

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ABSTRACT. Recently, interval-valued fuzzy graph is a growing research topic as it is the generalization of fuzzy graphs. The interval-valued fuzzy graphs are more flexible and compatible than fuzzy graphs due to the fact that they allowed the degree of membership of a vertex to an edge to be represented by interval values in [0.1] rather than the crisp values between 0 and 1. In this paper, we introduce the concepts of regular and totally regular interval-valued fuzzy graphs and discusses some properties of the μ-complement of interval-valued fuzzy graph. Self μ-complementary interval-valued fuzzy graphs and self-weak μ-complementary interval-valued fuzzy graphs are defined and a necessary condition for an interval valued fuzzy graph to be self μ-complementary is discussed. We define busy vertices and free vertices in interval valued fuzzy graph and study their image under an isomorphism.

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1. Introduction

Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, and transportation. Rosenfeld [13] introduced the notion of fuzzy graphs in 1975 and proposed another definitions including paths, cycles, connectedness and etc. The complement of a fuzzy graph was defined by Mordeson and Nair [8] and further studied by Sunitha and Vijayakumar [21]. The concept of weak isomorphism, co-weak isomorphism and isomorphism between fuzzy graphs was introduced by Bhutani in [5]. Nagoorgani and Chandrasekaran [12], defined complement of a fuzzy graph. In 1975, Zadeh [25] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy sets [26] in which the values of the membership degree are intervals of numbers instead of the numbers. In

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2011, Akram and Dudek [1] defined interval-valued fuzzy graphs and give some operations on them.

Rashmanlou et al. [14, 15, 16, 17, 18, 19, 20] studied bipolar fuzzy graphs, balanced interval-valued fuzzy graph, complete interval-valued fuzzy graphs and some properties of highly irregular interval-valued fuzzy graphs. Talebi and Rashmanlou [22, 23] studied properties of isomorphism and complement on interval-valued fuzzy graphs and bipolar fuzzy graphs. Akram and Davvaz discussed the properties of strong intuitionistic fuzzy graphs and they introduced the concept of intuitionistic fuzzy line graphs in [2]. In this paper, we define regular and totally regular interval-valued fuzzy graphs and discusses some properties of them. The μ-complement of interval-valued fuzzy graphs, also self μ-complementary and self-weak μ-complementary interval-valued fuzzy graphs are defined. We define busy vertices and free vertices in interval- valued fuzzy graphs and investigate properties of their image under a weak isomorphism, co-weak isomorphism and isomorphism. For other notations, terminologies and applications, the readers are referred to [3, 4, 6, 7, 9, 10, 11, 20, 24].

2. Preliminaries

A graph is an ordered pair $G^* : (V, E)$, where V is the set of vertices of G^* and E is the set of edges of G^*.

A fuzzy graph with a non-empty finite set S as the underlying set is a pair $G : (\sigma, \mu)$, where $\sigma : V \to [0, 1]$ is a fuzzy subset of V, $\mu : V \times V \to [0, 1]$ is a symmetric fuzzy relation on the fuzzy subset σ, such that

$$\mu(x, y) \leq \sigma(x) \wedge \sigma(y), \ for\ all\ x, y \in V$$

where $\wedge$ stands for minimum. The underlying crisp graph of the fuzzy graph $G : (\sigma, \mu)$ is denoted as $G^* : (\sigma^*, \mu^*)$, where

$$\sigma^* = \{u \in V |\ \sigma(u) > 0\}\ and\ \mu^* = \{(u, v) \in V \times V |\ \mu(x, y) > 0\}.$$

A path ρ in a fuzzy graph $G : (\sigma, \mu)$ is a sequence of distinct nodes $v_0, v_1, \cdots, v_n$ such that $\mu(v_{i-1}, v_i) > 0,\ 1 \leq i \leq n$. Here n is called the length of the path. The consecutive pairs (v_{i-1}, v_i) are called arcs of the path. We use the notation xy for an element of E.

A fuzzy graph G is said to be a complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, it is denoted as $K_\sigma : (\sigma, \mu)$.

By an interval-valued fuzzy graph of a graph $G^* = (V, E)$ we mean a pair $G = (A, B)$, where $A = [\mu_{A^-}, \mu_{A^+}]$ is an interval-valued fuzzy set on V and $B = [\mu_{B^-}, \mu_{B^+}]$ is an interval-valued fuzzy relation on E, such that :

$$\mu_{B^-}(xy) \leq \min(\mu_{A^-}(x), \mu_{A^-}(y)),\ \mu_{B^+}(xy) \leq \min(\mu_{A^+}(x), \mu_{A^+}(y))\ for\ all\ xy \in E.$$

We call A the interval-valued fuzzy vertex set of V, B the interval-valued fuzzy edge set of E , respectively. Note that B is a symmetric interval-valued fuzzy relation on A. Thus, $G = (A, B)$ is an interval-valued fuzzy graph of $G^* = (V, E)$ if

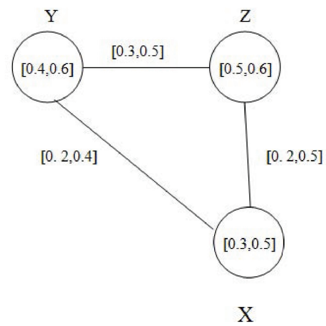

FIGURE 1. Interval-valued fuzzy graph G

$\mu_{B^-}(xy) \leq \min(\mu_{A^-}(x), \mu_{A^-}(y))$ *and* $\mu_{B^+}(xy) \leq \min(\mu_{A^+}(x), \mu_{A^+}(y))$ for all $xy \in E$.

Example 2.1 ([1])**.** Consider a graph $G^* = (V, E)$ such that $V = \{x, y, z\}$, $E = \{xy, yz, zx\}$. Let A be an interval-valued fuzzy set of V and let B be an interval-valued fuzzy set of $E \subseteq V \times V$ defined by

$$A = \Big\langle \Big(\frac{x}{0.3}, \frac{y}{0.4}, \frac{z}{0.5}\Big), \Big(\frac{x}{0.5}, \frac{y}{0.6}, \frac{z}{0.6}\Big)\Big\rangle,\ B = \Big\langle \Big(\frac{xy}{0.2}, \frac{yz}{0.3}, \frac{zx}{0.2}\Big), \Big(\frac{xy}{0.4}, \frac{yz}{0.5}, \frac{zx}{0.5}\Big)\Big\rangle$$

By routine computations, it is easy to see that $G = (A, B)$ is an interval-valued fuzzy graph of G^*.

Let G_1 and G_2 be two fuzzy graphs. A homomorphism $h : G_1 \to G_2$ is a map from V_1 to V_2 which satisfies $\sigma_1(x) \leq \sigma_2(h(x))$ for all $x \in V_1$ and $\mu_1(x, y) \leq \mu_2(h(x), h(y))$ for all $x, y \in V_1$.
A weak isomorphism $h : G_1 \to G_2$ is a bijective homomorphism that satisfies $\sigma_1(x) = \sigma_2(h(x))$ for all $x \in V_1$.
A co-weak isomorphism $h : G_1 \to G_2$ is a bijective homomorphism that satisfies $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.
An isomorphism $h : G_1 \to G_2$ is a bijective homomorphism that satisfies $\sigma_1(x) = \sigma_2(h(x))$ for all $x \in V_1$, $\mu_1(x, y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$ and we denote $G_1 \cong G_2$.
A fuzzy graph G is said to be a self- complementary fuzzy graph if $G \cong \overline{G}$.

