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RADIO NUMBER OF TRANSFORMATION GRAPHS OF A PATH

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ABSTRACT. A radio labeling of a graph G is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ with the property that $| f(u) - f(v) | \geq 1 + diam(G) - d(u, v)$ for every pair of vertices $u, v \in V(G)$, where $diam(G)$ and $d(u, v)$ are diameter and distance between u and v in the graph G respectively. The radio number of a graph G, denoted by $rn(G)$, is the smallest integer k such that G admits a radio labeling. In this paper, we completely determine radio number of all transformation graphs of a path.

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1. Introduction

All graphs in this paper are finite, simple, connected, and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d_G(u, v)$ or simply $d(u, v)$. The maximum of distance between any two vertices in G is called the diameter of G and is denoted by $diam(G)$.

A labeling of a connected graph G is an injection $f : V(G) \rightarrow Z^+$, while a radio labeling is a labeling with an additional condition that $|f(u) - f(v)| \geq 1 + diam(G) - d(u, v)$ for every pair of vertices $u, v \in V(G)$. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* $rn(G)$ of G is $min\{rn(f)\}$ over all radio labelings f of G. A radio labeling f of G is a *minimal radio labeling* of G if $rn(f) = rn(G)$.

Radio labeling is motivated by the channel assignment problem introduced by W.K. Hale et al [8] in 1980. The radio labeling of a graph is most useful in FM radio channel restrictions to overcome from the effect of noise. This problem turns out to find the minimum of maximum frequencies of all the radio stations considered under the network.

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The notion of radio labeling was introduced by G. Chartrand, David Erwin, Ping Zhang, and F. Harary in [5]. Since the introduction of radio labeling, several authors investigated the radio number of various networks [17, 9, 11, 6, 10, 12].

In 2005, Daphne Der-Fen Liu and Xuding Zhu [11] completely determined the radio numbers for paths and cycles. The results of D. D. F. Liu generalizes the radio number for paths obtained in [11]. Further D.D.F. Liu and M. Xie obtained radio labeling of square of paths in [10]. The results of [10] is now completely generalised for any k^{th} power of a path by P. Devadasa Rao, B. Sooryanarayana, and Chandru Hegde in [3, 7] for any $k \in Z^+$. In 2013, S.K.Vaidya and D.D.Bantva [13] determined the radio number of total graph of a path. The total graph is a particular case of transformation graph as well as square of a path [11]. In this paper, we completely determine the radio number of all transformation graphs of paths.

In 2001, Wu and Meng [14] introduced some new graphical transformations which generalize the concept of total graph. In 2005, a particular case when $xyz = -++$ was studied by Baoyindureng Wu, Li Zhang, and Zhao Zhang [15] and in the year 2008, the transformation graph G^{-+-} studied by Lan Xu and Baoyindureng Wu[16]. The transformation graph G^{xyz} is defined as follows[16].

2. Main results

Let $G = (V, E)$ be a finite and simple graph and α, β be two elements of $V(G) \cup E(G)$. Then associativity of α and β is taken as $+$ if they are adjacent or incident in G, otherwise $-$. Let xyz be a 3-permutation of the set $\{+, -\}$. The pair α and β is said to correspond to x or y or z of xyz if α and β are both in $V(G)$ or both are in $E(G)$, or one is in $V(G)$ and the other is in $E(G)$. The transformation graph G^{xyz} of G is the graph whose vertex set is $V(G) \cup E(G)$ two of its vertices α and β adjacent if and only if their associativity in G is consistent with the corresponding element of xyz.

There are eight transformation graphs of G corresponding to eight distinct 3-permutations of $\{+, -\}$. In particular, G^{+++} is exactly the total graph $T(G)$ of G and G^{---} is the complement of $T(G)$. The other six graphs G^{++-} and G^{--+}; G^{+-+} and G^{-+-}; and G^{-++} and G^{+--} forms three pairs of complementary graphs.

It follows immediately by the definition of a radio labeling that $rn(G) \geq |V(G)|$ for every graph G.

We recall the following results for immediate reference;

Theorem 2.1 ([11])**.** *For any integer $n \geq 4$,*

$$rn(P_n) = \begin{cases} 2k^2 + 3, & \text{if} \quad n = 2k+1 \\ 2k(k-1) + 2, & \text{if} \quad n = 2k \end{cases}$$

Theorem 2.2 ([4])**.** *For any $n \in Z^+$,*

$$rn(P_n) \leq \begin{cases} 2k^2 + k + 1, & \text{if} \quad n = 2k+1 \\ 2(k^2 - k) + 2, & \text{if} \quad n = 2k \end{cases}$$

Moreover, the bound is sharp when $n \leq 5$.

Theorem 2.3 ([4]). *Let C_n be the n-vertex cycle, $n \geq 3$. Then*

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 2, & if \quad n \equiv 0, 2(mod\ 4) \\ \frac{n-1}{2}\phi(n) + 1, & if \quad n \equiv 1, 3(mod\ 4) \end{cases}$$

where $\phi(n) = \begin{cases} k+1, & if\ n = 4k+1 \\ k+2, & if\ n = k + r\ for\ some\ r = 0, 2, 3 \end{cases}$

Theorem 2.4 ([10]). *Let P_n^2 be a square path on n vertices and let $k = \lfloor \frac{n}{2} \rfloor$. Then*

$$rn(P_n^2) = \begin{cases} k^2 + 2, & if\ n \equiv 1(mod\ 4),\ and\ n \geq 9 \\ k^2 + 1, & otherwise \end{cases}$$

In the next section of this paper, we completely determine the radio number of all the transformation graphs except the total graph which is covered in the Theorem 2.4. Our main results in the next sections are:

Theorem 2.5. *For any integer $n \geq 2$,* $rn(P_n^{+-+}) = \begin{cases} 3n-3, & if \quad n = 2, 3 \\ 3n, & if \quad 4 \leq n \leq 6 \\ 3n-2, & if \quad n \geq 7 \end{cases}$

