J. Appl. Math. & Informatics Vol. **35**(2017), No. 1 - 2, pp. 75 - 82
https://doi.org/10.14317/jami.2017.075

ON THE SHAPE OF MAXIMUM CURVE OF e^{az^2+bz+c}

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Abstract. In this paper, we investigate the proper shape and location of the maximum curve of transcendental entire functions e^{az^2+bz+c}. We show that the alpha curve of e^{az^2+bz+c} is a subset of a rectangular hyperbola, and the maximum curve is the connected set originating from the origin as a subset of the alpha curve.

AMS Mathematics Subject Classification : 30D20.
Key words and phrases : beta curve, alpha curve, maximum curve of an entire function

1. Intorduction

For an entire function $f(z)$, we define the maximum curve of $f(z)$ by the set of all $z \in \mathbb{C}$ such that

$$|z| = r, \ |f(z)| = \max_{|\zeta|=r} |f(\zeta)| = M(r, f), \ \ r \geq 0.$$

Our concerns are finding the proper shape and location of the maximum curve of the function $f(z) = e^{az^2+bz+c}$.

We begin with two known results related to the maximum curve of $f(z)$. W. K. Hayman found the number of candidates for the maximum curve of $e^{p(z)}$ near the origin.

Theorem 1 ([1]). *Suppose that*

$$f(z) = 1 + a_k z^k + \cdots, \quad (a_k \neq 0)$$

is analytic at $z = 0$. Then, for some $\epsilon > 0$, the points z with $|z| \leq \epsilon$, such that $|z| = \rho$, $|f(z)| = M(\rho, f)$, form at most k regular arcs, which make angles of $\frac{2p\pi}{k}$ with each other at $z = 0$, where p is a positive integer.

Received August 8, 2016. Revised October 31, 2016. Accepted November 3, 2016.
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If $p(z)$ is a polynomial of degree two, then we may write

$$f(z) = e^{p(z)} = a_0 + a_k z^k + \cdots, \quad (k = 1 \text{ or } 2).$$

So the function $f(z) = e^{p(z)}$ has at most two maximum curves starting from the origin.

To state the second known result, we let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad (a_n \neq 0,\ n \geq 1),$$

where

$$a_k = s_k e^{i\alpha_k},\ z = re^{i\theta}, \quad (k = 0, 1, 2, \cdots, n).$$

We write

$$\tau_j = -\frac{\alpha_n}{n} + (2j-1)\frac{\pi}{2n} \text{ and } L_j = \{re^{i\tau_j} : r > 0\},$$

where $j = 0, 1, 2, \cdots, 2n-1$. We divide the complex plane into $2n$ open sectors

$$S_j := \{z : \tau_j < \text{Arg } z < \tau_{j+1}\}, \quad (j = 0, 1, 2, \cdots, 2n-1)$$

sharing the same vertex at the origin.

From the following theorem, we can guess the location of the maximum curve of $e^{p(z)}$.

Theorem 2 ([2]). *The function $f(z) = e^{p(z)}$ has radial limits on each sector S_j:*

$$\lim_{\substack{|z|=r\to\infty \\ z\in S_j}} |f(z)| = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \infty & \text{if } j \text{ is even.} \end{cases}$$

Furthermore, the limits are uniform on any closed subsector of S_j.

The above theorem says that radial limits tend to infinity on some sectors and that sectors are determined by the argument of the leading coefficient of $p(z)$. From Theorem 2, we may assume that maximum curves of $e^{p(z)}$ are located in some sectors S_{2j} for sufficiently large r.

Previous two theorems give us rough and limited information on the maximum curve of $e^{p(z)}$ near the origin and the infinity. In this paper we study entire shape and proper location of the maximum curve of $e^{p(z)}$, where $p(z)$ is a polynomial of degree two.