Definition 2.2. Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ respectively. A homomorphism $f : G_1 \to G_2$ is a mapping $f : V_1 \to V_2$ such that:

(a) $\mu_{A_1^-}(x_1) \leq \mu_{A_2^-}(f(x_1)),\ \mu_{A_1^+}(x_1) \leq \mu_{A_2^+}(f(x_1)),\ for\ all\ x_1 \in V_1$

(b) $\mu_{B_1^-}(x_1y_1) \leq \mu_{B_2^-}(f(x_1)f(y_1)),\ \mu_{B_1^+}(x_1y_1) \leq \mu_{B_2^+}(f(x_1)f(y_1))$
$for\ all\ x_1y_1 \in E_1.$

A bijective homomorphism with the property

$(c)\ \mu_{A_1^-}(x_1)=\mu_{A_2^-}(f(x_1)),\ \mu_{A_1^+}(x_1)=\mu_{A_2^+}(f(x_1)),\ for\ all\ x_1\in V_1$

is called a weak isomorphism.

A bijective homomorphism $f:G_1\to G_2$ such that

$(d)\ \mu_{B_1^-}(x_1y_1)=\mu_{B_2^-}(f(x_1)f(y_1)),\ \mu_{B_1^+}(x_1y_1)=\mu_{B_2^+}(f(x_1)f(y_1))$
$for\ all\ x_1y_1\in E_1$

is called a co weak-isomorphism.

A bijective mapping $f:G_1\to G_2$ satisfying (c) and (d) is called an isomorphism.

Definition 2.3. Given an interval-valued fuzzy graph $G=(A,B)$, with the underlying set V, the order of G is defined and denoted as

$$O(G)=\left(\sum_{x\in V}\mu_{A^-}(x),\sum_{x\in V}\mu_{A^+}(x)\right).$$

The size of an interval-valued fuzzy graph G is

$$S(G)=\big(S^-(G),S^+(G)\big)=\left(\sum_{\substack{x\neq y\\ x,y\in V}}\mu_{B^-}(xy),\ \sum_{\substack{x\neq y\\ x,y\in V}}\mu_{B^+}(xy)\right).$$

Definition 2.4. Let $G=(A,B)$ be an interval-valued fuzzy graph on G^*. The open degree of a vertex u is defined as $deg(u)=(d^-(u),d^+(u))$, where $d^-(u)=\sum_{\substack{u\neq v\\ v\in V}}\mu_{B^-}(uv)$ and $d^+(u)=\sum_{\substack{u\neq v\\ v\in V}}\mu_{B^+}(uv)$. If all the vertices have the same open neighborhood degree $n=(n_1,n_2)$, then G is called an n-regular interval-valued fuzzy graph.

Definition 2.5. A path in an interval-valued fuzzy graph is a sequence of distinct vertices $v_1,v_2,\cdots,v_{n+1}$ such that $\mu_{B^+}(v_iv_{i+1})>0,\ 1\leq i\leq n$.

Definition 2.6. The length of a path $\rho=v_1v_2\cdots v_{n+1}$ $(n>0)$ is n.

Now, we give some new definitions of interval-valued fuzzy graphs.

Definition 2.7. In an interval-valued fuzzy graph G we have:

$$\mu^k_{B^-}(uv)=\sup\{\mu_{B^-}(uv_1)\wedge\mu_{B^-}(v_1v_2)\wedge\mu_{B^-}(v_2v_3),\cdots,\wedge\mu_{B^-}(v_{k-1}v)|\ u,v_1,v_2,v_{k-1},v\in V\}$$
$$\mu^k_{B^+}(uv)=\sup\{\mu_{B^+}(uv_1)\wedge\mu_{B^+}(v_1v_2)\wedge\mu_{B^+}(v_2v_3),\cdots,\wedge\mu_{B^+}(v_{k-1}v)|\ u,v_1,v_2,v_{k-1},v\in V\}.$$

Also we have $\mu^\infty_{B^-}(uv)=\sup\{\mu^k_{B^-}(uv)|\ k=1,2,3,\cdots\}$ and

$$\mu^\infty_{B^+}(uv)=\sup\{\mu^k_{B^+}(uv)|\ k=1,2,3,\cdots\}.$$

Definition 2.8. An interval-valued fuzzy graph $G=(A,B)$ is connected if $\mu^\infty_{B^+}(xy)>0$ for all $x,y\in V$.

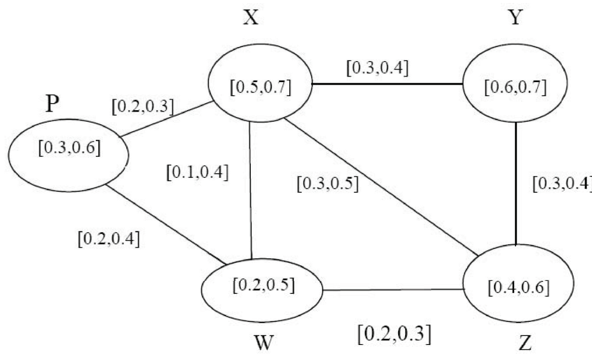

FIGURE 2. Interval-valued fuzzy graph G with strong arcs

Definition 2.9. In an interval-valued fuzzy graph $G = (A, B)$ an arc (u, v) is said to be a strong arc, if $\mu_{B^-}(uv) \geq \mu^{\infty}_{B^-}(uv)$, $\mu_{B^+}(uv) \geq \mu^{\infty}_{B^+}(uv)$.

Example 2.10. In this example it is obvious that (x, z), (y, z) and (p, w) are strong arcs.

Definition 2.11. A vertex u of an interval-valued fuzzy graph $G = (A, B)$ is said to be an isolated vertex if $\mu_{B^+}(uv) = 0$ for all $v \in V$.

Definition 2.12. Let u be a vertex in an interval-valued fuzzy graph $G = (A, B)$. Then $N(u) = \{v : v \in V \& (u, v) \textit{ is a strong arc}\}$ is called neighborhood of u.

Example 2.13. In Example 2.10 z is a neighborhood of x and y.

Definition 2.14. Two vertices u and v are said to be neighbors in an interval valued fuzzy graph if $\mu_{B^+}(uv) > 0$.

