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BIVARIATE DYNAMIC CUMULATIVE RESIDUAL TSALLIS ENTROPY†

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Abstract. Recently, Sati and Gupta (2015) proposed two measures of uncertainty based on non-extensive entropy, called the dynamic cumulative residual Tsallis entropy (DCRTE) and the empirical cumulative Tsallis entropy. In the present paper, we extend the definition of DCRTE into the bivariate setup and study its properties in the context of reliability theory. We also define a new class of life distributions based on bivariate DCRTE.

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1. Introduction

The concept of entropy was introduced by Shannon [21]. Let X be a continuous random variable having probability density function $f(x)$, cumulative distribution function $F(x)$, survival function $\bar{F}(x)$ and hazard rate $r(x) = f(x)/\bar{F}(x)$. Then the differential form of Shannon entropy is defined as

$$H(X) = -\int_0^\infty f(x)\log\left(f(x)\right)dx. \tag{1}$$

The entropy measure (1) is not useful for a system which has survived up to age t, Ebrahimi [5] modified (1) to measure uncertainty for the residual life time $X_t = [X - t|X > t]$, where $t > 0$ given by

$$H(X;t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\ \log\left(\frac{f(x)}{\bar{F}(t)}\right)\ dx. \tag{2}$$

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Rao et al. [16] introduced a new measure of uncertainty based on survival function instead of probability density function of a random variable X as follows:

$$\xi(X) = -\int_0^\infty \bar{F}(x) \log\left(\bar{F}(x)\right) dx, \tag{3}$$

and called it the Cumulative Residual Entropy (CRE).
Asadi and Zohrevand [2] proposed a new measure of uncertainty for residual lifetime $X_t = [X - t | X > t]$, which is given as

$$\xi(X;t) = -\int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right) dx, \tag{4}$$

and called it Dynamic Cumulative Residual Entropy (DCRE).
The properties and applications of DCRE have been studied extensively by Asadi and Zohervand [2], Di Crescenzo and Longobardi [4], Abbasnejad et al. [1], Navarro et al. [11], Sunoj and Linu [22], Kumar and Taneja [9], Sati and Gupta [20].

In the literature several generalization of Shannon entropy are available such as Renyi entropy [17], Varma entropy [24], Tsallis entropy [23] etc. In this article, we focus on non-extensive entropy.

Tsallis [23] defined the generalized non-expansive entropy of order α as

$$S_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^\infty (f(x))^\alpha dx\right), \ \alpha > 0, \ \alpha \neq 1. \tag{5}$$

In a recent work, Sati and Gupta [20] have proposed Dynamic Cumulative Residual Tsallis Entropy (DCRTE) of order α as follows:

$$\eta_\alpha(X;t) = \frac{1}{\alpha - 1}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right), \alpha > 0, \alpha \neq 1, \tag{6}$$

and studied its properties and applications.
The multivariate life distributions are used for studying the reliability characteristics of multi-component system with each component having a lifetime depending on the next component. In the univariate case, the reliability characteristics can be extended to higher dimensions. Although, a lot of work has been done on information measures in the univariate case, but very limited work has been done in higher dimensions. For more details, we refer to Rajesh and Nair [12], Nadarajah and Zografos [10], Ebrahimi et al. [6], Sathar et al. [19] and Rajesh et al. [13], [14], [15].

The main objective of the paper is to extend DCRTE defined in (6) to bivariate setup and study its properties and connect it to some well known reliability models. In section 2, we propose a bivariate dynamic cumulative residual Tsallis entropy (BDCRTE) of order α and characterize some well known bivariate models using the BDCRTE. In section 3, we define new classes of life distributions based on BDCRTE and study their properties.

2. Bivariate Dynamic Cumulative Residual Tsallis Entropy(BDCRTE)

In this section, we extend the definition of DCRTE to the bivariate setup known as the bivariate cumulative residual Tsallis entropy (BDCRTE) and we also give some characterization results of well known bivariate distributions in term of BDCRTE.

Definition 2.1. Let $X = (X_1, X_2)$ be a bivariate random vector admitting an absolutely continuous probability density function $f(x_1, x_2)$, cumulative density function $F(x_1, x_2)$ and survival function $\bar{F}(x_1, x_2)$ with respect to Lebesgue measure in the positive octant $R_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two-dimensional Euclidean space R_2. We define the bivariate DCRTE as

$$\eta_\alpha(X; t_1, t_2) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{\infty}\int_{t_2}^{\infty}\left(\frac{\bar{F}(x_1, x_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} dx_2 dx_1\right), \alpha > 0, \alpha \neq 1. \tag{7}$$

Ebrahimi [6] has proved that the bivariate residual entropy is not invariant under non singular transformations. Similarly we can show that bivariate DCRTE defined in equation (7) is not invariant under non singular transformations.