Theorem 2.6. *For any integer $n \geq 2$,* $rn(P_n^{--+}) = \begin{cases} 2n, & if \quad n = 2 \\ 2n-1, & if \quad n \geq 3 \end{cases}$

Theorem 2.7. *For any integer $n \geq 3$, $rn(P_n^{++-}) = 2n - 1$.*

Theorem 2.8. *For any integer $n \geq 2$,* $rn(P_n^{-++}) = \begin{cases} 2n, & if \quad n = 2 \\ 2n-1, & if \quad n = 3, 4 \\ 3n+1, & if \quad n = 5, 6 \\ 3n, & if \quad n = 7 \\ 3n-1, & if \quad n \geq 8 \end{cases}$

Theorem 2.9. *For any integer $n \geq 3$,* $rn(P_n^{+--}) = \begin{cases} 4n-1, & if \quad n = 3 \\ 2n-1, & if \quad n \geq 4 \end{cases}$

Theorem 2.10. *For any integer $n \geq 4$,* $rn(P_n^{-+-}) = \begin{cases} 3n+1, & if \quad n = 4 \\ 2n, & if \quad n = 5 \\ 2n-1, & if \quad n \geq 6 \end{cases}$

Theorem 2.11. *For any integer $n \geq 4$,* $rn(P_n^{---}) = \begin{cases} 3n, & if \quad n = 4 \\ 2n-1, & if \quad n \geq 5 \end{cases}$

Remark 2.1. For those values of n, not indicated in the above theorems, the graph G is disconnected or trivial.

Throughout this paper, let $v_0, v_1, v_2, \ldots, v_{n-1}$ denote the vertices of the path P_n with the edges $e_i = v_{i-1}v_i$ for each i, $1 \leq i \leq n-1$.

For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3. Diameter and Lower bounds

For each of main theorems stated in the previous section, we now establish the lower bounds for their radio number.

3.1. For $xyz = + - +$.

Lemma 3.1. *For any integer $n \geq 2$,*

$$diam(P_n^{xyz}) = \begin{cases} n-1, & if \quad n = 2, 3 \\ 3, & if \quad n \geq 4 \end{cases}$$

Proof. Let $G = P_n^{+-+}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| = 1$.
(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| > 1$.
(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j = i - 1$ or $j = i$.

Now the cases $n = 2$ and $n = 3$ are obvious. Consider $n \geq 4$ and let u and v be any vertices of G. If $u = v_i$ and $v = v_j$ (without loss of generality, we take $i \leq j$), then $d_G(v_i, v_j) = 1$ if $|i - j| = 1$ (since v_i and v_j are adjacent in this case), $d_G(v_i, v_j) = 2$ if $|i - j| = 2$ (since $v_i - v_{i+1} - v_{i+2} = v_j$ is a shortest path), and $d_G(v_i, v_j) = 3$ otherwise (since $v_i - e_{i+1} - e_j - v_j$ is a shortest path), so $d_G(v_i, v_j) \leq 3$. Similarly, if $u = e_i$ (or $u = v_i$) and $v = e_j$ with $i \leq j$, then we observe that $d_G(e_i, e_j) \leq 2$ and $d_G(v_i, e_j) \leq 2$. Thus $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 3$. □

Lemma 3.2. *For any integer $n \geq 2$, $rn(P_n^{xyz}) \geq \begin{cases} 3, & if \quad n = 2 \\ 6, & if \quad n = 3 \\ 3n, & if \quad n = 4, 5, 6 \\ 3n - 2, & if \quad n \geq 7 \end{cases}$*

Proof. Let $G = P_n^{+-+}$ and f be any radio labeling of G. We first prove the lemma for the case $n \geq 7$. In this case, as $d_G(e_i, e_j) \leq 2$ and $diam(G) = 3$, $|f(e_i) - f(e_j)| \geq 2$ (since f can assign two consecutive integers only for the diametrically opposite vertices). But then as G has $n - 1$ vertices that are the edges of P_n, to label $n - 1$ vertices $e_1, e_2, \ldots, e_{n-1}$, we require at least $(n-1)+(n-2)$ integers. Further as $d(e_i, v_j) \leq 2$, f cannot assign two consecutive integers for a vertex and an edge of P_n in G. So, f should leave at least one integer at this stage. Thus to label n vertices and $n - 1$ edges of P_n in G, f requires at least $(n) + (1) + ((n - 1) + (n - 2)) = 3n - 2$ integers. Hence $rn(P_n^{+-+}) = \max\{f(v) : v \in V(G)\} \geq 3n - 2$ for $n \geq 7$. However, to label n vertices of P_n in G we require more than n integers whenever $2 < n \leq 6$. If $n = 2$, then the result follows immediately as $rn(G) \geq |V(G)| = 3$. When $n = 3$, the vertex v_1 is adjacent to every other vertex u in G and hence $|f(u) - f(v_2)| \geq$

2. Therefore, f should leave at least one integer while labeling v_1 and hence $rn(G) \geq |V(G)| + 1 = 5 + 1 = 6$.

Let us relabel the vertices of G as $x_1, x_2, x_3, \ldots, x_n$ such that $f(x_i) < f(x_{i+1})$ for each $i, 1 \leq i \leq n$. Now in the case $n = 4$, for at most one i, $d(x_i, x_{i+1}) = diam(G)$ and hence $f(x_{i+1}) - f(x_i) \geq 2$ except for one $i \leq 7$. Therefore, $f(x_7) = [f(x_7) - f(x_6)] + [f(x_6) - f(x_5)] + \cdots + [f(x_2) - f(x_1)] + f(x_1) \geq 2 \times 5 + 1 + f(x_1) = 2 \times 5 + 1 + 1 = 12$. Similarly, for the cases $n = 5$ and $n = 6$, there are at most 2 and 3 possible values for i, respectively, such that $d(x_i, x_{i+1}) = diam(G)$. Hence in these cases, respectively, f requires at least $2 \times 6 + 1 \times 2 + 1 = 15$ and $2 \times 7 + 1 \times 3 + 1 = 18$ integers. □

3.2. For $xyz = --+$.

Lemma 3.3. *For any integer $n \geq 2$, $diam(P_n^{xyz}) = 2$.*

Proof. Let $G = P_n^{--+}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| > 1$.