Lemma 2.15. *Let G_1 and G_2 be two interval- valued fuzzy graphs and $h : G_1 \to G_2$ be an isomorphism. Then, for every $x, y \in V_1$ we have*

$$\mu^{\infty}_{B_1^-}(xy) = \mu^{\infty}_{B_2^-}(h(x)h(y)),$$
$$\mu^{\infty}_{B_1^+}(xy) = \mu^{\infty}_{B_2^+}(h(x)h(y)).$$

Proof. By definition, for every $x, y \in V_1$, we have $\mu_{A_1^-}(x) = \mu_{A_2^-}(h(x))$, $\mu_{A_1^+}(x) = \mu_{A_2^+}(h(x))$ for all $x \in V_1$ and $\mu_{B_1^-}(xy) = \mu_{B_2^-}(h(x), h(y))$, $\mu_{B_1^+}(xy) = \mu_{B_2^+}(h(x), h(y))$ for all $xy \in E_1$. Hence,

$$\begin{aligned}
\mu^{\infty}_{B_1^+}(xy) &= \sup\{\mu^{k}_{B_1^+}(xy) |\ k = 1, 2, 3, \cdots\} \\
&= \sup\{\bigwedge_{i=1}^{k} \mu_{B_1^+}(x_{i-1}x_i) |\ x = x_0, x_1, \cdots, x_k = y \in V_1,\ k = 1, 2, \cdots\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup\{\bigwedge_{i=1}^{k} \mu_{B_2^+}(h(x_{i-1})h(x_i))|\ x = x_0, x_1, \cdots, x_k = y \in V_1, k = 1, \cdots\} \\
&= \sup\{\mu^k_{B_2^+}(h(x)h(y))|\ k = 1, 2, \cdots\} \\
&= \mu^\infty_{B_2^+}(h(x)h(y)).
\end{aligned}
$$

Thus, $\mu^\infty_{B_1^+}(xy) = \mu^\infty_{B_2^+}(h(x)h(y))$. Similarly, we can prove that $\mu^\infty_{B_1^-}(xy) = \mu^\infty_{B_2^-}(h(x)h(y))$. □

Theorem 2.16. *Let G_1 and G_2 be interval-valued fuzzy graphs and G_1 be isomorphic to G_2. Then G_1 is connected if and only if G_2 is also connected.*

Proof. Let G_1 be isomorphic to G_2. Then there exists an isomorphism $h : G_1 \to G_2$, such that for every $x, y \in V$, $\mu_{A_1^-}(x) = \mu_{A_2^-}(h(x))$, $\mu_{B_1^-}(xy) = \mu_{B_2^-}(h(x)h(y))$ and $\mu_{A_1^+}(x) = \mu_{A_2^+}(h(x))$, $\mu_{B_1^+}(xy) = \mu_{B_2^+}(h(x)h(y))$. Now we have G_1 is connected if and only if $\mu^\infty_{B_1^+}(xy) > 0$ for all $x, y \in V_1$ if and only if $\mu^\infty_{B_2^+}(h(x)h(y)) > 0$ for all $x, y \in V_1$ (By Lemma 2.15) if and only if G_2 is connected. □

Theorem 2.17. *Let G_1 and G_2 be interval-valued fuzzy graphs. If $G_1 \cong G_2$, an arc in G_1 is strong if and only if the corresponding image arc in G_2 is also strong.*

Proof. Let $h : G_1 \to G_2$ be an isomorphism, and (x, y) be a strong arc in G_1 so

$$\mu_{B_1^-}(xy) \geq \mu^\infty_{B_1^-}(xy). \tag{1}$$

Since h is an isomorphism between G_1 and G_2, $\mu_{B_1^-}(xy) = \mu_{B_2^-}(h(x)h(y))$ and $\mu_{B_1^+}(xy) = \mu_{B_2^+}(h(x)h(y))$. By (1) and Lemma 2.15, $\mu_{B_2^-}(h(x)h(y)) = \mu_{B_1^-}(xy) \geq \mu^\infty_{B_1^-}(xy) = \mu^\infty_{B_2^-}(h(x)h(y))$. Similarly, we have $\mu_{B_2^+}(h(x)h(y)) \geq \mu^\infty_{B_2^+}(h(x)h(y))$. Therefore, $(h(x)h(y))$ is a strong arc in G_2. Conversely, by bijectivity and isomorphism property of h, strong arc in G_2 implies its pre image in G_1 is also strong. □

3. μ-complement and self μ-complement interval-valued fuzzy graphs

In this section we define $G^\mu : (A, B^\mu)$, μ-complement of an interval valued fuzzy graph G. We need $G^\mu : (A, B^\mu)$ be an interval valued fuzzy graph, thus in this section, we suppose that $G = (A, B)$ is an interval valued fuzzy graph that satisfies the following condition:

$$\mu_{A^-}(x) \wedge \mu_{A^-}(y) - \mu_{B^-}(xy) \leq \mu_{A^+}(x) \wedge \mu_{A^+}(y) - \mu_{B^+}(xy)\ for\ all\ x, y \in V.$$

Definition 3.1. Let $G = (A, B)$ be an interval-valued fuzzy graph. The μ-complement of G is defined as $G^\mu : (A, B^\mu)$, where $B^\mu = (\mu^\mu_{B^-}, \mu^\mu_{B^+})$ and we have

$$\mu^{\mu}_{B^-}(xy) = \begin{cases} \mu_{A^-}(x) \wedge \mu_{A^-}(y) - \mu_{B^-}(xy) & if\ \mu_{B^-}(xy) > 0 \\ 0 & if\ \mu_{B^-}(xy) = 0 \end{cases}$$

$$\mu^{\mu}_{B^+}(xy) = \begin{cases} \mu_{A^+}(x) \wedge \mu_{A^+}(y) - \mu_{B^+}(xy) & if\ \mu_{B^+}(xy) > 0 \\ 0 & if\ \mu_{B^+}(xy) = 0. \end{cases}$$

Proposition 3.2. *Let G_1 and G_2 be interval-valued fuzzy graphs, If G_1 and G_2 are isomorphic, then their μ-complements, G_1^{μ} and G_2^{μ}, are also isomorphic.*

Proof. Let $G_1 \cong G_2$, and $f : G_1 \to G_2$ be an isomorphism. Then, we have $\mu_{A_1^-}(x) = \mu_{A_2^-}(f(x))$ for all $x \in V_1$,
$\mu_{B_1^-}(xy) = \mu_{B_2^-}(f(x)f(y))$, $\mu_{B_1^+}(xy) = \mu_{B_2^+}(f(x)f(y))$ for all $xy \in E_1$.
If $\mu_{B_1^-}(xy) > 0$, then $\mu_{B_2^-}(f(x)f(y)) > 0$ and

$$\begin{aligned} \mu^{\mu}_{B_1^-}(xy) &= \mu_{A_1^-}(x) \wedge \mu_{A_1^-}(y) - \mu_{B_1^-}(xy) = \mu_{A_2^-}(f(x)) \wedge \mu_{A_2^-}(f(y)) \\ &- \mu_{B_2^-}(f(x)f(y)) = \mu^{\mu}_{B_2^-}(f(x)f(y)). \end{aligned}$$

If $\mu_{B_1^-}(xy) = 0$, then $\mu_{B_2^-}(f(x)f(y)) = 0$ and $\mu^{\mu}_{B_1^-}(xy) = 0 = \mu^{\mu}_{B_2^-}(f(x)f(y))$.

