J. Appl. Math. & Informatics Vol. **35**(2017), No. 1 - 2, pp. 25 - 32
https://doi.org/10.14317/jami.2017.025

REMARKS ON THE INNER POWER OF GRAPHS†

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Abstract. Let G be a graph and k is a positive integer. Hammack and Livesay in [The inner power of a graph, Ars Math. Contemp., **3** (2010), no. 2, 193–199] introduced a new graph operation $G^{(k)}$, called the k^{th} inner power of G. In this paper, it is proved that if G is bipartite then $G^{(2)}$ has exactly three components such that one of them is bipartite and two others are isomorphic. As a consequence the edge frustration index of $G^{(2)}$ is computed based on the same values as for the original graph G. We also compute the first and second Zagreb indices and coindices of $G^{(2)}$.

AMS Mathematics Subject Classification : 05C35, 05C12, 05A20, 05C05.
Key words and phrases : Inner power, Zagreb index, edge frustration index, Zagreb coindex.

1. Introduction

All graphs in this paper are finite without multiple edges. A **graph invariant** is any function on a graph that does not depend on a labeling of its vertices. If a graph invariant has application in chemistry, it is called **topological index**. Some of these topological indices are defined by graph distance and some others by vertex degrees and so on. Among degree−based topological indices two that are known as first and second Zagreb indices are the subject of numerous papers in the chemical literature [4, 9, 10, 13].

Let G be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. For every vertex $u \in V(G)$, the edge connecting u and v is denoted by uv and $deg_G(u)$ denotes the **degree** of u in G. We will omit the subscript G when the graph is clear from the context.

Received Febuary 5, 2016. Revised November 14, 2016. Accepted November 17, 2016.
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†The research of the second author is partially supported by the University of Kashan under grant no 364988/12, and, The research of the third author is partially supported by the University of Kashan under grant no 572763/11.

The first and second Zagreb indices were originally defined as $M_1(G) = \sum_{u\in V(G)} deg(u)^2$ and $M_2(G) = \sum_{uv\in E(G)} deg(u)deg(v)$, respectively. The first Zagreb index can be also expressed as a sum over edges of G [10],

$$M_1(G) = \sum_{uv\in E(G)} [deg(u) + deg(v)].$$

The readers interested in more information on Zagreb indices can be referred to [2, 4, 7, 9, 10, 11] and references therein. The first and second Zagreb coindices of a graph G are defined as $\overline{M}_1(G) = \sum_{uv\notin E(G)}[deg(u) + deg(v)]$ and $\overline{M}_2(G) = \sum_{uv\notin E(G)} deg(u)deg(v)$, respectively.

We now state the exact definition of graph power. Given a graph G, and a positive integer k, the k^{th} inner power of G is the graph $G^{(k)}$ defined as follows:

$$\begin{aligned} V(G^{(k)}) &= \{(x_0, x_1, \cdots, x_{k-1}) \mid x_i \in V(G) \ for\ 0 \le i < k\}, \\ E(G^{(k)}) &= \{(x_0, x_1, \cdots, x_{k-1})(y_0, y_1, \cdots, y_{k-1}) \mid \\ &\quad x_i y_{i\pm1} \in E(G) \ for\ 0 \le i < k\}, \end{aligned}$$

where arithmetic on the indices is done modulo k [5].

A graph G with vertex set $V(G)$ is bipartite if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that all edges have one endpoint in V_1 and the other in V_2. The smallest number of edges that have to be deleted from a graph to obtain a bipartite spanning subgraph is called the bipartite edge frustration of G and denoted by $\varphi(G)$ [3]. It is easy to see that G is bipartite if and only if $\varphi(G) = 0$.

A graph G is called $(n, m)-$graph, if it has n vertices and m edges. Throughout this paper our notation is standard. For terms and concepts not defined here we refer the reader to any of several standard monographs such as, e.g., [6] or [8].

2. Main results

In this section some new mathematical properties of the inner power of graphs are obtained. We begin by computing some topological indices of this new proposed graph operation.

2.1. The Components of $G^{(2)}$. In this section it is proved that $G^{(2)}$ has exactly three components such that one of them is bipartite and two others are isomorphic. We first calculate the number of edges of this graph.