(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| > 1$.

(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j = i - 1$ or $j = i$.

Let u and v be any vertices of G. If $u = v_i$ and $v = v_j$ (without loss of generality, we take $i \leq j$), then $d_G(v_i, v_j) = 1$ if $|i - j| > 1$ (since v_i and v_j are adjacent in this case), and $d_G(v_i, v_j) = 2$ if $|i - j| = 1$ (since $v_i - e_{i+1} - v_j$ is a shortest path). Hence $d_G(v_i, v_j) \leq 2$. Similarly, we observe that $d_G(e_i, e_j) \leq 2$ and $d_G(v_i, e_j) \leq 2$. Thus $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 2$. □

Lemma 3.4. *For any positive integer $n \geq 4$, $rn(P_n^{xyz}) \geq 2n - 1$.*

Proof. A direct consequence of $rn(G) \geq |V(G)|$. □

3.3. For $xyz = ++-$.

Lemma 3.5. *For any integer $n \geq 3$, $diam(P_n^{xyz}) = 2$.*

Proof. Let $G = P_n^{++-}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| = 1$.

(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| = 1$.

(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j \neq i - 1$ or $j \neq i$.

Let u and v be any vertices of G. If $u = v_i$ and $v = v_j$ (without loss of generality, we take $i \leq j$), then $d_G(v_i, v_j) = 1$ if $|i - j| = 1$ (since v_i and v_j are adjacent in this case) and $d_G(v_i, v_j) = 2$ if $|i - j| > 1$ (since $v_i - e_{i+2} - v_j$ is a shortest path if $|i - j| > 3$ and $v_i - v_{i+1} - v_j$ is a shortest path if $|i - j| = 2$). Hence

$d_G(v_i, v_j) \leq 2$. Similarly, we observe that $d_G(e_i, e_j) \leq 2$, and $d_G(v_i, e_j) \leq 2$. Thus $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 2$. □

Lemma 3.6. *For any integer $n \geq 4$, $rn(P_n^{xyz}) \geq 2n - 1$.*

Proof. A direct consequence of $rn(G) \geq |V(G)|$. □

3.4. For $xyz = -++$.

Lemma 3.7. *For any integer $n \geq 2$,* $diam(P_n^{xyz}) = \begin{cases} 2, & \text{if} \quad n = 2, 3, 4 \\ 3, & \text{if} \quad n \geq 5 \end{cases}$

Proof. Let $G = P_n^{-++}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| > 1$.
(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| = 1$.
(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j = i - 1$ or $j = i$.

The result follows immediately for the cases $n = 2$, $n = 3$, and $n = 4$ by Figure 1.

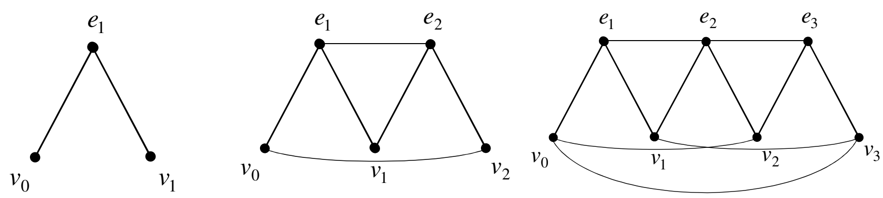

FIGURE 1. The Transformation graphs P_n^{-++} for the cases $n = 2, 3, 4$.

Consider $n \geq 5$ and let u and v be any vertices of G. If $u = v_i$ and $v = v_j$ (without loss of generality, we take $i \leq j$), then $d_G(v_i, v_j) = 1$ if $|i - j| > 1$ (since v_i and v_j are adjacent in this case) $d_G(v_i, v_j) = 2$ if $|i - j| = 1$ (since $v_i - e_{i+1} - v_j$ is a shortest path), so $d_G(v_i, v_j) \leq 2$.

If $u = e_i$ and $v = e_j$ with $i \leq j$ we observe that $d_G(e_i, e_j) = 1$ if $j = i + 1$, $d_G(e_i, e_j) = 2$ if $j = i + 2$ (since $e_i - e_{i+1} - e_j$ is a shortest path in this case), and $d_G(e_i, e_j) = 3$ if $j > i + 2$ (since $e_i - v_i - v_{j-1} - e_j$ is a shortest path in this case). So, $d_G(e_i, e_j) \leq 3$.

If $u = v_i$ and $v = e_j$, then we observe that $d_G(v_i, e_j) = 1$ if $|i - j| \leq 1$, and $d_G(v_i, e_j) = 2$ if $|j - i| > 1$ (since $v_i - v_{j-1} - e_j$ is a shortest path if $j > i + 2$ and $v_i - v_j - e_j$ is a shortest path if $i > j + 2$).

Thus $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 3$. □

Lemma 3.8. *For any integer $n \geq 3$,* $rn(P_n^{xyz}) \geq \begin{cases} 2n - 1 & \text{if} \quad n = 3, 4 \\ 3n + 1 & \text{if} \quad n = 5, 6, 7 \\ 3n - 1 & \text{if} \quad n \geq 8 \end{cases}$

Proof. Let $G = P_n^{-++}$ and f be a radio labeling of G. For $n = 3, 4$, result follows by the fact that $rn(G) \geq |V(G)|$. Let us relabel the vertices of G as $x_1, x_2, x_3, \ldots, x_n$ such that $f(x_i) < f(x_{i+1})$ for each $i, 1 \leq i \leq n$. Now in the case $n = 5$, for at most one i, $d(x_i, x_{i+1}) = diam(G) = 3$ (only the possibility is $x_i = e_1$ and $x_{i+1} = e_4$) and hence $f(x_{i+1}) - f(x_i) \geq 2$ except for one $i \leq 9$. Therefore, $f(x_9) = [f(x_9) - f(x_8)] + [f(x_8) - f(x_7)] + \cdots + [f(x_2) - f(x_1)] + f(x_1) \geq 2 \times 7 + 1 + f(x_1) = 2 \times 7 + 1 + 1 = 16$.