Thus $\mu^{\mu}_{B_1^-}(xy) = \mu^{\mu}_{B_2^-}(f(x)f(y))$ for all $xy \in E_1$. Similarly, we can prove that $\mu^{\mu}_{B_1^+}(xy) = \mu^{\mu}_{B_2^+}(f(x)f(y))$ for all $xy \in E_1$. Therefore $f : G_1^{\mu} \to G_2^{\mu}$ is an isomorphism, hence $G_1^{\mu} \cong G_2^{\mu}$.

The following example show that the converse of Proposition 3.2 is not true. □

Example 3.3. The following figures shows interval-valued fuzzy graphs G_1, G_2, G_1^{μ} and G_2^{μ} in which G_1 and G_2 are not isomorphism but $G_1^{\mu} \cong G_2^{\mu}$. By definition of μ-complement of an interval valued fuzzy graph G we have the following.

Theorem 3.4. *Let $G_1 = (A, B_1)$ and $G_2 = (A, B_2)$ be two interval-valued fuzzy graphs on $G^* = (V, E)$. Then, $G_1^{\mu} = G_2^{\mu}$ if and only if every arc (x, y) satisfying in one of the following conditions.*

(1) $\mu_{B_1^-}(xy) = \mu_{B_2^-}(xy),\ \mu_{B_1^+}(xy) = \mu_{B_2^+}(xy)$,

(2) $\mu_{B_1^-}(xy) = \mu_{B_2^-}(xy),\ \mu_{B_2^+}(xy) = 0,\ \mu_{B_2^+}(xy) = \mu_{A^+}(x) \wedge \mu_{A^+}(y)$,

(3) $\mu_{B_1^-}(xy) = \mu_{B_2^-}(xy),\ \mu_{B_1^+}(xy) = \mu_{A^+}(x) \wedge \mu_{A^+}(y),\ \mu_{B_2^+}(xy) = 0$,

(4) $\mu_{B_1^-}(xy) = 0,\ \mu_{B_2^-}(xy) = \mu_{A^-}(x) \wedge \mu_{A^-}(y),\ \mu_{B_1^+}(xy) = \mu_{B_2^+}(xy)$,

(5) $\mu_{B_1^-}(xy) = 0,\ \mu_{B_2^-}(xy) = \mu_{A^-}(x) \wedge \mu_{A^-}(y),\ \mu_{B_1^+}(xy) = 0$,
$\mu_{B_2^+}(xy) = \mu_{A^+}(x) \wedge \mu_{A^+}(y)$,

(6) $\mu_{B_1^-}(xy) = \mu_{A^-}(x) \wedge \mu_{A^-}(y),\ \mu_{B_2^-}(xy) = 0,\ \mu_{B_1^+}(xy) = \mu_{B_2^+}(xy)$,

(7) $\mu_{B_1^-}(xy) = \mu_{A^-}(x) \wedge \mu_{A^-}(y),\ \mu_{B_2^-}(xy) = 0,\ \mu_{B_1^+}(xy) = \mu_{A^+}(x) \wedge \mu_{A^+}(y)$,
$\mu_{B_2^+}(xy) = 0$.

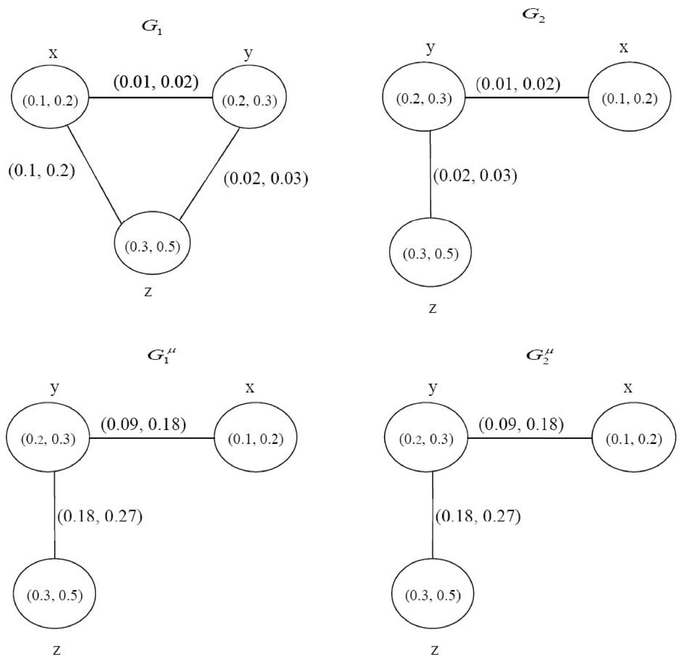

FIGURE 3. Interval-valued fuzzy graphs G_1, G_2, G_1^μ and G_2^μ

Theorem 3.5. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \varnothing$. Then, $(G_1 + G_2)^\mu \cong G_1^\mu \cup G_2^\mu$.*

Proof. Let $I : V_1 \cup V_2 \to V_1 \cup V_2$ be the identity map. We prove that for all $x, y \in V$, $(\mu_{A_1^-} + \mu_{A_2^-})^\mu(x) = \mu^\mu_{A_1^-}(x) \cup \mu^\mu_{A_2^-}(x)$, $(\mu_{A_1^+} + \mu_{A_2^+})^\mu(x) = \mu^\mu_{A_1^+}(x) \cup \mu^\mu_{A_2^+}(x)$ and $(\mu_{B_1^-} + \mu_{B_2^-})^\mu(xy) = \mu^\mu_{B_1^-} \cup \mu^\mu_{B_2^-}(xy)$, $(\mu_{B_1^+} + \mu_{B_2^+})^\mu(xy) = \mu^\mu_{B_1^+} \cup \mu^\mu_{B_2^+}(xy)$. For all $x, y \in V$ we have

$$\begin{aligned}(\mu_{A_1^-} + \mu_{A_2^-})^\mu(x) = (\mu_{A_1^-} \cup \mu_{A_2^-})(x) &= \begin{cases} \mu_{A_1^-}(x) & if\ x \in V_1 \\ \mu_{A_2^-}(x) & if\ x \in V_2 \end{cases} \\ &= \begin{cases} \mu^\mu_{A_1^-}(x) & if\ x \in V_1 \\ \mu^\mu_{A_2^-}(x) & if\ x \in V_2 \end{cases} \\ &= (\mu^\mu_{A_1^-} \cup \mu^\mu_{A_2^-})(x),\end{aligned}$$

$$(\mu_{A_1^+} + \mu_{A_2^+})^\mu(x) = (\mu_{A_1^+} \cup \mu_{A_2^+})(x) = \begin{cases} \mu_{A_1^+}(x) & if\ x \in V_1 \\ \mu_{A_2^+}(x) & if\ x \in V_2 \end{cases}$$

$$= \begin{cases} \mu^{\mu}_{A_1^+}(x) & if\ x \in V_1 \\ \mu^{\mu}_{A_2^+}(x) & if\ x \in V_2 \end{cases}$$
$$= (\mu^{\mu}_{A_1^+} \cup \mu^{\mu}_{A_2^+})(x).$$