If $Y_j = \phi_j(X_j)$, $j = 1, 2$ are one to one transformations, then

$$\eta_\alpha(Y_j; \phi_1(t_1), \phi_2(t_2)) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{\infty}\int_{t_2}^{\infty}\left(\frac{\bar{F}(x_1, x_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} J\, dx_2 dx_1\right),$$

where $J = |\frac{\partial}{\partial x_1}\phi_1(x_1) \times \frac{\partial}{\partial x_2}\phi_2(x_2)|$ is the absolute value of the Jacobian of transformation. In particular, if we take $\phi_j(X_j) = a_j X_j + b_j$, then we get

$$\eta_\alpha(Y_j; \phi_1(t_1), \phi_2(t_2)) = \frac{(1 - a_1 a_2)}{(\alpha - 1)} + a_1 a_2\, \eta_\alpha(X; t_1, t_2).$$

Now we take into account the behavior of the dynamic cumulative residual Tsallis entropy for the conditional distributions. Let us consider the random variables $Y_j = (X_j|X_i > t_i,\ i, j = 1, 2; i \neq j)$, where Y_j, $j = 1, 2$ corresponds to the conditional distributions of X_j given that X_i has survived up to time t_i, $i = 1, 2$ and have the survival functions $\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}$ for $x_1 \geq t_1$ & $\frac{\bar{F}(t_1, x_2)}{\bar{F}(t_1, t_2)}$ for $x_2 \geq t_2$, respectively. The DCRTE for the random variables Y_j, $j = 1, 2$ are defined as follows:

$$\eta_{1\alpha}(X; t_1, t_2) = \frac{1}{\alpha - 1}\left(1 - \int_{t_1}^{\infty}\left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} dx_1\right), \alpha > 0, \alpha \neq 1, \tag{8}$$

and

$$\eta_{2\alpha}(X;t_1,\,t_2) = \frac{1}{\alpha-1}\left(1 - \int\limits_{t_2}^{\infty}\left(\frac{\bar{F}(t_1,x_2)}{\bar{F}(t_1,t_2)}\right)^{\alpha} dx_2\right),\ \alpha>0\,,\ \alpha\neq 1, \tag{9}$$

respectively.

For a bivariate random vector $X = (X_1,\, X_2)$, Johnson and Kotz [8] defined the bivariate hazard rate as

$$r(t_1,\,t_2) = (\,r_1(t_1,\,t_2),\, r_2(t_1,\,t_2)\,),$$

where

$$r_i(t_1,\,t_2) = -\frac{\partial}{\partial t_i}\log \bar{F}(t_1,t_2),\ \ i=1,2. \tag{10}$$

For a bivariate random vector $X = (X_1,\, X_2)$, Zahedi [25] defined the bivariate mean residual life function (MRLF) as

$$m(t_1,\,t_2) = (\,m_1(t_1,\,t_2),\, m_2(t_1,\,t_2)\,),$$

where

$$m_i(t_1,\,t_2) = \left(\bar{F}(t_1,\,t_2)\right)^{-1}\int\limits_{t_i}^{\infty}\bar{F}(t_1,\,t_2)\,dt_i,\ \ i=1,2. \tag{11}$$

The following theorem shows that the bivariate dynamic cumulative residual Tsallis entropy (BDCRTE) uniquely determines the survival function $\bar{F}(t_1,t_2)$.

Theorem 2.2. *Let $X = (X_1,\, X_2)$ be a non-negative random vector admitting continuous distribution function with respect to Lebsegue measure. Let $\eta_{i\alpha}(X;t_1,\,t_2) < \infty\,;\ i=1,\,2,\ t=(t_1,\,t_2)\geq 0;\ \forall\,\alpha>0\,(\neq 1)$. Then for each α, $\eta_{i\alpha}(X;t_1,\,t_2)$ (where as $\frac{\partial}{\partial t_i}\eta_{i\alpha}(X;\,t_1,\,t_2)\neq 0,\,\forall\, i=1,\,2$) uniquely determines the survival function $\bar{F}(t_1,\,t_2)$.*

Proof. From the equation (8), we have

$$(\alpha-1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) = 1 - \frac{\int_{t_1}^{\infty}\left(\bar{F}(x_1,\,t_2)\right)^{\alpha}\,dx_1}{\left(\bar{F}(t_1,\,t_2)\right)^{\alpha}}. \tag{12}$$