Similarly, for the case $n = 6$, for at most two i, $d(x_i, x_{i+1}) = diam(G) = 3$ (the possibilities are $(x_i, x_{i+1}) = (e_1, e_4), (e_2, e_5)$) and hence $f(x_{i+1}) - f(x_i) \geq 2$ except for two $i \leq 11$. Therefore, $f(x_{11}) = [f(x_{11}) - f(x_{10})] + [f(x_{10}) - f(x_9)] + \cdots + [f(x_2) - f(x_1)] + f(x_1) \geq 2 \times 8 + 1 \times 2 + f(x_1) = 2 \times 8 + 2 + 1 = 19$.

And for case $n = 7$, for at most three i, $d(x_i, x_{i+1}) = diam(G) = 3$ and hence $f(x_{13}) \geq 2 \times 9 + 1 \times 3 + 1 = 22$.

We now consider the case $n \geq 8$. Since the diameter of G is 3, $d_G(v_i, v_j) \leq 2$ and $d_G(e_l, v_m) \leq 2$, for all possible i, j, l, m, we see that $|f(v_i) - f(v_j)| \geq 2$ and $|f(e_l) - f(v_m)| \geq 2$. Therefore, it follows that no consecutive integers can be assigned for two distinct vertices or an edge and a vertex of P_n in G and hence $f(x_{2n-1}) > 1 \times |E(P_n)| + 2 \times (|V(P_n)| - 1) + 1$ (here the last 1 is the minimum requirement for labeling a vertex after labeling an edge or vice versa)$\Rightarrow rn(G) = f(x_{2n-1}) > (n-1) + 2 \times (n-1) + 1 = 3n - 2 \Rightarrow rn(G) \geq 3n - 1$. $\square$

3.5. For $xyz = + - -$.

Lemma 3.9. *For any integer $n \geq 3$, $diam(P_n^{xyz}) = \begin{cases} 4, & \text{if } n = 3 \\ 2, & \text{if } n \geq 4 \end{cases}$*

Proof. Let $G = P_n^{+--}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| = 1$.

(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| > 1$.

(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j \neq i - 1$ or $j \neq i$.

When $n = 3$, the result follows immediately as $G \cong P_5$. Let $n \geq 4$ and let u and v be any vertices of G. If $u = v_i$ and $v = v_j$ (without loss of generality, we take $i \leq j$), then $d_G(v_i, v_j) = 1$ if $|i - j| = 1$ (since v_i and v_j are adjacent in this case), $d_G(v_i, v_j) = 2$ if $|i - j| \geq 2$ (since $v_i - v_{i+1} - v_j$ is a shortest path if $|j - i| = 2$; $v_i - e_{i+2} - v_j$ is a shortest path if $|i - j| > 2$), so $d_G(v_i, v_j) \leq 2$.

If $u = e_i$ and $v = e_j$ with $i \leq j$, then we observe that $d_G(e_i, e_j) = 1$ if $j \neq i+1$ and $d_G(e_i, e_j) = 2$ if $j = i + 1$ (since $e_i - v_{i+2(mod\ n)} - e_j$ is a shortest path in this case).

If $u = v_i$ and $v = e_j$, then we observe that $d_G(e_i, v_j) = 1$ if $j \neq i - 1$ or $j \neq i$ and $d_G(e_i, v_j) = 2$ if $|i - j| \leq 1$ (since $e_i - e_{j(mod\ n)} - v_i$ is a shortest path if $j > i + 1$). Thus $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 2$. $\square$

Lemma 3.10. *For any integer $n \geq 3$, $rn(P_n^{xyz}) \geq \begin{cases} 4n-1, & \text{if } n=3 \\ 2n-1, & \text{if } n \geq 4 \end{cases}$*

Proof. When $n = 3$, $G \cong P_5$ and hence the equality by Theorem 2.1. The case $n \geq 4$ is a direct consequence of $rn(G) \geq |V(G)|$. □

3.6. For $xyz = -+-$.

Lemma 3.11. *For any integer $n \geq 4$, $diam(P_n^{xyz}) = \begin{cases} 3, & \text{if } n=4 \\ 2, & \text{if } n \geq 5 \end{cases}$*

Proof. Let $G = P_n^{-+-}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i-j| \geq 2$.
(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i-j| = 1$.
(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j \neq i-1$ or $j \neq i$.

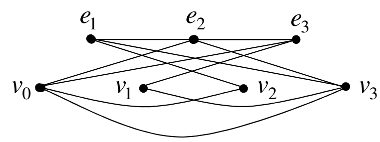

FIGURE 2. The Transformation graph P_4^{-+-}.

The case $n = 4$, follows by Figure 2. When $n \geq 5$, for $i \leq j$, $v_i - e_{i+3(mod\ n)} - v_j$(or $v_i - v_{i+3(mod\ n)} - v_j$), $0 \leq i, j \leq n-1$ is a shortest path if $j - i = 1$ and hence $d(v_i, v_j) = 2$ whenever v_i and v_j are not adjacent in G. Similarly, $e_i - e_{i+1} - e_j$ (or $e_i - v_{i+1(mod\ n)} - e_j$) is a shortest path if $|i-j| = 2$ (or $|i-j| > 2$), hence $d(e_i, e_j) = 2$ whenever e_i and e_j are not adjacent in G. Finally, $e_i - v_{i+2(mod\ n)} - v_j$ is a shortest path if $j = i-1$. Hence $d(e_i, v_j) = 2$, whenever e_i and v_j are non-adjacent in G.