Moreover,

$$(\mu_{B_1^-} + \mu_{B_2^-})^{\mu}(xy) = (\mu_{A_1^-} + \mu_{A_2^-})(x) \wedge (\mu_{A_1^-} + \mu_{A_2^-})(y) - (\mu_{B_1^-} + \mu_{B_2^-})(xy)$$
$$= \begin{cases} (\mu_{A_1^-} \cup \mu_{A_2^-})(x) \wedge (\mu_{A_1^-} \cup \mu_{A_2^-})(y) - (\mu_{B_1^-} \cup \mu_{B_2^-})(xy) & if\ xy \in E_1 \cup E_2 \\ (\mu_{A_1^-} \cup \mu_{A_2^-})(x) \wedge (\mu_{A_1^-} \cup \mu_{A_2^-})(y) - \mu_{A_1^-}(x) \wedge \mu_{A_2^-}(y) & if\ xy \in E' \end{cases}$$
$$= \begin{cases} \mu_{A_1^-}(x) \wedge \mu_{A_1^-}(y) - \mu_{B_1^-}(xy) & if\ xy \in E_1 \\ \mu_{A_2^-}(x) \wedge \mu_{A_2^-}(y) - \mu_{B_2^-}(xy) & if\ xy \in E_2 \\ \mu_{A_1^-}(x) \wedge \mu_{A_1^-}(y) - \mu_{A_1^-}(x) \wedge \mu_{A_1^-}(y) & if\ xy \in E' \end{cases}$$
$$= \begin{cases} \mu^{\mu}_{B_1^-}(xy) & if\ xy \in E_1 \\ \mu^{\mu}_{B_2^-}(xy) & if\ xy \in E_2 \\ 0 & if\ xy \in E' \end{cases}$$
$$= (\mu^{\mu}_{B_1^-} \cup \mu^{\mu}_{B_2^-})(xy),$$

$$(\mu_{B_1^+} + \mu_{B_2^+})^{\mu}(xy) = (\mu_{A_1^+} + \mu_{A_2^+})(x) \wedge (\mu_{A_1^+} + \mu_{A_2^+})(y) - (\mu_{B_1^+} + \mu_{B_2^+})(xy)$$
$$= \begin{cases} (\mu_{A_1^+} \cup \mu_{A_2^+})(x) \wedge (\mu_{A_1^+} \cup \mu_{A_2^+})(y) - (\mu_{B_1^+} \cup \mu_{B_2^+})(xy) & if\ xy \in E_1 \cup E_2 \\ (\mu_{A_1^+} \cup \mu_{A_2^+})(x) \wedge (\mu_{A_1^+} \cup \mu_{A_2^+})(y) - \mu_{A_1^+}(x) \wedge \mu_{A_2^+}(y) & if\ xy \in E' \end{cases}$$
$$= \begin{cases} \mu_{A_1^+}(x) \wedge \mu_{A_1^+}(y) - \mu_{B_1^+}(xy) & if\ xy \in E_1 \\ \mu_{A_2^+}(x) \wedge \mu_{A_2^+}(y) - \mu_{B_2^+}(xy) & if\ xy \in E_2 \\ \mu_{A_1^+}(x) \wedge \mu_{A_1^+}(y) - \mu_{A_1^+}(x) \wedge \mu_{A_1^+})(y) & if\ xy \in E' \end{cases}$$
$$= \begin{cases} \mu^{\mu}_{B_1^+}(xy) & if\ xy \in E_1 \\ \mu^{\mu}_{B_2^+}(xy) & if\ xy \in E_2 \\ 0 & if\ xy \in E' \end{cases}$$
$$= (\mu^{\mu}_{B_1^+} \cup \mu^{\mu}_{B_2^+})(xy).$$

□

Theorem 3.6. *Let $G_1 = (A_1, B_1)$ and $G_2 = (A_2, B_2)$ be two interval-valued fuzzy graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$ such that $V_1 \cap V_2 = \varnothing$. Then, $(G_1 \cup G_2)^{\mu} = G_1^{\mu} \cup G_2^{\mu}$.*

Proof. We shall prove that the identity map is the required isomorphism. If $x \in V_1$

$$(\mu_{A_1^-} \cup \mu_{A_2^-})^{\mu}(x) = (\mu_{A_1^-} \cup \mu_{A_2^-})(x) = \mu_{A_1^-}(x) = \mu^{\mu}_{A_1^-}(x) = (\mu^{\mu}_{A_1^-} \cup \mu^{\mu}_{A_2^-})(x),$$
$$(\mu_{A_1^+} \cup \mu_{A_2^+})^{\mu}(x) = (\mu_{A_1^+} \cup \mu_{A_2^+})(x) = \mu_{A_1^+}(x) = \mu^{\mu}_{A_1^+}(x) = (\mu^{\mu}_{A_1^+} \cup \mu^{\mu}_{A_2^+})(x).$$

If $x \in V_2$, then

$$(\mu_{A_1^-} \cup \mu_{A_2^-})^{\mu}(x) = (\mu_{A_1^-} \cup \mu_{A_2^-})(x) = \mu_{A_2^-}(x) = \mu^{\mu}_{A_2^-}(x) = (\mu^{\mu}_{A_1^-} \cup \mu^{\mu}_{A_2^-})(x),$$
$$(\mu_{A_1^+} \cup \mu_{A_2^+})^{\mu}(x) = (\mu_{A_1^+} \cup \mu_{A_2^+})(x) = \mu_{A_2^+}(x) = \mu^{\mu}_{A_2^+}(x) = (\mu^{\mu}_{A_1^+} \cup \mu^{\mu}_{A_2^+})(x).$$

If $xy \in E_1$, then

$$\begin{aligned}(\mu_{B_1^-} \cup \mu_{B_2^-})^{\mu}(xy) &= (\mu_{A_1^-} \cup \mu_{A_2^-})(x) \wedge (\mu_{A_1^-} \cup \mu_{A_2^-})(y) - (\mu_{B_1^-} \cup \mu_{B_2^-})(xy) \\ &= \mu_{A_1^-}(x) \wedge \mu_{A_1^-}(y) - \mu_{B_1^-}(xy) \\ &= \mu^{\mu}_{B_1^-}(xy) = (\mu^{\mu}_{B_1^-} \cup \mu^{\mu}_{B_2^-})(xy) \\ &= (\mu^{\mu}_{B_1^-} \cup \mu^{\mu}_{B_2^-})(xy).\end{aligned}$$

$$\begin{aligned}(\mu_{B_1^+} \cup \mu_{B_2^+})^{\mu}(xy) &= (\mu_{A_1^+} \cup \mu_{A_2^+})(x) \wedge (\mu_{A_1^+} \cup \mu_{A_2^+})(y) - (\mu_{B_1^+} \cup \mu_{B_2^+})(xy) \\ &= \mu_{A_1^+}(x) \wedge \mu_{A_1^+}(y) - \mu_{B_1^+}(xy) \\ &= \mu^{\mu}_{B_1^+}(xy) = (\mu^{\mu}_{B_1^+} \cup \mu^{\mu}_{B_2^+})(xy) \\ &= (\mu^{\mu}_{B_1^+} \cup \mu^{\mu}_{B_2^+})(xy).\end{aligned}$$