Differentiating (12) with respect to t_1 and simplifying, we obtain

$$(\alpha-1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\,t_1,\,t_2) = 1+\alpha\, r_1(X;t_1,t_2)\,[(\alpha-1)\eta_{1\alpha}(X;t_1,t_2)-1]. \tag{13}$$

Similarly for $i=2$, we also get

$$(\alpha-1)\,\frac{\partial}{\partial t_2}\eta_{2\alpha}(X;\,t_1,\,t_2) = 1+\alpha\, r_2(X;t_1,t_2)\,[(\alpha-1)\eta_{2\alpha}(X;t_1,t_2)-1]. \tag{14}$$

Let $\bar{F}_X(t_1,\,t_2)$ and $\bar{F}_Y(t_1,\,t_2)$ be two survival functions having bivariate dynamic entropies $\eta_{i\alpha}(X;\,t_1,\,t_2)$ and $\eta_{i\alpha}(Y;\,t_1,\,t_2)$ with hazard rates $r_i(X;\,t_1\,t_2)$ and $r_i(Y;\,t_1\,t_2)$, $i=1,2$ respectively.

Consider the following relationship between entropies of random vector X and Y :

$$\eta_{i\alpha}(X;\, t_1,\, t_2) \;=\; \eta_{i\alpha}(Y;\, t_1,\, t_2)\,,\;\; i = 1, 2. \tag{15}$$

Taking $i = 1$ and differentiating (15) with respect to t_1, we get

$$\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\, t_1,\, t_2) \;=\; \frac{\partial}{\partial t_1}\eta_{1\alpha}(Y;\, t_1,\, t_2).$$

$$(\alpha - 1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\, t_1,\, t_2) \;=\; (\alpha - 1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(Y;\, t_1,\, t_2). \tag{16}$$

Using (13), the equation (16) becomes

$$1 + \alpha\, r_1(X;\, t_1, t_2)\,[(\alpha - 1)\eta_{1\alpha}(X; t_1, t_2) - 1] = 1 + \alpha\, r_1(Y;\, t_1, t_2)\,[(\alpha - 1)\eta_{1\alpha}(Y; t_1, t_2) - 1]. \tag{17}$$

Since $\eta_{1\alpha}(X;\, t_1,\, t_2) \;=\; \eta_{1\alpha}(Y;\, t_1,\, t_2)$, therefore the equation (17) reduces to

$$r_1(X;\, t_1\, t_2) \;=\; r_1(Y;\, t_1\, t_2).$$

Similarly for $i = 2$, we get

$$r_2(X;\, t_1\, t_2) \;=\; r_2(Y;\, t_1\, t_2).$$

Thus, we have

$$\bar{F}_X(t_1,\, t_2) \;=\; \bar{F}_Y(t_1,\, t_2).$$

Hence, $\eta_{i\alpha}(X;\, t_1,\, t_2)$ uniquely determines the survival function $\bar{F}(t_1,\, t_2)$. □

The following theorem characterize some well known bivariate distributions using relationship between BDCRTE and bivariate mean residual life function $m(X; t_1, t_2)$.

Theorem 2.3. *For the random vector $X = (X_1,\, X_2)$ admitting continuous distribution function with respect to Lebsegue measure, a relationship of the form*

$$(\alpha - 1)\,\eta_{i\alpha}(X;\, t_1,\, t_2) = 1 - K m_i\,(X; t_1,\, t_2)\,,\;\; i = 1,\, 2,\; \alpha > 0\,,\, \alpha \neq 1, \tag{18}$$

where $m_i\,(X; t_1,\, t_2)\,, i = 1, 2$ are the components of the bivariate mean residual life function and K is a constant independent of t_i, holds for all $t_i \geq 0$, if and only if X follows any one of the three distributions:

(i) the bivariate Pareto distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = (1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c}\,;\; a_1, a_2, c, t_1, t_2 > 0; \quad 0 < b < (c+1)a_1 a_2, \tag{19}$$

(ii) the Gumbel's bivariate exponential distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right);\; \lambda_1, \lambda_2, t_1, t_2 > 0; \quad 0 < \theta < \lambda_1 \lambda_2, \tag{20}$$

and

(iii) the bivariate finite range distribution with joint survival function

$$\bar{F}(t_1,\, t_2) = (1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)^d\,;\quad p_1, p_2, d > 0;\;\; 0 < t_1 < \frac{1}{p_1};$$
$$0 < t_2 < \frac{1 - p_1 t_1}{p_2 - q t_1}, \tag{21}$$

according as $K\alpha < 1,\;\, K\alpha = 1$ *and* $K\alpha > 1$*, respectively.*

Proof. Differentiating equation (18) with respect to t_1 by taking $i = 1$, we obtain

$$(\alpha - 1)\,\frac{\partial}{\partial t_1}\eta_{1\alpha}(X;\, t_1,\, t_2) \;=\; -K\frac{\partial}{\partial t_1} m_1(X; t_1,\, t_2).$$