Thus $diam(G) = \max\{d_G(u,v) : u, v \in V(G)\} = 2$. □

Lemma 3.12. *For any integer $n \geq 4$, $rn(P_n^{xyz}) \geq \begin{cases} 3n+1, & \text{if } n=4 \\ 2n, & \text{if } n=5 \\ 2n-1, & \text{if } n \geq 6 \end{cases}$*

Proof. Let $G = P_n^{xyz}$ and f be a radio labeling of G.

For $n = 4$, let $x_1, x_2, \ldots, x_7$ be the rearrangement of the vertices of P_4^{xyz} such that $f(x_i) < f(x_{i+1})$, $1 \leq i \leq 7$. We know that, for $n = 4$, $diam(P_4^{xyz}) = 3$ and there exists only one pair of vertices at distance 3. Let x and y be the two antipodal vertices in P_4^{xyz}.

Case 1: If x and y uses consecutive labels, then $V - \{x\}$ or $V - \{y\}$ has exactly four pair of vertices of the form (x_i, x_{i+1}) such that $d(x_i, x_{i+1}) = 2$.

Thus, the sequence $x_1, x_2, \ldots, x_7$ has at least one pair of adjacent vertices of the form (x_i, x_{i+1}). Thus in this case $rn(P_n^{xyz}) \geq f(x_n) = \sum_{i=1}^{6}[f(x_{i+1}) - f(x_i)] + f(x_1) = (1 + 4 \times 2 + 3) + 1 = 13$.

Case 2: If no two vertices receive consecutive labels, then there exist exactly 6 pairs of vertices of the form (x_i, x_{i+1}) in the sequence $x_1, x_2, \ldots, x_7$ such that $d(x_i, x_{i+1}) = 2$. Thus in this case, $rn(P_n^{xyz}) \geq f(x_1) + (6 \times 2) = 13$.

Thus in any case, for $n = 4$, $rn(P_n^{xyz}) \geq 13 = 3n + 1$.

For $n = 5$, $diam(G) = 2$. Let $x_1, x_2, \ldots, x_9$ be the arrangement of vertices of P_5^{xyz} such that $f(x_i) < f(x_{i+1})$, $1 \leq i \leq 9$. There are at most 7 pairs of vertices of the form (x_i, x_{i+1}) such that $d(x_i, x_{i+1}) = 2 = diam(G)$. Thus $rn(P_5^{xyz}) \geq |V| + 1$. That is, $rn(P_5^{xyz}) \geq 10 = 2n$. When $n \geq 6$, a direct consequence of $rn(G) \geq |V(G)|$.

□

3.7. For $xyz = - - -$.

Lemma 3.13. *For any integer $n \geq 4$, $diam(P_n^{xyz}) = \begin{cases} 3, & if \quad n = 4 \\ 2, & if \quad n \geq 5 \end{cases}$*

Proof. Let $G = P_n^{---}$. Then in G,

(i) the vertex v_i is adjacent to the vertex v_j if and only if $|i - j| > 1$.
(ii) the vertex e_i is adjacent to the vertex e_j if and only if $|i - j| > 1$'
(iii) the vertex e_i is adjacent to the vertex v_j if and only if either $j \neq i - 1$ or $j \neq i$.

The case $n = 4$ is again easy to verify. When $n \geq 5$, for $i \leq j$, $v_i - v_{i+3(mod\ n)} - v_j$ is a shortest path if $|i - j| = 1$ and hence $d(v_i, v_j) = 2$ whenever v_i and v_j are not adjacent in G, Similarly, $e_i - v_{i+2(mod\ n)} - e_j$ is a shortest path if $|i - j| = 1$, hence $d(e_i, e_j) = 2$ whenever e_i and e_j are not adjacent in G. Finally, $e_i - e_{i+3(mod\ n)} - v_j$ is a shortest path if $|i - j| = 1$ hence $d(e_i, v_j) = 2$, whenever e_i and v_j are non-adjacent in G, hence $d(e_i, v_j) = 2$. Thus, $diam(G) = \max\{d_G(u, v) : u, v \in V(G)\} = 2$. □

Lemma 3.14. *For any integer $n \geq 4$, $rn(P_n^{xyz}) \geq \begin{cases} 3n, & if \quad n = 4 \\ 2n - 1, & if \quad n \geq 5 \end{cases}$*

Proof. Let $G = P_n^{xyz}$ and f be a radio labeling of G. For $n = 4$, let $x_1, x_2, \ldots, x_7$ be the vertices of P_4^{xyz} such that $f(x_i) < f(x_{i+1})$, $1 \leq i \leq 7$. There exists only one pair of vertices at distance 3 and there exist exactly 5 pairs of vertices in the sequence $x_1, x_2, \ldots, x_7$ such that $d(x_i, x_{i+1}) = 2$. Thus $rn(P_4^{xyz}) \geq f(x_1) + 1 + (2 \times 5) \geq 12 = 3n$. When $n \geq 5$, result follows by the direct consequence of $rn(G) \geq |V(G)|$. □

4. Upper Bounds and a radio labeling

In this section we actually show the upper limit, established in the previous section, for each of the transformation graphs is tight by executing a radio labeling.

4.1. For $xyz = + - +$.

Lemma 4.1. *For any integer* $n \geq 2$, $rn(P_n^{xyz}) \leq \begin{cases} 3n-3, & if \quad n = 2,3 \\ 3n, & if \quad 4 \leq n \leq 6 \\ 3n-2, & if \quad n \geq 7 \end{cases}$.

Proof. Let $G = P_n^{+-+}$. When $n = 2$, the graph $G \cong K_3$ and hence $rn(G) = 3$. The case $n = 3, 4$ follows respectively by the radio labelings of G shown in the Figure 3 and Figure 4.

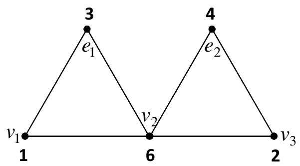

FIGURE 3. A radio labeling of P_3^{+-+}.