Lemma 2.1. *Suppose G is a simple $(n, m)-$graph and $(x, y) \in G^{(2)}$. Then $deg(x, y) = deg(x).deg(y)$ and $G^{(2)}$ is an $(n^2, 2m^2 + m)-$graph containing $2m$ loops.*

Proof. Suppose $x_1, ..., x_n$ and $y_1, ..., y_m$ are adjacent vertices of x and y, respectively. Then adjacent vertices of (x, y) are as follows:

$$\begin{array}{c}(y_1, x_1), \cdots, (y_m, x_1),\\ (y_1, x_2), \cdots, (y_m, x_2),\\ \vdots \qquad\qquad \vdots\\ (y_1, x_n), \cdots, (y_m, x_n).\end{array}$$

Therefore, $deg(x, y) = deg(x).deg(y)$. If $uv \in E(G)$ then $(u, v)(u, v)$, $(v, u)(v, u) \in E(G^{(2)})$ and so for each edge in G we have two loops in $G^{(2)}$. On the other hand, if $uv \notin E(G)$ then there is not a loop in $G^{(2)}$ containing (u, v). Therefore, $G^{(2)}$ has exactly $2m$ loops. Finally, $|E(G^{(2)})| = \frac{1}{2}\sum_{(x,y)\in V(G^{(2)})} deg(x, y) + m = 2m^2 + m$, as desired. □

Theorem 2.2. *If G is bipartite and connected then $G^{(2)}$ has exactly three components such that one of them is bipartite and two others are isomorphic.*

Proof. Suppose $x, x_i, x_{i+1} \in V(G)$ and $x_i x_{i+1} \in E(G)$. We prove that there are no paths connecting (x, x) to (x_i, x_{i+1}), (x, x) to (x_{i+1}, x_i) and (x_i, x_{i+1}) to (x_{i+1}, x_i).

(i). There exists a path $(x, x)(x_{i_1}, x_{j_1}) \cdots (x_{i_k}, x_{j_k})(x_i, x_{i+1})$ in $G^{(2)}$ connecting (x, x) to (x_i, x_{i+1}). We consider two cases that k is even or odd. We first assume that $k = 2n$. Thus, $C : xx_{j_1}x_{i_2}x_{j_3} \cdots x_{i_{2n}}x_{i+1}x_i x_{j_{2n}} \cdots x_{i_3}x_{j_2}x_{i_1}x$ is an odd cycle in G which is impossible. If $k = 2n + 1$, then $C' : xx_{j_1}x_{i_2}x_{j_3} \cdots x_{j_{2n+1}}x_i x_{i+1}x_{i_{2n+1}} \cdots x_{i_3}x_{j_2}x_{i_1}x$ is a odd cycle in G, leads to another contradiction.

(ii). *There exists a path*

$$(x, x)(x_{i_1}, x_{i_2}) \cdots (x_{i_{2n-1}}, x_{i_{2n}})(x_{i+1}, x_i)$$

in $G^{(2)}$ *connecting* (x, x) *to* (x_{i+1}, x_i). In this case, a similar argument as (i) leads to contradiction.

(iii). *There exists a path*

$$(x_i, x_{i+1})(x_{i_1}, x_{j_1}) \cdots (x_{i_k}, x_{j_k})(x_{i+1}, x_i)$$

in $G^{(2)}$ *connecting* (x_i, x_{i+1}) *to* (x_{i+1}, x_i). We consider two cases that k is even or odd. We first assume that $k = 2n$. Then the sequence

$$(x_i, x_{i+1})(x_{i_1}, x_{j_1}) \cdots (x_{i_{2n}}, x_{j_{2n}})(x_{i+1}, x_i)$$

is a path in $G^{(2)}$. Thus,

$$x_i x_{j_1} x_{i_2} x_{j_3} \cdots x_{i_{2n}} x_i$$

is a cycle of length $2n+1$ in G which is impossible. If $k = 2n+1$ then $x_i x_{j_1} x_{i_2} x_{j_3} \cdots x_{j_{2n+1}} x_{i+1} x_i$ is a cycle in G of length $2n+3$, leads to another contradiction.