Using the equation (13), we get

$$1 - K\alpha\, r_1(X; t_1, t_2)\, m_1(X; t_1, t_2) = \; -K\frac{\partial}{\partial t_1} m_1(X; t_1,\, t_2). \tag{22}$$

Using the relation $r_1(X; t_1, t_2)\, m_1(X; t_1, t_2) = 1 + \frac{\partial}{\partial t_1} m_1(X; t_1,\, t_2)$, the equation (22) reduces to

$$\frac{\partial}{\partial t_1} m_1(X; t_1,\, t_2) = \frac{(K\alpha - 1)}{K(1 - \alpha)} \;=\; C.$$

Integrating on both side with respect to t_1, we get

$$m_1(X; t_1,\, t_2) = \; C t_1 + D_1(t_2), \tag{23}$$

where D_1 is independent of t_1.
Similarly for $i = 2$, we have

$$m_2(X; t_1,\, t_2) = \; C t_2 + D_2(t_1). \tag{24}$$

Hence

$$m_i(X; t_1,\, t_2) = \; C t_i + D_i(t_j)\,,\quad i \neq j,\; i, j = 1,\, 2,$$

where $C = \frac{(K\alpha - 1)}{K(1-\alpha)}$ and $D_i(t_j)$ is a function of t_j only. Based on the characterization theorem given by Sankaran and Nair [18], we can easily prove that X follows bivariate Pareto distribution with survival function (19) when $C > 0$, Gumbel's exponential distribution with survival function (20) when $C = 0$ and bivariate finite range distribution with survival function (21) when $C < 0$.

Converse:

(i) When X follows bivariate Pareto distribution with survival function (19), then using the equation (8), we get

$$\begin{aligned}(\alpha - 1)\,\eta_{1\alpha}(X;\, t_1,\, t_2) \;&= 1 - \frac{\int_{t_1}^{\infty} (1 + a_1 x_1 + a_2 t_2 + b x_1 t_2)^{-c\alpha}\; dx_1}{(1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c\alpha}} \\ &= 1 - \left[\frac{(c-1)}{(c\alpha - 1)}\,\frac{(1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)}{(c-1)(a_1 + b t_2)}\right].\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha - 1)\,\eta_{i\alpha}(X;\,t_1,\,t_2) = 1 - Km_i\,(X; t_1,\,t_2)\,,$$

where $K = \frac{(c-1)}{(c\alpha-1)}$, such that $K\alpha < 1$.

(ii) When X follows Gumbel's exponential distribution with survival function (20), then using the equation (8), we get

$$\begin{aligned}(\alpha - 1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) &= 1 - \frac{\int_{t_1}^{\infty}\left(e^{-\lambda_1 x_1-\lambda_2 t_2-\theta x_1 t_2}\right)^{\alpha}\,dx_1}{\left(e^{-\lambda_1 t_1-\lambda_2 t_2-\theta t_1 t_2}\right)^{\alpha}}\\ &= 1 - \frac{1}{\alpha(\lambda_1+\theta t_2)}.\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha - 1)\,\eta_{i\alpha}(X;\,t_1,\,t_2) = 1 - Km_i\,(X; t_1,\,t_2)\,,$$

where $K = \frac{1}{\alpha}$, such that $K\alpha = 1$.

(iii) When X follows bivariate finite range distribution with survival function (21), then using the equation (8), we get

$$\begin{aligned}(\alpha - 1)\,\eta_{1\alpha}(X;\,t_1,\,t_2) &= 1 - \frac{\int_{t_1}^{\infty}\left(1-p_1x_1-p_2t_2+qx_1t_2\right)^{d\alpha}\,dx_1}{\left(1-p_1t_1-p_2t_2+qt_1t_2\right)^{d\alpha}}\\ &= 1 - \left[\frac{(d+1)}{(d\alpha+1)}\,\frac{(1-p_1t_1-p_2t_2+qt_1t_2)}{(d+1)(p_1+qt_2)}\right].\end{aligned}$$

Similar result holds for $i = 2$. Hence

$$(\alpha - 1)\,\eta_{i\alpha}(X;\,t_1,\,t_2) = 1 - Km_i\,(X; t_1,\,t_2)\,,$$

where $K = \frac{(d+1)}{(d\alpha+1)}$, such that $K\alpha > 1$.