If $xy \in E_2$, then

$$\begin{aligned}(\mu_{B_1^-} \cup \mu_{B_2^-})^{\mu}(xy) &= (\mu_{A_1^-} \cup \mu_{A_2^-})(x) \wedge (\mu_{A_1^-} \cup \mu_{A_2^-})(y) - (\mu_{B_1^-} \cup \mu_{B_2^-})(xy) \\ &= \mu_{A_2^-}(x) \wedge \mu_{A_2^-}(y) - \mu_{B_2^-}(xy) \\ &= \mu^{\mu}_{B_2^-}(xy) = (\mu^{\mu}_{B_1^-} \cup \mu^{\mu}_{B_2^-})(xy) \\ &= (\mu^{\mu}_{B_1^-} \cup \mu^{\mu}_{B_2^-})(xy).\end{aligned}$$

$$\begin{aligned}(\mu_{B_1^+} \cup \mu_{B_2^+})^{\mu}(xy) &= (\mu_{A_1^+} \cup \mu_{A_2^+})(x) \wedge (\mu_{A_1^+} \cup \mu_{A_2^+})(y) - (\mu_{B_1^+} \cup \mu_{B_2^+})(xy) \\ &= \mu_{A_2^+}(x) \wedge \mu_{A_2^+}(y) - \mu_{B_2^+}(xy) \\ &= \mu^{\mu}_{B_2^+}(xy) = (\mu^{\mu}_{B_1^+} \cup \mu^{\mu}_{B_2^+})(xy) \\ &= (\mu^{\mu}_{B_1^+} \cup \mu^{\mu}_{B_2^+})(xy).\end{aligned}$$

□

Definition 3.7. An interval-valued fuzzy graph G is said to be a self μ- complementary interval-valued fuzzy graph if $G \cong G^{\mu}$.

Theorem 3.8. *If G is a self μ-complementary interval-valued fuzzy graph, then*

$$(1) S^{-}(G) = \frac{1}{2}\left(\sum \mu_{A^-}(x) \wedge \mu_{A^-}(y)\right),$$
$$(2) S^{+}(G) = \frac{1}{2}\left(\sum \mu_{A^+}(x) \wedge \mu_{A^+}(y)\right).$$

Proof. Let $G \cong G^{\mu}$ then there exist a bijective map $h : V \to V$ such that for all $x, y \in V$ we have

$$\mu_{A^-}(x) = \mu^{\mu}_{A^-}(h(x)) = \mu_{A^-}(h(x)),\ \mu_{A^+}(x) = \mu^{\mu}_{A^+}(h(x)) = \mu_{A^+}(h(x)) \quad (2)$$

and

$$\mu_{B^-}(xy) = \mu^{\mu}_{B^-}(h(x)h(y)),\ \mu_{B^+}(xy) = \mu^{\mu}_{B^+}(h(x)h(y)). \quad (3)$$

Let $\mu_{B^-}(xy) \neq 0$. Using (3), $\mu^{\mu}_{B^-}(h(x)h(y)) \neq 0$. Thus $\mu_{B^-}(h(x)h(y)) \neq \mu_{A^-}(h(x)) \wedge \mu_{A^-}(h(y))$ and $\mu_{B^-}(h(x)h(y)) \neq 0$. Now,

$$\mu^{\mu}_{B^-}(h(x)h(y)) = \mu_{A^-}(h(x)) \wedge \mu_{A^-}(h(y)) - \mu_{B^-}(h(x)h(y)).$$

By (3), $\mu_{B^-}(xy) = \mu_{A^-}(h(x)) \wedge \mu_{A^-}(h(y)) - \mu_{B^-}(h(x)h(y))$. Thus $\mu_{B^-}(xy) + \mu_{B^-}(h(x)h(y)) = \mu_{A^-}(x) \wedge \mu_{A^-}(y)$, by (2). As h is a bijective map on taking summation, $2\sum \mu_{B^-}(xy) = \sum \mu_{A^-}(x) \wedge \mu_{A^-}(y)$. Therefore, $S^-(G) = \frac{1}{2}\left(\sum \mu_{A^-}(x) \wedge \mu_{A^-}(y)\right)$. Similarly, we can show that

$$S^+(G) = \frac{1}{2}\left(\sum \mu_{A^+}(x) \wedge \mu_{A^+}(y)\right)$$

. □

Definition 3.9. An interval-valued fuzzy graph $G = (A, B)$ is said to be a self weak μ-complementary interval-valued fuzzy graph if G is weak isomorphic with G^{μ}.

Theorem 3.10. *In an interval-valued fuzzy graph $G = (A, B)$, if for all $xy \in E$, $\mu_{B^-}(xy) \leq \frac{1}{2}(\mu_{A^-}(x) \wedge \mu_{A^-}(y))$ and $\mu_{B^+}(xy) \leq \frac{1}{2}(\mu_{A^+}(x) \wedge \mu_{A^+}(y))$, then G will be a self -weak μ-complementary interval valued fuzzy graph.*

Proof. The identity map $h : V \to V$ is a weak isomorphism from G to G^{μ}. □

4. Busy vertices and free vertices in interval valued fuzzy graphs

Definition 4.1. A vertex v in an interval-valued fuzzy graph $G = (A, B)$ is said to be a busy vertex if $\mu_{A^-}(v) \leq d^-(v)$ and $\mu_{A^+}(v) \leq d^+(v)$, otherwise it is called a free vertex.

Lemma 4.2. *Let $G_1 \cong G_2$ and $h : G_1 \to G_2$ be an isomorphism. Then $deg(x) = deg(h(x))$ for all $x \in V$.*

Proof. Since $G_1 \cong G_2$, we have $\mu_{B_1^-}(x_1y_1) = \mu_{B_2^-}(h(x_1)h(y_1))$, $\mu_{B_1^+}(x_1y_1) = \mu_{B_2^+}(h(x_1)h(y_1))$ for all $x_1y_1 \in E_1$. Hence,

$$d^-(x) = \sum_{x \neq y} \mu_{B^-}(xy) = \sum_{x \neq y} \mu_{B^-}(h(x)h(y)) = d^-(h(x))$$

$$d^+(x) = \sum_{x \neq y} \mu_{B^+}(xy) = \sum_{x \neq y} \mu_{B^+}(h(x)h(y)) = d^+(h(x)).$$

Also, we know that

$$deg(x) = (d^-(x), d^+(x))$$

for all $x \in V$. Thus, $deg(x) = deg(h(x))$ for all $x \in V$. □

Theorem 4.3. *If $G_1 \cong G_2$, then the busy vertices are preserved under isomorphism.*

Proof. Let $h : V_1 \to V_2$ be an isomorphism between G_1 and G_2. Then $\mu_{A_1^-}(x) = \mu_{A_2^-}(h(x))$, $\mu_{A_1^+}(x) = \mu_{A_2^+}(h(x))$ for all $x \in V_1$ and $\mu_{B_1^-}(xy) = \mu_{B_2^-}(h(x)h(y))$ and $\mu_{B_1^+}(xy) = \mu_{B_2^+}(h(x)h(y))$ for all $xy \in E_1$.