This shows that $G^{(2)}$ has at least 3 components. The components of $G^{(2)}$ containing (x,x), (x_i, x_{i+1}) and (x_{i+1}, x_i) are denoted by A, B and C, respectively. Assume that (a,b) is an arbitrary vertex of $G^{(2)}$. If $b = a$ then $(a,b) \in A$ and if a and b are adjacent in G then $(a,b) \in B \cup C$. If a and b are not adjacent in G then there is a path $P : a x_1 x_2 \cdots x_r b$ connecting a to b. If r is even then $(a,b)(x_r,x_1)(x_2,x_{r-1})\cdots(x_{\frac{r}{2}}, x_{\frac{r}{2}})$ is a path in $G^{(2)}$ connecting (a,b) to $(x_{\frac{r}{2}}, x_{\frac{r}{2}})$. Since $(x_{\frac{r}{2}}, x_{\frac{r}{2}}) \in A$, $(a,b) \in A$. If r is odd then $(a,b)(x_r,x_1)(x_2,x_{r-1})\cdots(x_{\frac{r-1}{2}}, x_{\frac{r+1}{2}})$ is a path in $G^{(2)}$ connecting (a,b) to $(x_{\frac{r-1}{2}}, x_{\frac{r+1}{2}})$. Since $(x_{\frac{r-1}{2}}, x_{\frac{r+1}{2}}) \in B \cup C$, $(a,b) \in B \cup C$. This proves that $G^{(2)}$ has exactly three components.

We claim that A is bipartite and B and C are isomorphic. Suppose A has an odd cycle, say $(x,x)(x_1,y_1)(x_2,y_2)\cdots(x_{2n},y_{2n})(x,x)$. Then $x x_1 y_2 x_3 \cdots y_{2n} x$ is an odd cycle in G which is impossible. Thus, A is bipartite. Finally, $B = \{(a,b) \mid (b,a) \in C\}$ and so B and C are isomorphic. This completes the proof. □

2.2. Computing Some Topological Indices of $G^{(2)}$. The aim of this section is to compute exact formulas for the edge frustration index, the first and second Zagreb indices and the first and second Zagreb coindices of $G^{(2)}$. The degree of a vertex (u,v) in $G^{(2)}$ is defined as the number of loops and the number of edges incident to (u,v).

Theorem 2.3. *$\varphi\left(G^{(2)}\right) = \frac{1}{2}\left(M_1(G) - 4m + n - k\right)$, where k is the number of vertices of odd degrees.*

Proof. It is easy to see that the edge and vertex frustration indices of a given graph G is the summation of this number in each component of G. Apply Theorem 1 to prove that $\varphi\left(G^{(2)}\right) = 2\varphi(B)$, where B is one of the components of $G^{(2)}$ introduced in the proof of Theorem 1. Suppose $V(G) = \{x_1, \cdots, x_n\}$, where $x_1, \cdots, x_k$ have odd degree, $x_{k+1}, \cdots, x_n$ have even degree and $n_i = deg(x_i)$. Therefore,

$$\begin{aligned}
\varphi\left(G^{(2)}\right) &= 2\left(\sum_{i=1}^{k} \frac{1}{4} n_i (n_i - 2) + \sum_{i=k+1}^{n} \frac{1}{4}(n_i - 1)^2\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{k} n_i (n_i - 2) + \sum_{i=k+1}^{n} (n_i - 1)^2\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} (n_i - 1)^2 - \sum_{i=1}^{k} 1\right) \\
&= \frac{1}{2}\sum_{i=1}^{n} (n_i - 1)^2 - \frac{k}{2}
\end{aligned}$$

$$= \frac{1}{2}\left(M_1\left(G\right) - 4m + n - k\right),$$

as desired. □

Theorem 2.4. $M_1\left(G^{(2)}\right) = \left(M_1\left(G\right)\right)^2$.

Proof. By Lemma 1 and definition, we have:

$$\begin{aligned} M_1\left(G^{(2)}\right) &= \sum_{(u,v)\in V\left(G^{(2)}\right)} deg\left(u,v\right)^2 \\ &= \sum_{u,v\in V(G)} \left(deg\left(u\right) deg\left(v\right)\right)^2 \\ &= \sum_{u,v\in V(G)} deg\left(u\right)^2 deg\left(v\right)^2 \\ &= \sum_{u\in V(G)} \sum_{v\in V(G)} deg\left(u\right)^2 deg\left(v\right)^2 \\ &= \sum_{u\in V(G)} deg\left(u\right)^2 \sum_{v\in V(G)} deg\left(v\right)^2 \\ &= M_1\left(G\right) M_1\left(G\right) = \left(M_1\left(G\right)\right)^2, \end{aligned}$$

proving the result. □

Theorem 2.5. $M_2(G^{(2)}) = 2\left(M_2(G)\right)^2 - \sum_{uv\in E(G)} deg(u)^2 deg(v)^2$.