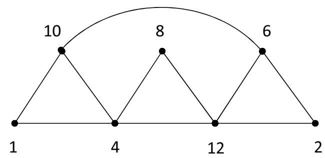

FIGURE 4. A radio labeling of P_4^{+-+}.

Now for the case $n = 5, 6$, define a function $f : V(G) \rightarrow Z^+$ as, $f(v_i) = 1 + \lceil \frac{n}{2} \rceil i$, for $0 \leq i \leq 2$, $f(v_j) = f(v_{j-3}) + 1$, for $3 \leq j \leq n-1$, and $f(e_1) = n+4$, $f(e_j) = f(e_{j-1}) + 2$, for $2 \leq j \leq n-1$. Hence, it follows that f is a radio labeling of G and $rn(G) \leq \text{span} f = n + 2 + 1 + 2n - 3 = 3n$. We now consider the case $n \geq 7$.

Case 1: If $n \equiv 0 (mod\, 3)$, and $n \equiv 2 (mod\, 3)$, then define a function $f : V(G) \rightarrow Z^+$ by $f(v_i) = 1 + \frac{n}{3} i$, for $0 \leq i \leq 2$; $f(v_i) = f(v_{i-3}) + 1$, for $3 \leq i \leq n-1$; $f(e_1) = n + 2$, and $f(e_i) = f(e_{i-1}) + 2$, for $2 \leq i \leq n-1$.

Case 2: If $n \equiv 1 (mod\ 3)$, then define a function $f : V(G) \rightarrow Z^+$ by $f(v_i) = 1 + \frac{n}{3} i$, for $0 \leq i \leq 1$; $f(v_i) = 3 + \frac{n}{3}$, for $i = 2$, $f(v_i) = f(v_{i-3}) + 1$, for $3 \leq i \leq n-1$; $f(e_1) = n + 2$, and $f(e_i) = f(e_{i-1}) + 2$, for $2 \leq i \leq n-1$.

Clearly f is a radio labeling. In fact, (i) $|f(v_i) - f(v_j)| = 1 \Leftrightarrow j = i + 3 (mod\ n) \Leftrightarrow d(v_i, v_j) \geq 3$ for all $i, 1 \leq i \leq n$, (ii) $|f(u) - f(v)| = 2 \Leftrightarrow u = e_i$ and $v = e_{i+1}$, or u and v are non-incident pairs in $G \Leftrightarrow d(u, v) = 2$, and, (iii) $|f(u) - f(v)| \geq 3 \Leftrightarrow u$ and v are adjacent vertices or, non-incident edges or, incident pair of vertex and edges $\Leftrightarrow d(u, v) = 1$. Hence, as $diam(G) = 3$ (by Lemma 3.1) it follows that f is a radio labeling of G and hence $rn(G) \leq$ span $f = f(e_{n-1}) = f(e_{j-1}) + 2 = \cdots = f(e_1) + 2 \times (n-2) = n+2+2n-4 = 3n-2$. $\square$

Lemma 3.2 and Lemma 4.1 together prove the Theorem 2.5.

4.2. For $xyz = --+$.

Lemma 4.2. *For any integer $n \geq 2$, $rn(P_n^{xyz}) \leq \begin{cases} 2n, & \text{if } n = 2 \\ 2n-1, & \text{if } n \geq 3 \end{cases}$*

Proof. Let $G = P_n^{--+}$. When $n = 2$, the graph $G \cong P_3$ and hence it follows from the Theorem 2.2. Consider $n \geq 3$, define a function $f : V(G) \to Z^+$ by $f(v_i) = i+1$, for $0 \leq i \leq n-1$ and $f(e_i) = n+i$, $1 \leq i \leq n-1$. The function f is a radio labeling for $n \geq 3$ because by Lemma 3.3 $diam(G)+1 = 3$ and

(1) $|f(v_i) - f(v_j)| + d(v_i, v_j) = \begin{cases} |i-j|+2 = 3, & \text{if } v_iv_j \notin E(G) \\ |i-j|+1 \geq 3, & \text{if } v_iv_j \in E(G) \end{cases}$

(2) $|f(e_i) - f(e_j)| + d(e_i, e_j) = \begin{cases} |i-j|+2 = 3, & \text{if } e_ie_j \notin E(G) \\ |i-j|+1 \geq 3, & \text{if } e_ie_j \in E(G) \end{cases}$

(3) $|f(e_i) - f(v_j)| = 1 \Leftrightarrow i = 1$ and $j = n-1$ (since $f(e_i) > f(v_j)$) and in this case $d(e_i, v_j) = 2$. So, $|f(e_i) - f(v_j)| + d(e_i, v_j) \geq 3$.

Thus $rn(G) \leq \text{span } f = f(e_{n-1}) = 2n-1$. $\square$

Theorem 2.2, Theorem 2.3, Lemma 3.4 and Lemma 4.2 together prove the Theorem 2.6.

4.3. For $xyz = ++-$.

Lemma 4.3. *For any integer $n \geq 3$, $rn(P_n^{xyz}) \leq 2n-1$.*

Proof. Let $G = P_n^{++-}$. When $n = 3$, the graph $G \cong C_5$ and hence the result follows by Theorem 2.3. When $n = 4$, the result follows by the radio labeling given in Figure 5.

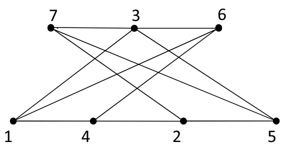

FIGURE 5. A radio labeling of P_4^{++-}.