□

Now we provide characterization result in terms of relationship between bivariate DCRTE and bivariate hazard rate function.

Theorem 2.4. *For the random vector $X = (X_1,\,X_2)$ admitting an absolutely continuous function with respect to Lebesgue measure. Then the following relationship of the form*

$$(\alpha - 1)\,\frac{\partial}{\partial t_i}\eta_{i\alpha}(X;\,t_1,\,t_2) = c\,r_i\,(X; t_1,\,t_2)\,,\quad i = 1,\,2,\;\; \alpha > 0,\,\alpha \neq 1, \tag{25}$$

hold for all $t_1,\,t_2 \geq 0$, then X follows the Gumbel's bivariate exponential distribution with joint survival function

$$\bar{F}(t_1,\,t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right);\; \lambda_1, \lambda_2, t_1, t_2 > 0;\; 0 < \theta < \lambda_1\lambda_2,\, when\, c = 0. \tag{26}$$

Proof. When equation (25) hold for $i = 1$, then using the equation (12) and equation (13), we get

$$1 - \alpha\, r_1(X;t_1,t_2)\,\frac{\int_{t_1}^{\infty}\big(\bar{F}(x_1,\,t_2)\big)^{\alpha}\,dx_1}{\big(\bar{F}(t_1,\,t_2)\big)^{\alpha}} = c\, r_1(X;t_1,t_2),$$

or equivalently

$$\frac{1}{r_1(X;t_1,t_2)} = c + \alpha\,\frac{\int_{t_1}^{\infty}\big(\bar{F}(x_1,\,t_2)\big)^{\alpha}\,dx_1}{\big(\bar{F}(t_1,\,t_2)\big)^{\alpha}}. \tag{27}$$

Differentiating equation (27) with respect to t_1 and simplifying, we obtain

$$\frac{-1}{\big(r_1(X;t_1,t_2)\big)^2}\,\frac{\partial}{\partial t_1}\{r_1(X;t_1,t_2)\} = -\,\alpha c\, r_1(X;t_1,t_2).$$

$$\frac{\partial}{\partial t_1}\{\log\left(r_1(X;t_1,t_2)\right)\} = \alpha c\, r_1^2(X;t_1,t_2). \tag{28}$$

For simplification, we assume that $\log\left(r_1(X;t_1,t_2)\right) = y_1(t_1,t_2)$; that is, $r_1(X;t_1,t_2) = e^{y_1(t_1,t_2)}$, then the equation (28) reduces to

$$\frac{\partial}{\partial t_1}\{(y_1(t_1,t_2))\} = \alpha c\, e^{2y_1(t_1,t_2)}.$$

Integrating on both side with respect to t_1, we get

$$r_1(X;t_1,t_2) = \frac{1}{\sqrt{K_1(t_2) - 2\alpha c\, t_1}}\,.$$

Similarly for $i = 2$, we get

$$r_2(X;t_1,t_2) = \frac{1}{\sqrt{K_2(t_1) - 2\alpha c\, t_2}}\,.$$

Hence

$$r_i(X;t_1,t_2) = \frac{1}{\sqrt{K_i(t_j) - 2\alpha c\, t_i}}\,,\quad i \neq j\,,\quad i,j = 1,\,2, \tag{29}$$

where $K_i(t_j) > 0$ is constant and independent of t_i.
When $c = 0$, then from the equation (29), we get $r_i(X;t_1,t_2) = \frac{1}{\sqrt{K_i(t_j)}}$ or equivalently

$$-\frac{\partial}{\partial t_i}\{\log\ \bar{F}(t_1,t_2)\} = \frac{1}{\sqrt{K_i(t_j)}}.$$

Integrating both side with respect to t_i, we get

$$-\log\ \bar{F}(t_1,t_2) = \frac{t_i}{\sqrt{K_i(t_j)}} + Q_i(t_j)$$

$$\bar{F}(t_1,t_2) = e^{-\left[\frac{t_i}{\sqrt{K_i(t_j)}} + Q_i(t_j)\right]},\quad i \neq j\,,\quad i = 1,\,2. \tag{30}$$

Applying for $i = 1, 2$ and equating (30) we get

$$\frac{t_1}{\sqrt{K_1(t_2)}} + Q_1(t_2) = \frac{t_2}{\sqrt{K_2(t_1)}} + Q_2(t_1) \tag{31}$$