2. Beta curve and alpha curve of $e^{p(z)}$

For an entire function $f(z)$, we define a new function

$$A(z) = z\frac{f'(z)}{f(z)}$$

as T. F. Tyler did.([3]) Using the polar form of the Cauchy-Riemann equations, we obtain

$$A(z) = r\frac{\partial}{\partial r}\log\left|f(re^{i\theta})\right| - i\frac{\frac{\partial}{\partial\theta}\left|f(re^{i\theta})\right|}{\left|f(re^{i\theta})\right|}.$$

We follow Tyler's phrase again.

Definition 3. We call the curve where $\frac{\partial}{\partial\theta}\log\left|f(re^{i\theta})\right| = 0$ *the beta curve* of $f(z)$, and those parts of the beta curve where $r\frac{\partial}{\partial r}\log\left|f(re^{i\theta})\right|$ is positive will be called *the alpha curve* of $f(z)$.

From the definition we know that the alpha curve is a subset of beta curve, and the maximum curve is a subset of alpha curve.

Since

$$\frac{\partial}{\partial\theta}|f(re^{i\theta})| = \frac{\partial}{\partial\theta}[e^{\mathrm{Re}\,p(re^{i\theta})}] = \frac{\partial}{\partial\theta}[\mathrm{Re}\,p(re^{i\theta})]\cdot e^{\mathrm{Re}\,p(re^{i\theta})}$$

and $e^{\mathrm{Re}\,p(re^{i\theta})} \neq 0$, the beta curve of $e^{p(z)}$ is the set of all points $z = re^{i\theta}$ such that

$$\frac{\partial}{\partial\theta}[\mathrm{Re}\,p(re^{i\theta})] = 0.$$

We set

$$\begin{aligned} p(z) &= az^2 + bz + c \\ &= (a_1 + a_2 i)z^2 + (b_1 + b_2 i)z + (c_1 + c_2 i), \end{aligned}$$

where $a \neq 0$ and a_j, b_j, c_j $(j = 1, 2)$ are real numbers.

Theorem 4. *The beta curve of $f(z) = e^{p(z)}$ is a hyperbola.*

Proof. Let $z = re^{i\theta} = x + iy$. Then we have

$$\begin{aligned} \frac{\partial}{\partial\theta}\left[\mathrm{Re}\,p(re^{i\theta})\right] &= \frac{\partial}{\partial\theta}[a_1 r^2(\cos^2\theta - \sin^2\theta) - 2a_2 r^2\cos\theta\sin\theta \\ &\quad + b_1 r\cos\theta - b_2 r\sin\theta + c_1] \\ &= -2a_1 r^2\sin 2\theta - 2a_2 r^2\cos 2\theta - b_1 r\sin\theta - b_2 r\cos\theta \\ &= -\left[4a_1 xy + 2a_2(x^2 - y^2) + b_1 y + b_2 x\right]. \end{aligned}$$

So the beta curve can be written as a quadratic equation

$$4a_1 xy + 2a_2(x^2 - y^2) + b_1 y + b_2 x = 0. \tag{1}$$

Since we assumed $a = a_1 + a_2 i \neq 0$, the discriminant

$$D = \begin{vmatrix} 2a_2 & 2a_1 \\ 2a_1 & -2a_2 \end{vmatrix} = -4({a_1}^2 + {a_2}^2)$$

of the quadratic equation (1) always has negative value. Hence the beta curve of $f(z) = e^{p(z)}$ is a hyperbola. □

Here we state some properties of the beta curve of $e^{p(z)}$. We denote the center of the beta curve by O' in the plane. The coordinate of the center O' is given by

$$O' = (x_c,\, y_c) = \left(-\frac{a_1 b_1 + a_2 b_2}{4({a_1}^2 + {a_2}^2)}, -\frac{a_1 b_2 - a_2 b_1}{4({a_1}^2 + {a_2}^2)} \right).$$

And the beta curve of $e^{p(z)}$ is a rectangular(equilateral) hyperbola.

A quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$$

is said to be degenerated if it is a product of two linear equations. In this paper we do not consider the degenerated case which is relatively simple.

The alpha curve of $f(z)$ is a subset of the beta curve satisfying

$$r\frac{\partial}{\partial r}\log|f(re^{i\theta})| > 0.$$

In other words, the alpha curve of $f(z)$ is the set of all points $z = re^{i\theta}$ such that

$$\frac{\partial}{\partial \theta}|f(re^{i\theta})| = 0 \text{ and } r\frac{\partial}{\partial r}\log|f(re^{i\theta})| > 0.$$

From Theorem 4, we knew that the beta curve of $e^{p(z)}$ is a hyperbola of the form

$$4a_1 xy + 2a_2(x^2 - y^2) + b_1 y + b_2 x = 0.$$

And since

$$\begin{aligned} \log|f(re^{i\theta})| &= \log|e^{p(re^{i\theta})}| = \operatorname{Re} p(re^{i\theta}) \\ &= a_1 r^2 \cos 2\theta - a_2 r^2 \sin 2\theta + b_1 r \cos\theta - b_2 r \sin\theta + c_1, \end{aligned}$$

we obtain

$$\begin{aligned} r\frac{\partial}{\partial r}\log|f(re^{i\theta})| &= 2a_1 r^2 \cos 2\theta - 2a_2 r^2 \sin 2\theta + b_1 r \cos\theta - b_2 r \sin\theta \\ &= 2a_1(x^2 - y^2) - 4a_2 xy + b_1 x - b_2 y. \end{aligned}$$

Hence the alpha curve of $e^{p(z)}$ is a subset of the beta curve lying inside the region

$$2a_1(x^2 - y^2) - 4a_2 xy + b_1 x - b_2 y > 0. \tag{2}$$

The boundary of the region (2),

$$r\frac{\partial}{\partial r}\operatorname{Re} p(re^{i\theta}) = 2a_1(x^2 - y^2) - 4a_2 xy + b_1 x - b_2 y = 0 \tag{3}$$

is also a rectangular hyperbola. The beta curve of $e^{p(z)}$ and the boundary (3) of the region (2) share the same center $(x_c,\, y_c)$, and the hyperbola (3) also passes through the origin. The beta curve of $e^{p(z)}$ and the boundary for the alpha curve of $e^{p(z)}$ meet only at two points, the origin O and the point $Q = (2x_c,\, 2y_c)$, where $O' = (x_c,\, y_c)$. Hence in any case, $e^{p(z)}$ has two alpha curves, one starts from the origin and the other which is symmetric to the former with respect to O' starts from the point Q. Both of alpha curves eventually tend to infinity.

The beta curve of $e^{p(z)}$,

$$4a_1xy + 2a_2(x^2 - y^2) + b_1y + b_2x = 0$$

meets the coordinate axis at three points (0, 0), $(0,\, \frac{b_1}{2a_2})$, $(-\frac{b_2}{2a_2},\, 0)$. From the above arguments, we can determine which part of the beta curve is the alpha curve of the given function $e^{p(z)}$. If the function

$$A(x,\, y) := 2a_1(x^2 - y^2) - 4a_2xy + b_1x - b_2y \tag{4}$$

has positive sign at $(0,\, \frac{b_1}{2a_2})$ (or $(-\frac{b_2}{2a_2},\, 0)$), then the subset of beta curve starting from O or Q containing $(0,\, \frac{b_1}{2a_2})$ (or $(-\frac{b_2}{2a_2},\, 0)$) is the alpha curve of $e^{p(z)}$.

Lemma 5. *Suppose that (x, y) and (x', y') are symmetric w.r.t $O' = (x_c, y_c)$. Then*

$$q(\hat{x}, \hat{y}) := Re\, p(x + iy) - Re\, p(x' + iy') = b_1\hat{x} - b_2\hat{y}$$

where $\hat{x} = x - x_c,\ \hat{y} = y - y_c$.