For $n \geq 5$, define a function $f : V(G) \to Z^+$ as $f(v_0) = 1$, $f(v_1) = \lceil \frac{n}{2} \rceil + 1$; $f(v_i) = f(v_{i-2}) + 1, 2 \leq i \leq n-1$; and $f(e_{n-i}) = f(e_{n-i-2}) + 1$, for all $i, 3 \leq i \leq n-1$, with $f(e_{n-1}) = n+1$ and $f(e_{n-2}) = \begin{cases} 2n - \lceil \frac{n}{3} \rceil, & \text{if } n \text{ is even} \\ n + \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd} \end{cases}$

The function f is a radio labeling. In fact, in this case, $diam(G) + 1 = 2 + 1 = 3$, $|f(u) - f(v)| + d(u, v) \geq 3$ if u and v are adjacent pairs in P_n, $|f(u) - f(v)| + d(u, v) \geq 1$ if u and v are non-adjacent pairs in P_n, and, as $f(e_i) > f(v_j)$, $|f(e_i) - f(v_j)| = 1 \Leftrightarrow j = n-1$ for n even, and $j = n-2$ for n odd $\Leftrightarrow d(e_i, v_j) = 2$. Thus $rn(G) \leq \text{span } f = 2n-1$. $\square$

Theorem 2.3, Lemma 3.6 and Lemma 4.3 together prove the Theorem 2.7.

4.4. For $xyz = -++$.

Lemma 4.4. *For any integer $n \geq 2$, $rn(P_n^{xyz}) \leq$* $\begin{cases} 2n, & \textit{if} \quad n = 2 \\ 2n-1, & \textit{if} \quad n = 3,4 \\ 3n+1, & \textit{if} \quad n = 5,6 \\ 3n, & \textit{if} \quad n = 7 \\ 3n-1, & \textit{if} \quad n \geq 8 \end{cases}$

Proof. Let $G = P_n^{-++}$. When $n = 2$, the graph $G \cong P_3$ and hence the result follows by the Theorem 2.2. When $n = 3, 4$, and 5, the result follows respectively by the radio labelings in Figure 6, Figure 7, and Figure 8.

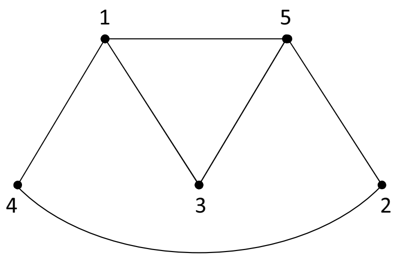

FIGURE 6. A radio labeling of P_3^{-++}.

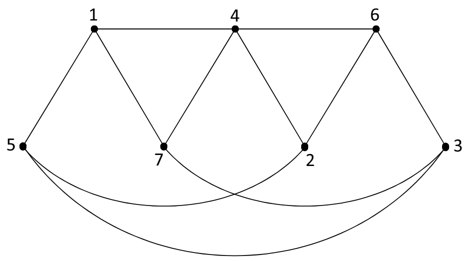

FIGURE 7. A radio labeling of P_4^{-++}.

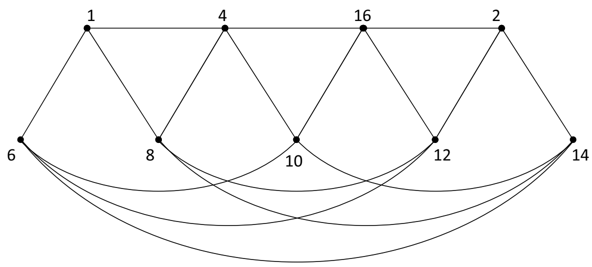

FIGURE 8. A radio labeling of P_5^{-++}.

When $n = 6, 7, 8$, it is easy to verify that the function $f : V(G) \to Z^+$ defined by $f(v_0) = 9; f(v_i) = f(v_{i-1}) + 2$, for all $i = 1, 2, \ldots, n-1$; and $f(e_1) = 3, f(e_2) = 6, f(e_3) = 1$, $f(e_j) = f(e_{j-3}) + 1$, for all $i = 4, 5, \ldots, n-1$.

When $n \geq 9$, define $f(e_i) = \frac{i+2}{3}$ if $i \equiv 1(mod\ 3)$, $f(e_i) = \lceil \frac{n+2}{3} \rceil + (\frac{i-2}{3})$ if $i \equiv 2(mod\ 3)$, $f(e_i) = \lceil \frac{2n+1}{3} \rceil + (\frac{i-3}{3})$ if $i \equiv 0(mod\ 3)$, and $f(v_0) = n+1, f(v_i) = f(v_{i-1}) + 2$, for $1 \leq i \leq n-1$. The function f is a radio labeling. In fact,

(1) $|f(v_i) - f(v_j)| + d(v_i, v_j) \geq 4$, if $v_i v_j$ are adjacent or non-adjacent in G.

(2) $|f(e_i) - f(e_j)| + d(e_i, e_j) \geq 4$, if $e_i e_j$ are adjacent or non-adjacent in G.
(3) $f(v_j) > f(e_i)$, it follows that $|f(v_j) - f(e_i)| = 2$, only if $i = n-3$ and $j = 0$. So, $|f(v_j) - f(e_i)| + d(v_j, e_i) \geq 4 = 1 + diam(G)$.

Thus $rn(G) \leq$ span $f = f(v_{n-1}) = 3n - 1$. □

Theorem 2.2, Lemma 3.8 and Lemma 4.4 together prove the Theorem 2.8.

4.5. For $xyz = +--$.

Lemma 4.5. *For any integer $n \geq 3$, $rn(P_n^{xyz}) \leq \begin{cases} 4n-1, & if \quad n = 3 \\ 2n-1, & if \quad n \geq 4 \end{cases}$.*

Proof. Let $G = P_n^{+--}$. When $n = 3, 4$, the result follows by the labeling in Figure 9 and Figure 10.

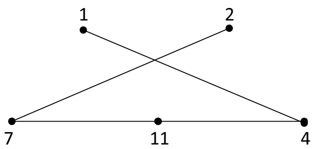

FIGURE 9. A radio labeling of P_3^{+--}.

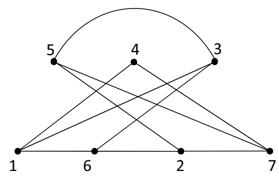

FIGURE 10. A radio labeling of P_4^{+--}.