As $t_1 \to 0$, equation (31) becomes

$$Q_1(t_2) = \frac{t_2}{\sqrt{K_2(0)}} + Q_2(0)$$

As $t_2 \to 0$, equation (31) becomes

$$Q_2(t_1) = \frac{t_1}{\sqrt{K_1(0)}} + Q_1(0)$$

Putting the value of $Q_1(t_2)$ and $Q_2(t_1)$ in the equation (31), we get

$$\frac{t_1}{\sqrt{K_1(t_2)}} + \frac{t_2}{\sqrt{K_2(0)}} + Q_2(0) = \frac{t_2}{\sqrt{K_2(t_1)}} + \frac{t_1}{\sqrt{K_1(0)}} + Q_1(0) \tag{32}$$

Since $Q_1(0) = Q_2(0) = \bar{F}(0, 0)$, equation (31) become

$$\frac{1}{t_2\sqrt{K_1(t_2)}} - \frac{1}{t_2\sqrt{K_1(0)}} = \frac{1}{t_1\sqrt{K_2(t_1)}} - \frac{1}{t_1\sqrt{K_2(0)}} = \theta(say), \tag{33}$$

which implies

$$\frac{1}{\sqrt{K_1(t_2)}} = \lambda_1 + \theta t_2$$

Similarly, we get

$$\frac{1}{\sqrt{K_2(t_1)}} = \lambda_2 + \theta t_1,$$

where $\frac{1}{\sqrt{K_1(0)}} = \lambda_1$ and $\frac{1}{\sqrt{K_2(0)}} = \lambda_2$. Substituting these value in the equation (31), after simplification we get

$$\bar{F}(t_1, t_2) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2\right).$$

The converse part is staightforward. □

3. New class of life distributions

In this section, we define a new class of life distributions based on proposed bivariate dynamic cumulative residual Tsallis entropy (BDCRTE).

Definition 3.1. The distribution function $F(t_1, t_2)$ is said to be increasing (decreasing) in Bivariate DCRTE, denoted by IBDCRTE (DBDCRTE), if $\eta_{i\alpha}(X; t_1, t_2)$ is an increasing (decreasing) function of t_i, $i = 1, 2$.

The following theorem gives the necessary and sufficient conditions for BDCRTE to be increasing(decreasing) BDCRTE.

Theorem 3.2. *The bivariate distribution function $F(t_1, t_2)$ is increasing (decreasing) BDCRTE if and only if for all $t_1, t_2 \geq 0$.*

$$\eta_{i\alpha}(X; t_1, t_2) \geq (\leq) \frac{1}{(\alpha-1)} \left(1 - \frac{1}{\alpha\, r_i(t_1, t_2)}\right), \quad i = 1, 2 \;\forall\; \alpha > 0\,,\, \alpha \neq 1.$$

Proof. The proof of the theorem follows from equation (13) and (14). □

The following theorem provides lower bound of BDCRTE based on bivariate mean residual life function.

Theorem 3.3. *Let $X = (X_1, X_2)$ be a non-negative random vector admitting absolute continuous distribution function with respect to Lebesgue measure and $m_i(t_1, t_2), i = 1, 2$ are the components of the bivariate mean residual life function, then*

$$\eta_{i\alpha}(X; t_1, t_2) \geq \frac{1}{(\alpha-1)} \left(1 - m_i(t_1, t_2)\right), \quad i = 1, 2\,, \forall\; \alpha > 0\,,\, \alpha \neq 1.$$

Proof. We know that

$$\left(\bar{F}(t_1, t_2)\right)^{\alpha} \leq (\geq) \bar{F}(t_1, t_2), \;\; \forall\; t_i > 0, \; i = 1, 2\,, \; \alpha > 1\, (0 < \alpha < 1).$$

$$\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} dx_1 \leq (\geq) \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right) dx_1\,, \; \alpha > 1\, (0 < \alpha < 1).$$

Case 1: When $\alpha > 1$

$$\frac{1}{(\alpha-1)} \left(1 - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} dx_1\right) \geq \frac{1}{(\alpha-1)} \left(1 - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right) dx_1\right).$$

$$\eta_{1\alpha}(X; t_1, t_2) \geq \frac{1}{(\alpha-1)} \left(1 - m_1(t_1, t_2)\right).$$