Proof. Since

$$\begin{aligned} \operatorname{Re} p(x + iy) =& \operatorname{Re} p((\hat{x} + x_c) + i(\hat{y} + y_c)) \\ =& a_1(\hat{x}^2 - \hat{y}^2) - 2a_2\hat{x}\hat{y} + \frac{b_1}{2}\hat{x} - \frac{b_2}{2}\hat{y} + K, \end{aligned}$$

we get

$$q(\hat{x}, \hat{y}) = b_1\hat{x} - b_2\hat{y},$$

where $K = c_1 - \frac{3a_1(b_1^2 - b_2^2) - 6a_2b_1b_2}{16(a_1^2 + a_2^2)}$. □

Let Ω be the set

$$\Omega := \{(\hat{x}, \hat{y}) : q(\hat{x}, \hat{y}) > 0\} = \{(x, y) : b_1(x - x_c) - b_2(y - y_c) > 0\} \tag{5}$$

in the plane and let L be the line

$$L:\ b_1\hat{x} - b_2\hat{y} = 0. \tag{6}$$

The beta curve passes through $(-\frac{b_2}{2a_2}, 0)$ and $(0, \frac{b_1}{2a_2})$. So the line L is parallel to the line passing through $(-\frac{b_2}{2a_2}, 0)$ and $(0, \frac{b_1}{2a_2})$. And if $(\hat{x}, \hat{y}) \in \Omega$, then $(-\hat{x}, -\hat{y}) \in \Omega^c$.

3. Maximum curve of e^{az^2+bz+c}

We call the arm of the beta curve of $e^{p(z)}$ passing through the origin curve A and the other arm curve B. We can determine the alpha curve by checking the sign of $A(x, y)$ at a proper point, where $A(x, y)$ is the same function as in (4).

In any case, the alpha curve consist of two separated curves, one starts from the origin and the other one starts from the point $Q = (2x_c,\, 2y_c)$, where $O' = (x_c,\, y_c)$. We divide the beta curve of $e^{p(z)}$ into four pieces by the origin and the point Q. We call each of four pieces as follows:

A_α: alpha curve originating from the origin ($A_\alpha \subset A$)
B_α: alpha curve originating from the point Q ($B_\alpha \subset B$)
A_β: rest of curve after excluding A_α from A (includes the origin)
B_β: rest of curve after excluding B_α from B (includes the point Q)

The curve A_α and $B_\alpha(A_\beta$ and $B_\beta)$ are symmetric with respect to the center O' of the beta curve.

From the definition of the beta curve, every point of the beta curve on the circle $|z| = r\,(r > 0)$ is a critical point of

$$\mathrm{Re}\, p(re^{i\theta}), \quad \theta \in [0,\, 2\pi]$$

where $r > 0$ is fixed. And $|f(z)| = e^{\mathrm{Re}\, p(z)}$ increases along the alpha curve A_α and B_α, and $|f(z)|$ decreases along the curve A_β and B_β as $r = |z|$ grows.

Lemma 6. *Suppose that P_1 is a point on the curve B_α and P_2 is the point that is symmetric to P_1 w.r.t O'. Then*

$$\overline{OP_1} > \overline{OP_2}.$$

Proof. The beta curve is rectangular hyperbola and two points $O,\ P_2$ are an the curve A_α, so $\angle OO'P_2 < \frac{\pi}{2}$. And since $\overline{O'P_1} = \overline{O'P_2}$, we have $\overline{OP_1} > \overline{OP_2}$. □

Now we state and prove the main Theorem. Here we prove the case, all of real and imaginary parts of a, b are positive. Other cases can be proved by slight modifications.