Also h preserves the degree of vertices, by Lemma 4.2, i.e. $d_1^-(x) = d_2^-(h(x))$, $d_1^+(x) = d_2^+(h(x))$. If x is a busy vertex in G_1, then $\mu_{A_1^-}(x) \le d_1^-(x)$ and $\mu_{A_1^+}(x) \le d_1^+(x)$. Then $\mu_{A_2^-}(h(x)) \le d_2^-(h(x))$ and $\mu_{A_2^+}(h(x)) \le d_2^+(h(x))$. Hence, $h(x)$ is a busy vertex in G_2. □

Theorem 4.4. *Let interval-valued fuzzy graph G_1 be co-weak isomorphic with G_2. If v is a free vertex in G_1, then its image under co-weak isomorphism is also a free vertex in G_2.*

Proof. Let v be a free vertex in G_1. Then $\mu_{A_1^-}(v) > d_1^-(v)$ or $\mu_{A_1^+}(v) > d_1^+(v)$. Let $h : V_1 \to V_2$ be a co-weak isomorphism between G_1 and G_2. Then for all $x, y \in V_1$ $\mu_{A_1^-}(x) \le \mu_{A_2^-}(h(x))$, $\mu_{A_1^+}(x) \le \mu_{A_2^+}(h(x))$ and $\mu_{B_1^-}(xy) = \mu_{B_2^-}(h(x)h(y))$, $\mu_{B_1^+}(xy) = \mu_{B_2^+}(h(x)h(y))$. If $\mu_{A_1^-}(v) > d_1^-(v)$, then $d_1^-(v) < \mu_{A_1^-}(v) \le \mu_{A_2^-}(h(v))$.

Hence $\mu_{A_2^-}(h(v)) > d_1^-(v) = \sum_{v \ne u} \mu_{B_1^-}(vu) = \sum_{v \ne u} \mu_{B_1^-}(h(v)h(u))$ $= d_2^-(h(v))$. Thus $\mu_{A_2^-}(h(v)) > d_2^-(h(v))$. Therefore $h(v)$ is a free vertex in G_2. If $\mu_{A_1^+}(v) > d_1^+(v)$, similarly we can show that $h(v)$ is a free vertex in G_2. □

Theorem 4.5. *Let an interval-valued fuzzy graph G_1 be weak isomorphic to G_2. If $u \in V_1$ is a busy vertex in G_1 then its image under a weak isomorphism in G_2 is also busy.*

Proof. Let $h : V_1 \to V_2$ be a weak isomorphism between G_1 and G_2. Then for all $x, y \in V_1$

$$\mu_{A_1^-}(x) = \mu_{A_2^-}(h(x)),\ \mu_{A_1^+}(x) = \mu_{A_2^+}(h(x)) \tag{4}$$

and

$$\mu_{B_1^-}(xy) \le \mu_{B_2^-}(h(x)h(y)),\ \mu_{B_1^+}(xy) \le \mu_{B_2^+}(h(x)h(y)). \tag{5}$$

Let u in V_1 be a busy vertex. Then

$$\mu_{A_1^-}(u) \le d_1^-(u),\ \mu_{A_1^+}(u) \le d_1^+(u). \tag{6}$$

From (4) and (6) we have $\mu_{A_2^-}(h(u)) = \mu_{A_1^-}(u) \le d_1^-(u) = \sum_{v \ne u} \mu_{B_1^-}(uv) \le \sum_{v \ne u} \mu_{B_2^-}(h(u)h(v)) = d_2^-(h(u))$. Hence $\mu_{A_2^-}(h(u)) \le d_2^-(h(u))$. Also, $\mu_{A_2^+}(h(u))$

	a	b	c	d
μ_{A^-}	0.5	0.6	0.5	0.5
μ_{A^+}	0.7	0.7	0.6	0.5

	ab	bc	cd	da
μ_{B^-}	0.3	0.5	0.3	0.5
μ_{B^+}	0.5	0.5	0.5	0.5

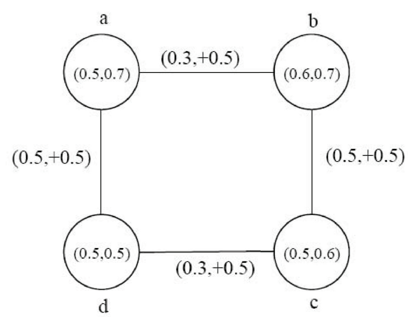

FIGURE 4. Regular interval-valued fuzzy graph G

$=\mu_{A_1^+}(u) \leq d_1^+(u) = \sum_{v\neq u} \mu_{B_1^+}(uv) \leq \sum_{v\neq u} \mu_{B_2^+}(h(u)h(v)) = d_2^+(h(u))$. Therefore $\mu_{A_2^+}(h(u)) \leq d_2^+(h(u))$. Hence $h(u)$ is a busy vertex in G_2. □

5. Regular interval-valued fuzzy graphs

In this section we introduce the concepts of regular and totally regular interval-valued fuzzy graphs and discusses some properties on them.

Definition 5.1. Let $G = (A, B)$ be an interval-valued fuzzy graph. If each vertex of G has same closed neighborhood degree $m = (m_1^*, m_2^*)$, then G is called m-totally regular interval-valued fuzzy graph. The closed neighborhood degree of a vertex x is defined by $deg[x] = (d^-[x], d^+[x])$, where $d^-[x] = d^-(x) + \mu_A^-(x)$, $d^+[x] = d^+(x) + \mu_A^+(x)$.

We show with the following examples that there is no relationship between n-regular interval-valued fuzzy graph and m-totally regular interval-valued fuzzy graph.

Example 5.2. Consider a graph G^* such that $V = \{a, b, c, d\}$, $E = \{ab, bc, cd, ad\}$. Let A be an interval-valued fuzzy subset of V and B be a interval-valued fuzzy subset of E defined by Routine computations show that $G = (A, B)$ is regular, but is not totally regular.

Example 5.3. Consider a graph G^* such that

$$V = \{v_1, v_2, v_3\}, E = \{v_1v_2, v_2v_3, v_3v_1\}.$$

Let A be an interval-valued fuzzy subsets of V and B be an interval-valued fuzzy subset of E defined by

$$\mu_{A^-}(v_1) = 0.4,\ \mu_{A^-}(v_2) = 0.4,\ \mu_{A^-}(v_3) = 0.4,$$
$$\mu_{A^+}(v_1) = +0.5,\ \mu_{A^+}(v_2) = +0.5,\ \mu_{A^+}(v_3) = 0.5,$$
$$\mu_{B^-}(v_1v_2) = 0.2,\ \mu_{B^-}(v_2v_3) = 0.2,\ \mu_{B^-}(v_3v_1) = 0.2,$$
$$\mu_{B^+}(v_1v_2) = +0.3,\ \mu_{B^+}(v_2v_3) = +0.3,\ \mu_{B^+}(v_3v_1) = 0.3.$$

Routine computations show that interval-valued fuzzy graph G is both regular and totally regular.