Case 2: When $0 < \alpha < 1$

$$\frac{1}{(1-\alpha)} \left(\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right)^{\alpha} dx_1 - 1\right) \geq \frac{1}{(1-\alpha)} \left(\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)}\right) dx_1 - 1\right).$$

$$\eta_{1\alpha}(X; t_1, t_2) \geq \frac{1}{(\alpha-1)} \left(1 - m_1(t_1, t_2)\right).$$

Thus

$$\eta_{1\alpha}(X; t_1, t_2) \geq \frac{1}{(\alpha-1)} \left(1 - m_1(t_1, t_2)\right), \forall\; \alpha > 0\,,\, \alpha \neq 1.$$

Similar result holds for $i = 2$. Therefore, we have

$$\eta_{i\alpha}(X; t_1, t_2) \geq \frac{1}{(\alpha-1)} \left(1 - m_i(t_1, t_2)\right), \quad i = 1, 2\,, \forall\; \alpha > 0\,,\, \alpha \neq 1.$$

□

We proposed the following theorem to obtain the bivariate hazard rate ordering based on BDCRTE.

Theorem 3.4. *Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be two non-negative random vector with survival functions $\bar{F}(t_1, t_2)$ and $\bar{G}(t_1, t_2)$, and hazard rates $r_F(t_1, t_2)$ and $r_G(t_1, t_2)$ respectively. If $X \geq^{hr} Y$, that is $r_F(t_1, t_2) \leq r_G(t_1, t_2)$, then*

$$\eta_{i\alpha}(X; t_1, t_2) \ \leq \ (\geq)\, \eta_{i\alpha}(Y; t_1, t_2) \ , \forall \ \alpha > 1\, (0 < \alpha < 1), \ \ i = 1, 2.$$

Proof. We know that $r_F(t_1, t_2) \leq r_G(t_1, t_2)$ which implies $\bar{F}(t_1, t_2) \geq \bar{G}(t_1, t_2)$. Therefore, we have for all $\alpha > 0$

$$\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \ \geq \ \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1.$$

$$1 - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \ \leq \ 1 - \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1.$$

For $\alpha > 1$

$$\frac{1}{(\alpha - 1)} \left(1 \ - \int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 \right) \ \leq \ \frac{1}{(\alpha - 1)} \left(1 \ - \int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1 \right)$$

$$\eta_{1\alpha}(X; t_1, t_2) \ \leq \ \eta_{1\alpha}(Y; t_1, t_2).$$

For $0 < \alpha < 1$

$$\frac{1}{(1 - \alpha)} \left(\int_{t_1}^{\infty} \left(\frac{\bar{F}(x_1, t_2)}{\bar{F}(t_1, t_2)} \right)^{\alpha} dx_1 - 1 \right) \ \geq \ \frac{1}{(1 - \alpha)} \left(\int_{t_1}^{\infty} \left(\frac{\bar{G}(x_1, t_2)}{\bar{G}(t_1, t_2)} \right)^{\alpha} dx_1 - 1 \right)$$

$$\eta_{1\alpha}(X; t_1, t_2) \ \geq \ \eta_{1\alpha}(Y; t_1, t_2).$$

Thus,

$$\eta_{1\alpha}(X; t_1, t_2) \ \leq \ (\geq)\, \eta_{1\alpha}(Y; t_1, t_2)\, , \ \forall \ \alpha > 1 (0 < \alpha < 1)$$

Similar result holds for $i = 2$. Therefore, we have

$$\eta_{i\alpha}(X; t_1, t_2) \ \leq \ (\geq)\, \eta_{i\alpha}(Y; t_1, t_2)\, , \ \forall \ \alpha > 1 (0 < \alpha < 1), \ \ i = 1, 2.$$

□

Gupta and Sankaran [7] proposed the bivariate equilibrium distribution. Let $X = (X_1, X_2)$ be a bivariate positive random vector admitting an absolute continuous survival function $\bar{F}(x_1, x_2)$. Then its bivariate equilibrium distribution is the distribution of a random vector $Y = (Y_1, Y_2)$ such that the density function and survival function of $(Y_i | \, Y_j > t_j)$, $i, j = 1, 2. i \neq j$ are of the form:

$$g_i(t_i | Y_j > t_j) \ = \ \frac{P(X_i > t_i | X_j > t_j)}{E(X_i | X_j > t_j)}$$

$$= \frac{\bar{F}(t_1,t_2)}{\bar{F}_j(t_j)\,E(X_i|X_j>t_j)}, \;\; i\neq j;\;\; i,j=1,2 \tag{34}$$

and

$$\begin{aligned}\bar{G}_i(t_i|Y_j>t_j) &= \int_{t_i}^{\infty} g_i(u|Y_j>t_j)du \\ &= \frac{\bar{F}(t_1,t_2)\,m_i(t_1,t_2)}{\bar{F}_j(t_j)\,E(X_i|X_j>t_j)}, \;\; i\neq j;\;\; i,j=1,2, \end{aligned}\tag{35}$$

respectively, for $t_1, t_2 \geq 0$.