For $n \geq 5$, define a function $f : V(G) \to Z^+$ as $f(e_{n-1}) = n+1; f(e_i) = f(e_{i+1}) + 1, n-2 \leq i \leq 1;$ $f(v_0) = 1; f(v_1) = 1 + \lceil \frac{n}{2} \rceil, f(v_i) = f(v_{i-2}) + 1$, $2 \leq i \leq n-1$. The function f is a radio labeling. In fact,

(1) $|f(v_i) - f(v_j)| + d(v_i, v_j) \geq 3 = 1 + diam(G)$,
(2) $|f(e_i) - f(e_j)| + d(e_i, e_j) \geq 3 = 1 + diam(G)$,
(3) $f(e_i) > f(v_j)$, it follows that $|f(e_i) - f(v_j)| = 1$, only if $i = j+1$.

Thus $rn(G) \leq$ span $f = f(e_1) = 2n - 1$. □

Lemma 3.10 and Lemma 4.5 together prove the Theorem 2.9.

4.6. For $xyz = -+-$.

Lemma 4.6. *For any integer $n \geq 4$, $rn(P_n^{xyz}) \leq \begin{cases} 3n+1, & if \quad n = 4 \\ 2n, & if \quad n = 5 \\ 2n-1, & if \quad n \geq 6 \end{cases}$*

Proof. Let $G = P_n^{-+-}$. When $n = 4, 5$, the result follows by the labeling in Figure 11 and Figure 12.

When $n \geq 6$, define a function $f : V(G) \to Z^+$ as $f(v_i) = i+1, 0 \leq i \leq n-1$ and $f(e_{n-1}) = n+1$, $f(e_{n-2}) = f(e_{n-1}) + \lfloor \frac{n}{2} \rfloor, f(e_i) = f(e_{i+2}) + 1, n-3 \leq i \leq 1$. The function f is a radio labeling for $n \geq 6$ because, by Lemma 3.11, $diam(G) + 1 = 3$ and,

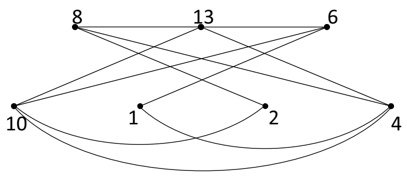

FIGURE 11. A radio labeling of P_4^{-+-}.

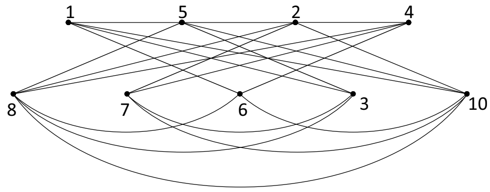

FIGURE 12. A radio labeling of P_5^{-+-}.

(1) $|f(v_i) - f(v_j)| + d(v_i, v_j) = \begin{cases} |i-j| + 2 = 3, & \text{if } v_iv_j \notin E(G) \\ |i-j| + 1 \geq 3, & \text{if } v_iv_j \in E(G) \end{cases}$

(2) $|f(e_i) - f(e_j)| + d(e_i, e_j) \geq \begin{cases} 2+1 = 3, & \text{if } e_ie_j \notin E(G) \\ 1+2 = 3, & \text{if } e_ie_j \in E(G) \end{cases}$

(3) $|f(e_i) - f(v_j)| = 1$, if $i = j = n-1$ and in this case $d(e_i, v_j) = 2$.
So, $|f(e_i) - f(v_j)| + d(e_i, v_j) \geq 3$.

Thus $rn(G) \leq \text{span } f = \begin{cases} f(e_1) = 2n-1, & \text{if } n \text{ is odd} \\ f(e_2) = 2n-1, & \text{if } n \text{ is even} \end{cases}$ □

Lemma 3.12 and Lemma 4.6 together prove the Theorem 2.10.

4.7. For $xyz = ---$.

Lemma 4.7. *For any integer $n \geq 4$, $rn(P_n^{xyz}) \leq \begin{cases} 3n, & \text{if } \ n = 4 \\ 2n-1, & \text{if } \ n \geq 5 \end{cases}$.*

Proof. Let $G = P_n^{---}$. For $n = 4$, the result follows by the labeling in Figure 13.

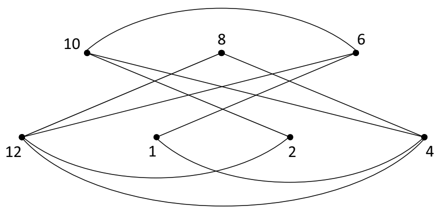

FIGURE 13. The graph P_4^{---}.

For $n \geq 5$, define a function $f : V(G) \to Z^+$ as $f(v_i) = i+1$, for $0 \leq i \leq n-1$ and $f(e_{n-1}) = n+1, f(e_i) = f(e_{i+1}) + 1$ for $n-2 \leq i \leq 1$. The function f is a radio labeling for $n \geq 5$ because, by Lemma 3.13, $diam(G) + 1 = 3$ and

(1) $|f(v_i) - f(v_j)| + d(v_i, v_j) = \begin{cases} |i-j| + 2 = 3, & \text{if } v_iv_j \notin E(G) \\ |i-j| + 1 \geq 3, & \text{if } v_iv_j \in E(G) \end{cases}$,

(2) $|f(e_i) - f(e_j)| + d(e_i, e_j) = \begin{cases} |i-j| + 2 = 3, & \text{if } e_ie_j \notin E(G) \\ |i-j| + 1 \geq 3, & \text{if } e_ie_j \in E(G) \end{cases}$,

(3) $|f(e_i) - f(v_j)| = 1 \Leftrightarrow i = j$ and $j = n-1$ (since $f(e_i) > f(v_j)$) and in this case $d(e_i, v_j) = 2$. So, $|f(e_i) - f(v_j)| + d(e_i, v_j) \geq 3$.

Thus $rn(G) \leq \text{span} f = f(e_1) = 2n - 1$. □

Lemma 3.14 and Lemma 4.7 together prove the Theorem 2.11.

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