Proposition 5.4. *The size of a n-regular interval-valued fuzzy graph G is $\frac{nk}{2}$, where $|V| = k$.*

Proof. The size of G is $S(G) = \left(\sum_{xy\in E}\mu_B^-(xy), \sum_{xy\in E}\mu_B^+(xy)\right)$. Since G is n-regular, $d_G(v) = n$ for all $v \in V$. We have

$$\sum_{v\in V} d_G(v) = 2\left(\sum_{xy\in E}\mu_B^-(xy), \sum_{xy\in E}\mu_B^+(xy)\right).$$

So, $2S(G) = \sum_{v\in V} d_G(v) = \sum_{v\in V} n = nk$. Hence $S(G) = \frac{nk}{2}$. □

Proposition 5.5. *Let $G_1 \cong G_2$. Then*
(i) If G_1 is regular interval-valued fuzzy graph, then G_2 is also. (ii) if G_1 is totally regular interval-valued fuzzy graph, then G_2 is also.

Proof. Let $G_1 \cong G_2$ and G_1 is $n = (n_1, n_2)$-regular interval-valued fuzzy graph. Since, $\deg(x) = (d^-(x), d^+(x)) = \left(\sum_{x\neq y}\mu_{B^-}(xy), \sum_{x\neq y}\mu_{B^+}(xy)\right) = (n_1, n_2)$, we have

$$n_1 = d^-(x) = \sum_{x\neq y}\mu_{B^-}(xy) = \sum_{x\neq y}\mu_{B^-}(h(x)h(y)) = d^-(h(x)),$$
$$n_2 = d^+(x) = \sum_{x\neq y}\mu_{B^+}(xy) = \sum_{x\neq y}\mu_{B^+}(h(x)h(y)) = d^+(h(x)).$$

Thus G_2 is n regular interval-valued fuzzy graph.

Now let $G_1 \cong G_1$ and G_1 is $m = (m_1, m_2)$- totally regular interval-valued fuzzy graph. By Definition 5.1 we have, $deg[x] = (d^-[x], d^+[x])$, where $d^-[x] = d^-(x) + \mu_A^-(x)$ and $d^+[x] = d^+(x) + \mu_A^+(x)$. Therefore,

$$m_1 = d^-(x) + \mu_{A^-}(x) = d^-(h(x)) + \mu_{A^-}(h(x)) = d^-[h(x)],$$
$$m_2 = d^+(x) + \mu_{A^+}(x) = d^+(h(x)) + \mu_{A^+}(h(x)) = d^+[h(x)].$$

It follows that G_2 is m -totally regular interval-valued fuzzy graph. □

Theorem 5.6. *Let $G = (A, B)$ be an interval-valued fuzzy graph of a graph G^*. If $A = (\mu_{A^-}, \mu_{A^+})$ is a constant function, the following are equivalent:*

(a) G is a regular interval-valued fuzzy graph,

(b) G is a totally regular interval-valued fuzzy graph.

Proof. Suppose that $A = (\mu_{A^-}, \mu_{A^+})$ is a constant function and $\mu_{A^-}(x) = c_1$, $\mu_{A^+}(x) = c_2$ for all $x \in V$.
$(a) \Longrightarrow (b)$: Assume that G is a n-regular interval-valued fuzzy graph, then $d^-(x) = n^-$, $d^+(x) = n^+$ for all $x \in V$. So, $d^-[x] = d^-(x) + \mu_{A^-}(x) = n^- + c_1$, $d^+[x] = d^+(x) + \mu_{A^+}(x) = n^+ + c_2$ for all $x \in V$. Hence, G is a totally regular interval-valued fuzzy graph.
$(b) \Longrightarrow (a)$: Suppose that G is a totally regular interval-valued fuzzy graph, then $d^-[x] = k_1$, $d^+[x] = k_2$ for all $x \in V$ or $d^-(x) + \mu_{A^-}(x) = k_1$, $d^+(x) + \mu_{A^+}(x) = k_2$ or $d^-(x) + c_1 = k_1$, $d^+(x) + c_2 = k_2$ for all $x \in V$ or $d^-(x) = k_1 - c_1$, $d^+(x) = k_2 - c_2$ for all $x \in V$. Thus, G is a regular interval-valued fuzzy graph. □

Proposition 5.7. *If an interval-valued fuzzy graph G is both regular and totally regular, then $A = (\mu_{A^-}, \mu_{A^+})$ is constant function.*

Proof. Let G be a regular and totally regular interval-valued fuzzy graph, then $d^-(x) = n_1$, $d^+(x) = n_2$ for all $x \in V_1$, $d^-[x] = k_1$, $d^+[x] = k_2$ for all $x \in V_1$. Now we have

$$\begin{aligned} d^-[x] = k_1 \quad &\Longleftrightarrow \quad d^-(x) + \mu_{A^-}(x) = k_1 \Longleftrightarrow n_1 + \mu_{A^-}(x) = k_1 \\ &\Longleftrightarrow \quad \mu_{A^-}(x) = k_1 - n,\ for\ all\ x \in V_1. \end{aligned}$$

Similarly, we can show that, $\mu_{A^+}(x) = k_2 - n_2$ for all $x \in V$. Hence $A = (\mu_{A^-}, \mu_{A^+})$ is a constant function. □

Remark 5.1. Let $G = (A, B)$ be an interval-valued fuzzy graph where crisp graph G^* is the cycle $C : v_0, v_1, \cdots, v_n = v_0$. Then, we have the followings.
(i) If n is odd, G is regular if and only if B is a constant function.
(ii) If n is even, G is regular if and only if $\mu_{B^-}(v_{i-1}, v_i) = \mu_{B^-}(v_{i+1}, v_{i+2})$, $\mu_{B^+}(v_{i-1}, v_i) = \mu_{B^+}(v_{i+1}, v_{i+2})$, $1 \leq i \leq n$, which $i+1, i+2$ are in module n.

6. Conclusions

Graph theory is an extremely useful tool in solving the combinatorial problems in different areas including geometry, algebra, number theory, topology, operations research, optimization and computer science. The interval-valued fuzzy sets constitute a generalization of the notion of fuzzy sets. The interval-valued fuzzy models give more precision, flexibility and compatibility to the system as compared to the classical and fuzzy models. In this paper, we introduced the concepts of regular and totally regular interval-valued fuzzy graphs and discussed some properties of the μ-complement of interval-valued fuzzy graph. Self

μ-complementary interval-valued fuzzy graphs and self- weak μ-complementary interval-valued fuzzy graphs are defined.

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