Remark 3.1. The residual Tsallis entropy can be expressed for the bivariate random vector $X = (X_1, X_2)$ as follows:

$$H_\alpha(X;t_1,t_2) = (\,H_{1\alpha}(X;t_1,t_2)\,,\; H_{2\alpha}(X;t_1,t_2)\,)\,,$$

where

$$H_{i\alpha}(X;t_1,t_2) = \frac{1}{(\alpha-1)}\left(1-\int_{t_i}^{\infty}\left(\frac{f_i(x_i|X_j>t_j)}{\bar{F}_i(t_i|X_j>t_j)}\right)^{\alpha}dx_i\right),\, i,j=1,2,\, i\neq j\,.$$

In the following theorem we establish a relation between BDCRTE and residual Tsallis entropy corresponding to the bivariate equilibrium random vector $Y = (Y_1, Y_2)$.

Theorem 3.5. *Let $X = (X_1, X_2)$ be a non-negative random vector and $Y = (Y_1, Y_2)$ be the equilibrium random vector associate with X, then*

$$H_{i\alpha}(Y;t_1,t_2) = \frac{\eta_{i\alpha}(X;t_1,t_2)}{m_i^{\alpha}(t_1,t_2)} + \frac{1-\,m_i^{-\alpha}(t_1,t_2)}{(\alpha-1)}, \;\; i=1,2,\; \forall\,\alpha>0\,,\,\alpha\neq 1, \tag{36}$$

where $H_{i\alpha}(Y;t_1,t_2)$ denote the bivariate residual Tsallis entropy corresponding to Y and $m_i(t_1,t_2), i = 1,2$ are the components of the bivariate mean residual life function.

Proof. When the equation (36) holds for $i = 1$, we have

$$H_{1\alpha}(Y;t_1,t_2) = \frac{1}{(\alpha-1)}\left(1-\int_{t_1}^{\infty}\left(\frac{g_1(y_1|Y_2>t_2)}{\bar{G}_1(t_1|Y_2>t_2)}\right)^{\alpha}dy_1\right).$$

Applying the results of the equations (34) and (35), we get

$$H_{1\alpha}(Y;t_1,t_2) = \frac{\eta_{1\alpha}(X;t_1,t_2)}{m_1^{\alpha}(t_1,t_2)} + \frac{1-\,m_1^{-\alpha}(t_1,t_2)}{(\alpha-1)}.$$

Similar result holds for $i = 2$. Therefore, we have

$$H_{i\alpha}(Y;t_1,t_2) \;=\; \frac{\eta_{i\alpha}(X;t_1,t_2)}{m_i^{\alpha}(t_1,t_2)} + \frac{1-\,m_i^{-\alpha}(t_1,t_2)}{(\alpha-1)},\;\; i=1,2,\;\forall\,\alpha>0\,,\,\alpha\neq 1.$$

□

Cox (1972) introduced the concept of proportional hazards model (PHM). Let X and X_θ be two continuous random variables with survival functions $\bar{F}_X(x)$ and $\bar{F}_{X_\theta}(x)$, respectively. The relation between survival functions of random life times is given by

$$\bar{F}_{X_\theta}(x) \;=\; \left[\bar{F}_X(x)\right]^{\theta},\;\; x\,\in R\;\;,\theta>0.$$

The following lemma compares the DCRTE of X and X_θ.

Lemma 3.6. *The following statement holds:*

(a) $\eta_\alpha(X_\theta;t)\;\geq\;\eta_\alpha(X;t)$ *for* $\theta\geq 1\,,\,\alpha>1$ *and* $\theta\leq 1\,,\;0<\alpha<1.$

(b) $\eta_\alpha(X_\theta;t)\;\leq\;\eta_\alpha(X;t)$ *for* $\theta\geq 1\,,\;0<\alpha<1$ *and* $\theta\leq 1\,,\,\alpha>1.$

The proof of the lemma is straightforward.

4. Conclusion

The definition of dynamic cumulative residual Tsallis entropy (DCRTE) have been extended to bivariate setup consequently proposed the Bivariate DCRTE. The monotonic behaviour in the context of bivariate random vector has also been studied. Some well known bivariate life time distributions are characterized. Additionally, we have defined some a new class of life distributions based on BDCRTE.

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