The beta curve passes through the point $(0, \frac{b_1}{2b_2})$ and since $x_c < 0$,

$$A(0, \frac{b_1}{2b_2}) = 2(a_1^2 + a_2^2)\frac{b_1}{a_2^2}x_c < 0.$$

If the curve A passes $(0, \frac{b_1}{2b_2})$, then A_α is in the fourth quadrant. If the curve A does not pass the point, then B_β passes $(0, \frac{b_1}{2b_2})$. So in any case A_α is in the fourth quadrant. And the line L has positive slope, where L is the line as in (6).

Theorem 7. *The maximum curve of $e^{p(z)}$ is A_α, the alpha curve originating from the origin.*

Proof. Let C_r be the circle $|z| = r(r > 0)$. And suppose that C_r intersects with A_α at z_r. If the circle C_r meets the curve B_α at z'_r, then it is enough to show that $|f(z_r)| > |f(z'_r)|$, since the maximum curve is a subset of the alpha curve. To show the curve A_α is the maximum curve, we consider two cases.

Case 1. $y_c > 0$.

In this case the line L, where L is the line as in (6), does not meet A_α and each point on A_α belongs to

$$\Omega = \{(x, y) : (y - y_c) < \frac{b_1}{b_2}(x - x_c)\}.$$

If $r < \overline{OQ}$, then C_r does not meet B_α. Hence

$$\max_{|z|=r} |f(z)| = |f(z_r)|.$$

If $r \geq \overline{OQ}$, then C_r intersects with B_α at one point, z'_r. Since $O \in \Omega$ and $|z'_r| \geq \overline{OQ}$, $Q \in \Omega^c$ and $z'_r \in \Omega^c$. Let z''_r be a point that is symmetric to z'_r w.r.t O'. Then z''_r is on A_α and $z''_r \in \Omega$. Since $|z''_r| < |z'_r| = |z_r|$ by Lemma 6, we have

$$|\mathrm{Re}(p(z'_r))| < |\mathrm{Re}(p(z''_r))| < |\mathrm{Re}(p(z_r))|$$

and

$$\max_{|z|=r} |f(z)| = |f(z_r)|.$$

Case 2. $y_c \leq 0$.

If A_α lies inside Ω, then it is the same case as in the above. Suppose that the line L intersects with A_α at S. Let V be a vertex of the beta curve. From lines of computations, we have

$$(2\overline{O'V})^2 - \overline{OS}^2 = \frac{b_2(a_1b_1 + a_2b_2) + b_1(a_1b_2 - a_2b_1)}{4(a_1^2 + a_2^2)^{3/2}} > 0.$$

The last inequality holds since $y_c \leq 0$. And since $\overline{OS} < 2\overline{O'V} < d((0,0), B)$, if $r \leq \overline{OS}$, then C_r does not meet the curve B_α, where $d((0,0), B)$ is the distance between the origin and the curve B.

If C_r does not meet B_α, then, we have

$$\max_{|z|=r} |f(z)| = |f(z_r)|.$$

Suppose C_r intersects with B_α at z'_r. Let z''_r be the point on B_α that is symmetric to z_r w.r.t O'. Since $|z_r| > \overline{OS}$, $z_r \in \Omega$ and $z''_r \in \Omega^c$. And $|z_r| = |z'_r| < |z''_r|$ by Lemme 6. So we have

$$\mathrm{Re}(p(z'_r)) < \mathrm{Re}(p(z''_r)) < \mathrm{Re}(p(z_r))$$

Hence

$$|f(z'_r)| < |f(z''_r)| < |f(z_r)|$$

and

$$\max_{|z|=r} |f(z)| = |f(z_r)|.$$

This completes the proof. □

The minimum curve of $f(z)$ is defined by the set of all $z \in \mathbb{C}$ such that

$$|z| = r, \ |f(z)| = \min_{|\zeta|=r} |f(\zeta)|, \ \ r \geq 0.$$

With similar arguments as in the proof of Theorem 7, we have the following result.

Theorem 8. *The minimum curve of* $e^{p(z)}$ *is* A_β*, the beta curve originating from the origin.*

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