Proof. Suppose (u,v) and (u',v') are adjacent vertices of $G^{(2)}$ then vu', $uv' \in E(G)$ and (v,u), (v',u') are adjacent in $G^{(2)}$. Therefore,

$$\begin{aligned} M_2(G^{(2)}) &= \sum_{(u,v)(u',v')\in E(G^{(2)}} deg(u,v)deg(u',v') \\ &= 2 \sum_{uv'\in E(G), vu'\in E(G)} deg(u)deg(v)deg(u')deg(v') - \sum_{uv\in E(G)} deg(u)^2 deg(v)^2 \\ &= 2 \sum_{uv'\in E(G)} deg(u)deg(v') \sum_{u'v\in E(G)} deg(u')deg(v) - \sum_{uv\in E(G)} deg(u)^2 deg(v)^2 \\ &= 2\left(M_2(G)\right)^2 - \sum_{uv\in E(G)} deg(u)^2 deg(v)^2. \end{aligned}$$

This completes our argument. □

Theorem 2.6. $\overline{M_1}(G^{(2)}) = 4m^2(n^2-1) + 2M_2(G) - \left(M_1(G)\right)^2$.

Proof. By definition, $M_1(G^{(2)}) + \overline{M}_1(G^{(2)})$ is equal to:

$$\sum_{(x,y)(x',y')\in E(G^{(2)})} [deg(x,y) + deg(x',y')]$$

$$+ \sum_{(x,y)(x',y')\notin E(G^{(2)})} [deg(x,y) + deg(x',y')]$$

$$
\begin{aligned}
&= \sum_{\{(x,y),(x',y')\}\in V(G^{(2)})} [deg(x,y) + deg(x',y')] \\
&- 2\sum_{(x,y)\in V(G^{(2)})} deg(x,y) + 2\sum_{xy\in E(G)} deg(x,y) \\
&= \frac{1}{2}\sum_{(x,y),(x',y')\in V(G^{(2)})} [deg(x,y) + deg(x',y')] - 4m^2 + 2M_2(G) \\
&= \frac{1}{2}\left(\sum_{(x,y),(x',y')\in V(G^{(2)})} deg(x,y) + \sum_{(x,y),(x',y')\in V(G^{(2)})} deg(x',y')\right) \\
&- 4m^2 + 2M_2(G) \\
&= \frac{1}{2}\times 2\sum_{(x,y),(x',y')\in V(G^{(2)})} deg(x,y) - 4m^2 + 2M_2(G) \\
&= |V(G^{(2)})|\sum_{(x,y)\in V(G^{(2)})} deg(x,y) - 4m^2 + 2M_2(G) \\
&= n^2\left(\sum_{xy\in V(G)} deg(x)deg(y)\right) - 4m^2 + 2M_2(G) \\
&= 4n^2m^2 - 4m^2 + 2M_2(G).
\end{aligned}
$$

Apply Theorem 3, to complete our argument. □

Theorem 2.7. $\overline{M_2}\left(G^{(2)}\right) = \sum_{xy\in E(G)} deg\,(x)^2\, deg\,(y)^2 + 8m^4 - \frac{1}{2}\left(M_1\,(G)\right)^2 - 2\left(M_2\,(G)\right)^2$.

Proof. By definition of prime power,

$$
\begin{aligned}
M_2\left(G^{(2)}\right) + \overline{M_2}\left(G^{(2)}\right) &= \sum_{(x,y)(x',y')\in E\left(G^{(2)}\right)} deg\,(x,y)\, deg\,(x',y') \\
&+ \sum_{(x,y)(x',y')\notin E\left(G^{(2)}\right)} deg\,(x,y)\, deg\,(x',y') \\
&= \frac{1}{2}\sum_{(x,y)(x',y')\in V\left(G^{(2)}\right)} deg\,(x,y)\, deg\,(x',y') \\
&- \sum_{(x,y)\in V\left(G^{(2)}\right)} deg\,(x,y)\, deg\,(x,y) \\
&= \frac{1}{2}\sum_{(x,y)\in V\left(G^{(2)}\right)} deg\,(x,y) \sum_{(x',y')\in V\left(G^{(2)}\right)} deg\,(x',y') \\
&- \frac{1}{2}\left(M_1\,(G)\right)^2 \\
&= \frac{1}{2}\times 4\times m^2\times 4\times m^2 - \frac{1}{2}\left(M_1\,(G)\right)^2
\end{aligned}
$$

$$= \quad 8 \times m^4 - \frac{1}{2}\left(M_1\left(G\right)\right)^2.$$

Apply Theorem 4 to complete our proof. □

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